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ABSTRACT

This document contains both volumes of the proceedings of a conference of the Mathematics Education Research Group of Australasia (MERGA). Three keynote papers by Gordon Knight, J. Michael Shaughnessy, and Robyn Zevenbergen, and a practical implications paper by Lyn English are included. Seventy-five conference papers are included in these proceedings that focus on people in mathematics education, interactions among them, and the instructional materials employed. In the Short Communication Papers section there are 28 brief papers and poster sessions pertaining to the same broad topic. Topics treated in this section include graphics calculators, numeracy, use of imagery, professional development, and teaching stochastics. (DDR)

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PEOPLE IN MATHEMATICS EDUCATION

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MERGA 20
AOTEAROA

Conference Proceedings, Volume 1

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PEOPLE IN MATHEMATICS EDUCATION

Fred Biddulph
Ken Carr
(Editors)

MERGA 20
AOTEAROA

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EDITORS' NOTE

MERGA 20

The Twentieth Annual Conference of the Mathematics Education Research Group of Australasia (MERGA) is being held in Rotorua, New Zealand, 7-11 July 1997.

Welcome

The Conference Committee welcomes you to Rotorua. This is the first time the MERGA annual conference has been held in New Zealand, and we trust you will enjoy your time. The quality and range of papers is substantial. There is something in this Conference for all mathematics educators.

The conference is organized by mathematics education staff at the University of Waikato, Hamilton. We acknowledge especially the work of Andy Begg as chief organizer.

Paper selection

For the first time these proceedings include abstracts of the short communication papers and posters - there are 28 papers in this category.

The guidelines for presenting papers at MERGA were used to select papers. Two independent reviewers were used for each paper. Where there was disagreement a third reviewer was consulted.

Papers have been published largely as submitted by authors. While care has been taken by the editors, some errors inevitably occur.

We offer sincere thanks to the reviewers of papers who were all mathematics educators from New Zealand. Your speedy and thorough attention to the task was appreciated.

Conference theme

The theme 'He Tangata, he tangata, he tangata' is Maaori for 'Tis people, tis people, tis people'. The expression comes from a proverb, a translation of which appears in the Programme booklet. A parallel is drawn between a flax bush and generations of people, and this symbolises a belief in the importance of human kind as a whole.

Fred Biddulph & Ken Carr
Editors
Hamilton
New Zealand
1997

Practical Implications Award

The Practical Implications award is sponsored by the Key Centre for Teaching and Research in School Science and Mathematics at Curtin University in Perth, Western Australia. It is open to all members of MERGA and is designed to stimulate the writing of papers on research related to mathematics teaching or learning or mathematics curricula.

The successful entrant this year is:

Lyn English

Seventh-grade students fail mathematics or does mathematics fail disadvantaged students?

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... I do and I understand, and then I forget. The role of memory in Mathematics Education

Gordon Knight
Massey University

Memory has received very little attention in mathematics education literature in recent years, except in the negative connotation of rote learning. However since memory and cognition are very closely linked memory has an important part to play in many, if not most aspects of learning mathematics. The relationships between memory and understanding, memory and assessment, memory and culture, and memory and problem solving are discussed and a case is made for a much more positive role for memory in mathematics education research.

Introduction

It seems to me that the task of a keynote speaker differs from that of other presenters at a conference like this. For those presenting research papers the task is to discuss their paper with an audience who have indicated their interest in the topic and the focus is principally on the written paper. For a keynote speaker, however, the emphasis is on the oral presentation, and the challenge is to provoke, interest, and, to some extent, entertain a large group of people with differing backgrounds and interests. Not unlike teaching mathematics classes really! In almost forty years of teaching mathematics in a variety of institutions, I have never written a mathematics lesson or lecture and for this reason I have found writing this paper for publication several months before I will speak on the topic a difficult and unsatisfactory process. I will certainly not read this paper at the conference, and reserve the right to change the contents significantly both before, and during, the address. However, I understand that I must not change the topic altogether and I will try to remember that.

The human memory is fascinating and amazing. Why is it that of all the millions of stimuli which we receive each day we remember some and forget the others? Why is it that if I wish to recall someone's name I very seldom can at the time, but if I have tried, the name will pop into my head much later, often at night and always long after the information is any use to me? I have much sympathy with the Russian poet Joseph Brodsky who wrote: "More than anything, memory resembles a library in alphabetical disorder, and with no collected works by anyone".

In my, and I suspect most of your, experiences of learning mathematics at school, memory played an important and explicit part. We were sent away to learn things for homework and were tested the next day. We were given seemingly endless lists of practice exercises so that procedures became semi-automatic. I am certainly not intending to argue that this was appropriate. We now know a good deal more about the way in which people learn mathematics. However, I do propose to argue that, in our anxiety to change this approach to learning we are in danger of underestimating the role which, in my view, memory must have in mathematics education.

In recent years, consideration of the role of memory in the learning of mathematics has been almost absent in mathematics education research and literature. There is, of course, a significant body of psychological research into memory, focussing principally on laboratory studies of semantic and episodic memory (Bjorklund, 1995). Little of this seems to have influenced mathematics education research, probably because of its lack of direct relevance to the mathematics classroom. Very little recent mathematics education research mentions memory at all. In preparing this paper I reviewed the past five years entries in Zentralblatt fur Didactic der Mathematik and found just six papers out of the hundreds reviewed which addressed the topic. Most of these related to learning disabled students. I looked in the indices of all of the recently published books on Mathematics Education on the shelves of the library of my University and none of them had a reference to memory. The abstracts of the papers published in the proceedings of last year's MERGA conference (Clarkson, 1996) also contained no reference to memory at all. Finally, I looked

through the Proceedings of ICME 7 (Gaulin *et al*, 1992) and found just two references which in both cases were negative, referring to the contrast between memorisation and other, more desirable, aspects of mathematics learning.

For example, Silver (1992, p 375) wrote "... mathematics (should) be taught through activities that invite students to think, reason, explain, and justify, rather than simply to memorise and imitate". I would, of course, have no quarrel with this statement at all. However, it is clear to me that memory does have a part to play in thinking, in reasoning, in explaining, and in justifying. In our anxiety to avoid the dangers of rote learning, we have, I believe, forgotten the essential role which memory plays in all cognitive activity.

Because of this critical relationship between memory and cognition, I believe that memory is an important aspect of many, if not most, facets of matheamtics education. For example, the initial relationship to constructivism is clear; children construct their own memories (Piaget and Inhelder) and we might well be interested in activities which are related to the construction process. There is also an obvious connection with assessment issues. It is in this context that much of the negative connotation of memorisation occurs. There are also, I believe, relationships to the general concepts of knowledge and learning, to metacognition, to motivation, to issues of culture and gender, and to problem solving and investigations. In this presentation, I would like to explore just one or two of these relationships and to make a general case for a more significant consideration of memory issues in mathematics education research.

Memory and Understanding

Understanding has been a major topic in the mathematics education literature for more than twenty years, much of the discussion having its origins in the work by Skemp (1976) on instrumental and relational understanding. The latest contribution is the substantial book by Anna Sierpinska (1994) entitled 'Understanding in Mathematics'. From the point of view of this presentation, it is significant that in the index of the book there is no reference to memory at all. Within the book itself the only mention I could find of memory were two rather dismissive references (p 7 and 70) to a cognitive science approach to understanding. Sierpinska writes:

> Among the many views of understanding, there is one which identifies an act of understanding with a retrieval of a 'frame' or 'script' from memory, sometimes called the 'computer metaphor approach'. (p 7)

> There are several deceiving aspects of this approach. One is that it represents the functioning of the human mind as mechanical, automatical... (p 70)

There is clearly more to understanding than retrieval, but it is this classification of anything to do with memory as mechanical and automatic, and consequently undesirable, which is at the heart of the rejection of memory as an important aspect of mathematics education.

It seems to me that implicit in much of the recent mathematics education literature are the following assumption concerning understanding:

- Understanding is permanent. Once you have understood something you will always understand it.
- If you understand something you will be able to remember it.

My own experience of learning mathematics and my observations of students learning mathematics at many levels lead me to reject both of these assumptions.

As an undergraduate student I learnt a great deal of mathematics and understood at least some of it quite well. A large part of that mathematics I have not used for more than forty years and it is now well and truly forgotten and, given any sensible definition of understanding, I certainly no longer understand it. Understanding is not permanent. There is no contradiction, in my view, in saying "I used to understand that". There have also been many times when I have been engaged in a mathematical task and I have been

completely unable to recall a fact, formula or procedure which I both understood when I learnt it and understand when I am reminded. Understanding is no guarantee of recall.

The same applies to my students. One of the most frustrating features of helping individual students with their problems arises not with students who have misunderstood the concept or procedure involved, at least then you have something to work on, but with those who show that they do understand the right procedure but continue to use a previously learned wrong procedure.

What then is the relationship between memory and understanding? Do we remember things differently if we understand them? Is there a step which we should be encouraging students to take after they understand so that they will remember? Are there activities which would promote both understanding and memory? All of these seem to me to be the basis for good research questions.

Memory and assessment

The relationship between memory and traditional forms of assessment by examination is obvious. It is the excesses of rote learning, imitation, and regurgitation associated with examinations which has led to the rejection of memorising as a valid activity in mathematics education. However, it is not possible to remove memory from assessment tasks altogether, even if this were seen as desirable.

Firstly, there are some features of mathematics which are matters of symbolism or definition which have to be remembered rather than understood. One simply has to learn that 5!! is the symbol for the product 5x3x1. There is no logic, or understanding, involved. A logical response would be extreme surprise that it was a five. The number words and symbols themselves are in this same category. How do we learn that the word eight or the symbol 8 is associated with certain sets of objects if not by rote? Having learnt this we can, of course, understand that, and why, 8 comes between 7 and 9. If this is the case it seems to me to be essential to assess whether or not the student can remember what the symbol 8 represents.The testing of recall is entirely appropriate in some circumstances.

Secondly, in order to reduce the emphasis on memory in traditional mathematics examinations, it has become common to provide information for students to use during the examination. This may range from a list of formulae to a completely open book examination where students may take in any reference material they wish. Do we really understand the effect of this on student learning? I have certainly not seen any research in this area. I do however have colleagues whose opinion I respect who are concerned that the message we are giving to students is that it is not necessary, and is perhaps even undesirable, to commit formulae and procedures to memory. My colleagues feel that this has a significant adverse effect on future learning. Are they right?

Thirdly, if we abandon traditional forms of assessment, there is still an important memory component. Ken Carr (1994) suggests interviewing, observation, student portfolios, self assessment, and cooperative assessment as alternative assessment procedures for teachers. At least three of these have important links to memory. In interviewing, or observing, a student one immediately becomes aware of what the student can remember. In self assessment, with its obvious links with metacognition, one would hope that the students themselves would become aware of what they can remember and to be able to reflect on that. The area of metamemory (Bjorkland, 1995, p 248), for knowledge of the workings of one's own memory, seems to me to be well worth researching in mathematics education.

Memory and Culture

There are significant differences in the value which cultures place on memory. For example, since Maori culture was originally oral, without a written language, the ability to remember and repeat information was essential to the preservation of the culture and was consequently highly valued. The ability to recite one's whakapapa, or geneology, for many generations and to be able to link oneself to others in this way remains an important feature of Maori culture. Many Asian cultures also seem to place a

high value on memory, at least as far as education is concerned. Asian students are often characterised as rote learners. Wong (1996), in discussing the performance of Asian students, states that "lessons are structured as one continuous practice session for public examinations" and suggests that the origin of this examination culture lies in Confucian values and view of life.

What do we say to the Maori students and the significant, and growing, number of Asian students in New Zealand? Your values are wrong and we know best? All the lessons of Maori education for the past one hundred and fifty years indicate the futility and immorality of this approach. Do we need to reject their skills of practice and memorisation? Could we not, as Wong suggests, "Explore the Asian style of understanding through repetitive learning"? Again there is ample scope for research.

Memory and problem solving

The bad name which memory has in mathematics education is also to be found in writings on problem solving. Thomas and Kota (1996) writing on problem solving in algebra write:

> Brown et al (1988) expressed the opinion that students appear to cover their inability to translate the relationship between the words into symbolic form or to solve the resulting equations resorting to memorizing rules and procedures that they eventually come to believe represent the essence of algebra. (p 564)

They are right, of course. Memorization of rules and procedures is not the essence of algebra. However it is equally true that if we cannot remember any algebraic rules or procedures we are, it seems, to me most unlikely to be able to identify the relationship between the words of the statement of a problem and a symbolic form. Have we again thrown out the baby with the bath water?

What is the relationship between problem solving ability and memory? Certainly, memorization is no guarantee of problem solving ability, but without some recall we cannot begin to solve problems. Is there a qualitative difference between the recall ability of good and poor problem solvers? Are there activities which we could devise which would help students in this area?

Summary

I could continue and consider memory and motivation, memory and gender issues, etc., but I hope I have made what is a simple, but important, point. In recent research in mathematics education we have forgotten to include memory as a factor except in a negative way. Perhaps we need some new terms to enable us to distinguish between the positive and negative aspects of memory. A term for 'memory with understanding' might be helpful, and another to contrast 'memorization' (with its connotations of rote learning) with the more positive aim of using memory to provide access to our understanding.

In his book 'Forty year's on' (1969), Alan Bennet describes how Rumpole's teacher proclaimed that: "Education is what is left after you have forgotten all that you have ever learnt". The teacher went on to suggest that Rumpole was seeking to circumvent this process by learning as little as possible. Perhaps we, like Rumple, are also in danger of circumventing the process by not expecting our students to remember anything so that they have nothing to forget.

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Missed opportunities in research on the teaching and learning of data and chance

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This paper and the accompanying keynote talk for the 20th annual MERGA conference are dedicated to the memory of my friend and colleague Georg Schrage, August 9, 1940 - February 11, 1997.

Reflecting on a body of research work can sometimes lead to the recognition of areas of opportunity for research that have gone largely unnoticed. In this paper we consider three such opportunities in the area of research on the teaching and learning of probability and statistics: i) Following up on students' initial thinking to watch for future transitions; ii) Investigating students' thinking on variability; and iii) Posing research questions that begin with what students *can* do rather than pointing out what they *cannot* do. Situations from research tasks, past and future, are used as starting points the discussion.

Introduction

It is a great honor that you have invited me to give this address at the 20th MERGA conference. I hope very much that my thoughts and words today might plant some seeds for future research in the area of data and chance, seeds that hopefully will enable us to improve the teaching and learning of probability and statistics at various levels of education.

The area of probability and statistics within mathematics education, or stochastics as some of our European colleagues refer to it, has fascinated me ever since I first had the opportunity to teach probability to beginning college students. There are two things that have fascinated me most in my twenty odd year journey with students in stochastic reasoning. One is finding out what students think or believe about chance and data. How do they reason in situations that involve chance? The second involves documenting student growth or change as they are interacting with probability or data tasks. In reflecting on the current state of affairs of our research efforts in probability and statistics, I find that we have made a good deal of progress in the first arena--uncovering students' conceptions and beliefs about chance and data. However, we have made very little progress in the second arena, the documentation of student growth and change as they interact with chance and data tasks or curriculum materials. I believe this is a missed opportunity in our research.

In preparing for this talk, and reflecting on some of the recent literature in research on probability and statistics, there were several other themes that emerged as opportunities that we researchers in the area of chance and data may be missing . These themes became nagging ones and kept returning to me as I pondered the current research situation in our field. In particular, three themes strike me as missed, golden opportunities, and so I would like to bring them to your attention. One we have already mentioned above i) following student's thinking as they interact with materials. I was also struck by ii) the great absence of research on students' thinking about variability--where is it anyway? and, iii) the possibility of starting to do research based on what students *can* do rather than on documenting what they *cannot* do. And so, as fate would have it, I have chosen to try to plant some seeds today in this ground: a closer look at several of these "missed opportunities."

Some missed opportunities

For me it is helpful to think about these missed opportunities by starting from the context of several tasks, or situations, that have been used in research in probability and statistics.

Situation 1--Following up on Students' Thinking

In teaching probability and statistics we must not only deal with helping students to develop an understanding of and to apply some difficult mathematical and statistical ideas, but we must also deal with the psychological issues involving chance and data that can be deeply rooted in students' experiences or in their beliefs about chance phenomena. Sometimes the mathematical issues and the psychological issues are quite confounded, which makes our job all the more difficult. For example, consider the situation of drawing a sample of people from a large city in the U.S. to give their opinions on whether motorcyclists should be required to wear helmets for protective head gear. If we do a proper job of drawing our sample so that it is "representative" of the community, so that it doesn't have obvious bias, and so that it is as random as possible, we'd expect the opinion results from our sample of people to mirror the whole population, and thus to give us some realistic notion of the community's pulse on the wearing of helmets. This concept of a representative sample is a very powerful, and necessary one in the area of statistical decision making. On the other hand, consider the following version of an oft used task in research on probability and statistics.

> A fair coin is flipped 5 times in succession. Which do you feel is more likely to occur for the five flips?
>
> A) HTTHT B) HHHHH C) they have the same chance of happening.

Here are some responses to this task which came from a recent convenience sample of middle school mathematics teachers before they had a class in probability and statistics.

"I would go with A) only because it more closely approximates the ratio of 50-50, but in such a small *sample* anything is possible."

"I would say both are equally likely on any particular instance, although the long term results would, gravitate to a result more like A).

" To get either sequence you'd have to multiply the probability of each being a Head or Tail (1/2) times each (of the) other toss(es), so (1/2)^5 for both."

"...(After circling C--then circles A) I changed my mind again. The chances are 50/50 no matter what, and sequence A is more likely."

" A) because I think it would be more likely to have a series of two of the same than to have five of the same."

"B) because A) is too *specific."*

"C). It is a 50-50 chance for either an H or a T, so either sequence."

The responses from these teachers are very similar to the types of responses that have been found among middle school, secondary, and college students, and which have been well documented (Fischbein, 1996; Konold, 1983; Konold et. al., 1993; Shaughnessy, 1977, 1981). They exhibit a great variety of beliefs, conceptions, and interpretations of the problem. The notion of a "representative" sample that is so helpful in the helmet survey can cause problems when applied in this context. There is no "sample" in the above question, there are just several outcomes from five flips. And yet

some of my subjects felt that the "representative" notion should apply here, too, when they said "A) more closely approximates the 50-50 ratio" or "(we are)...more likely to have a series of two of the same, than to have five of the same."

Several of the responders even tried to superimpose the idea of sample on the problem. "In a small *sample* anything is possible" and "the *long term* results gravitate toward A)," as if there were an ongoing sample. Still other responses indicate what Cliff Konold has described as the *outcome approach* (Konold, 1989; Serrano, 1996). "I would say that both are equally likely in a particular instance..." and "it is a 50-50 chance for and H or a T so could be either (outcome)." Students may be just trying to predict what the next sequence would be if we tossed five coins, rather than considering these two sequences as only two among many possibilities. In fact, the question is framed in a way that might actually lead a subject to focus on single outcomes, rather than a range of possible outcomes. One of the teachers did use some knowledge about probability and did some calculations that indicate recognition of a whole range of outcomes, but was an exception.

So, here we have an example of a well documented misconception about the likelihood that certain binomial sequences will occur in a probability experiment. Now what? What are our next steps? It seems in research on probability and statistics, we have developed a tradition of putting tasks to students and documenting the types of answers and responses they give, classifying various beliefs or conceptions about chance, randomness. However, I think that all too often we stop at that point, describe what we have found, and lament the state of affairs and the difficulty students have with chance concepts. For us as mathematics educators, the variety of responses to the question above can be a good starting point for research, not just an ending point in itself. Here we have a golden opportunity to place students into an environment to investigate this type of problem over time, and to document any growth and changes in their thinking. It seems that we don't often use the fruits of our research efforts to spin off further investigations. In this we may be missing some golden opportunities to ask questions that can show what our students are capable of, rather than simply documenting what they are incapable of.

Suppose we took the point of view that our students' conceptions are *"transitional conceptions,* rather than *mis*-conceptions, that their thinking is always under construction. Where could we go from here with our questioning, what types of tasks might we pose for students, to help them begin transitioning in their thinking about binomial situations? This may be a slightly different way for some of us to think about research. The nature of evidence in our research on learning need not be static, it can be dynamic. We have an opportunity to do research on the *process* of learning.

Here is an example (Figure 1) of a visual environment for binomial experiments that I have used as a point of departure for investigating student's growth in thinking. It is an activity taken from the *Math and the Mind's Eye* series (Shaughnessy & Arcidiacono, 1993). The *Math and the Mind's Eye* series is the parent project that has led to the development of a middle school mathematics curriculum, based on visual thinking, that is currently underway in Portland, Oregon (Foreman and Bennett, 1995, 1996).

In this checkerboard environment, we place a marker on the start square. As we flip a coin, an H indicates a diagonal move on the checkerboard one square up and left, and a T a move up and right, so that we stay on the shaded squares. I generally ask students to play this game several times, flipping a coin five times and moving the marker so that they can see the path of the marker up the board as they record the string of five H's and T's. In this way, sequences of tosses can be visualized on the checkerboard as "paths" up the board. For example, in Figure 2 the path HTTHT is drawn in on the board, as is the path HHHHH.

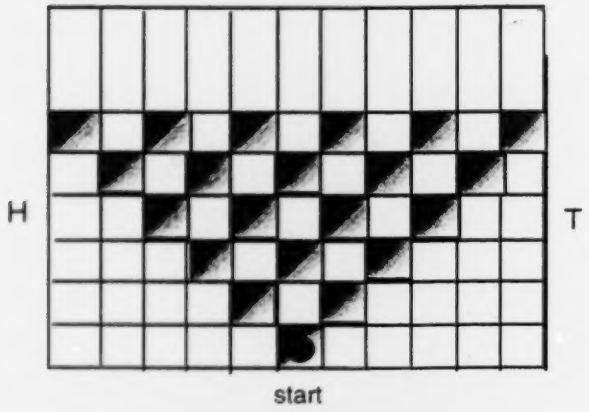


Figure 1

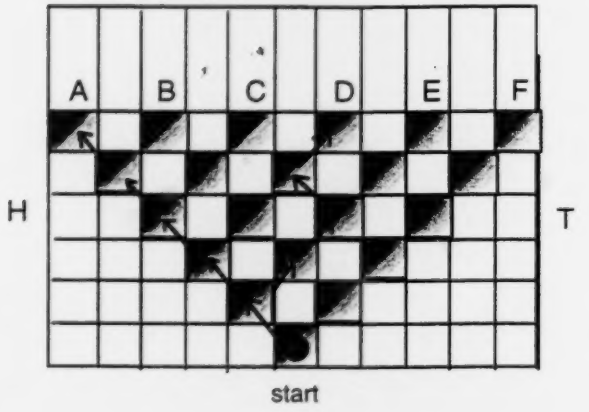


Figure 2

Here are examples of questions that I have used to investigate the development of transitional conceptions in students' thinking about binomial experiments.

1) How many paths can you find that correspond to HHHHH? How many correspond to HTTHT?

2) Altogether, how many different paths are possible on the checkerboard for five tosses?

3) How many different paths lead to each of the possible six finishing areas A - F?

As students work through this visual environment, some things inevitably come up as they discuss and analyze the game. Students' have:

a) At first predicted equal #'s for the finishing places , then, changed their mind because they see that there are more paths that lead to some finishing places than to others as they are playing it.

b) Made connections between the paths and the sequences of H's and T's. Students can draw in all possible paths (there are 32 of them) which corresponds to listing all the possible sequences of 5 H's and T's.

c) Built their own theoretical model for the chance of a given outcome, like HTTHT, by counting paths. In doing this, they often mention that their model does depend on each path having an equal chance of occurring, but this seems reasonable to assume when using a fair coin. The branching odds are 50-50 at each juncture.

d) Made connections between the number of paths and the numbers in Pascal's triangle (Figure 3).

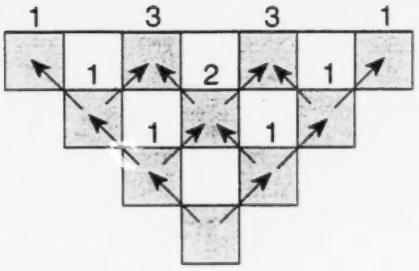


Figure 3

This particular visual model is quite powerful as a teaching tool as well as for use in a research environment, because it can be adopted for unequally likely branching, and for any path length. In other words, the two parameters in the binomial distribution, the number of trials "n", and the probability of success on a given trial "p", can be represented by the path length and the branching probabilities respectively. Students can *see* how HTTHT and HHHHH fit into the entire spectrum of possible outcomes. As researchers we can observe and document their thinking in transition. Growth and changes over time in students' thinking can also be documented using an approach like that of Watson, Collis & Moritz (1995). They investigated students' responses over grades 3, 5, 7, & 9 on various chance and data tasks and noted differences across ages. However, there are also opportunities for us to trace individual students or groups of students over time as they interact with problems and tasks. We do take advantage of these types of opportunities very often in our research, although recently some case studies of younger students engaged in probability tasks has been conducted by a group at Illinois State University (Jones et. al., 1997).

Situation 2--Where is the research on variability?

The two big concepts in the teaching and learning of data are central tendency and variability. Although there have been investigations into students' concepts and beliefs about "averages"(Russell and Mokros, 1996; Pollatsek et. al. 1981), there does not seem to be a similar tradition of research into students' ideas about variability or spread. One reason for the lack of research on variability, I believe, is that our research often mirrors the emphasis in our curricular materials, at least this seems to be the case in the area of chance and data. While there are a variety of models for centers (means, medians, modes) that have been developed and tried with students, there is not a corresponding rich array of models for spread or variability. For example, in research on averages, Russell and Mokros (1996) mention three different models of the mean that they have used to investigate elementary students' thinking about the mean--a fair divisions model, a center of balance model, and a dynamic distribution model in which students work backward from the mean to recreate various possibilities for the original distribution of the data. Russell and Mokros point out that each of these models has some limitations when used to introduce the concept of mean to children. Still another model that we have used for the mean (Figures 4 and 5) involves forming a stack of cubes to represent each data point, and then leveling off the stacks by making trades to find the mean (Foreman and Bennett, 1995).

This model works equally well when asking students to find means from distributions, or when asking them to reconstruct distributions from means--the converse problem that Russell and Mokros investigated. The cube columns can be represented on centimeter grid paper, and cubes can be "sliced" to represent trades that involve rational numbers. We are rich in models for centers in our curriculum materials and they have carried over into our research tasks, but I find us lacking in similar creative models or approaches to concepts of variability and spread.

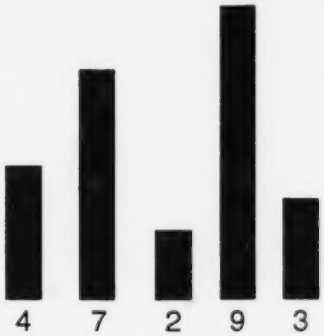


Figure 4

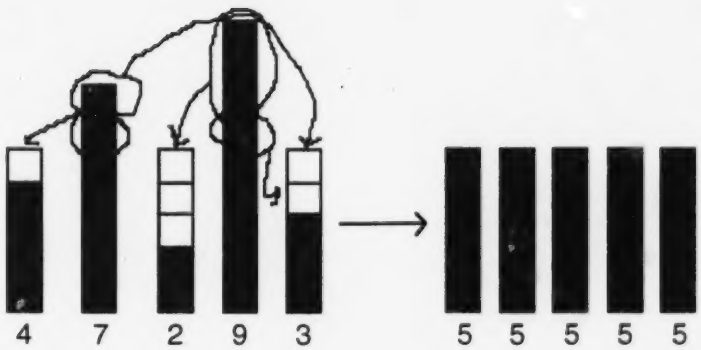


Figure 5

Batanero et. al. (1994) make this same point about the lack of research on variability, and they mention an article by Loosen et. al. (1985) which suggests that textbooks in statistics seem to put more emphasis on looking at heterogeneity in data (centers) rather than on variability in data (spreads). (Note: there is some preliminary work being undertaken by Watson & Moritz (1997) in which they solicit students' thoughts about variability by asking them to compare and analyze various hypothetical distributions of student test scores).

I have my own suspicions as to why variability has not received the attention that measures of central tendency have in our research. One reason is that statisticians have traditionally been very enamored with the standard deviation as *the* measure of spread or variability, and teachers and curriculum developers often avoided dealing with spread because they felt they couldn't do so without introducing standard deviation. Standard deviation is not only computationally messy, but difficult to motivate why it is a good choice for measuring spread, especially with beginning students. Another reason for lack of attention to variability is that centers or averages are often used to predict what *will* happen in the future, or to compare two different groups-- not always used correctly in this regard of course but nevertheless used. The incorporation of spreads or variation into these predicting or comparing processes only confounds people's ability to make clean predictions or comparisons. You see, where there is variability or spread, there must also be some waffling in our predictions or comparisons, and predictors do not like to waffle (unless of course they are politicians). Furthermore, this whole concept of variability is outside of many people's comfort zone, and may even outside their zone of belief. Jim Landwehr (1989) nicely summarized people's over reliance on means and their beliefs about variability when he noted that people

* believe that any difference in means between two groups is significant

* have unwarranted confidence in small samples

* have insufficient respect for small differences in large samples

* mistakenly believe there is no variability in the 'real world.'

Let us again look at an example, this time we consider some research evidence about peoples' intuitive conceptions of variability. Here is a version of another task that has often been used. This particular version was reported by Schrage (1983).

What would you think is more likely to occur?

A) 7 of 10 babies born are males?

B) 70 of 100 babies born are males?

C) These two have the same chance of occurring?

Here are some typical responses to this task, taken once again from the sample of middle school mathematics teachers:

C) Because the chance of getting boys or girls is the same in both situations, so both choice A) and choice B) have the same chance of occurring

C) The ratio of 7 out of 10 is the same as 70 out of 100, so it makes no difference, they are the same.

B) With 100 babies you will have a lot more chances of having males.

A) With less babies it seems more likely to be further away from 50-50.

A) The more babies you have, the closer you should come to a 50-50 split.

As in our first example we can find a variety of beliefs and interpretations of this problem. Some of the interpretations (1 and 2) focus on predicting from averages or ratios--using a 50:50 ratio of boys to girls or from the sample proportion which is .7 for both choices A and B. On the other hand, several of the responses indicate that there is an awareness that smaller samples can deviate from expectations, or conversely, that larger ones should be closer to expectations (4 & 5). Schrage (1983) reports that although 60% of his sample of 153 pre service mathematics teachers chose response A, his results are misleading because in his follow up questions most of them gave incorrect reasons for this correct response, indicating that they were totally ignoring the effect of sample size in the problem.

One difficulty I see with this item, and ones similar to it, is that it actually focuses the attention of the responders on centers, while dealing with the concept of spread. The problem is (implicitly at least) asking for the likelihood of a particular outcome (7 out of 10 boys vs. 70 out of 100 boys) rather than asking what the likely range of outcomes for having 10 children, or 100 children might be. If we want to know what our students *can* do with variability, I think we have to ask a different type of question here. I think we need to pose questions on variability that can be answered in a sampling context in which the concept of an "interval of likely values" arises, rather than in a context that forces people to compare the point value probabilities of two particular events. Here is an example of a task on variability that I have used with students.

Imagine you have a huge jar of M&M's with many different colors in it. We know that the manufacturer of M&M's puts in 40% browns. If you reached in and pulled samples of 20 M&M's at a time, what do you think would be the likely range for the number of browns you found in your samples?

Responses I have received to this question vary from 7-9 browns up to 5-12 browns. But no one yet has said "you will get 8 browns every time." So, the idea of a range of likely values in a sampling situation where the population proportion is known (40% in this case) is very accessible to students, what Fischbein might call a primary intuition (Fischbein, 1987). This can be a point of departure for further research on variability. What will students say if we change this question to samples of 100 M&M's? Then what will they say is the likely range for the browns? Nowadays it is very easy to actually "draw samples of 100 M&M's" by introducing a simulation of the M&M's experiment using some sort of sample-resample software. I find the little software package Prob-Sim (Konold & Miller, 1994) very useful for generating samples of this kind. The results from using Prob-Sim to draw 100 samples of 10 "babies" and 100 samples of 100 babies are presented in Figures 6a and 6b.

Key: Column totals indicate the number of times that n (horizontal axis) boys in 10 babies occurred.

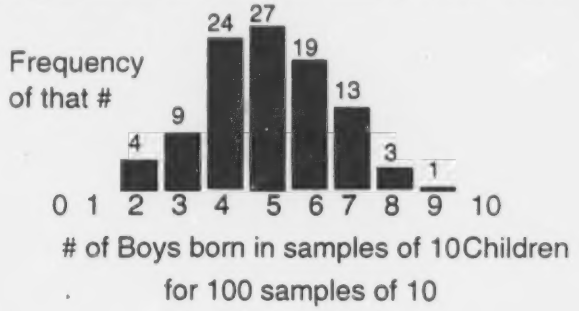


Figure 6a

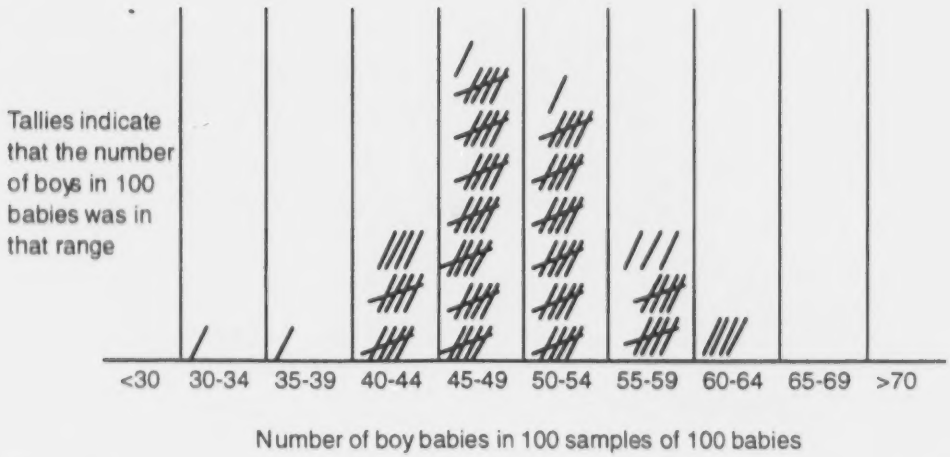


Figure 6b

Notice that there were 13 instances of 7 males born in 10 babies, however the outcome 70 (or even more) males in 100 babies did not ever occur in any of our samples of 100 babies.

Based on such resampling results students can begin to gain intuition on what the likely spread of outcomes is for a sample from a population. They can begin to gain some sense of what is likely and what is unlikely to occur by considering the entire distribution of outcomes. They can begin to develop some sense of the affect of sample size on the spread of likely outcomes. Neither a formal quantification of "spread" nor formal definitions for variability need to be used in order for us to begin to investigate students' conceptions of variability. Furthermore, if we begin our research investigations on variability based on something that students *can* do (like estimating

confidence intervals for the M&M's problem), rather than something they *cannot* do (like comparing binomial probabilities in the baby problem) we may find that research on variability is an untapped well in research on data and chance. (By the way, the baby problem and the M&M problem can also be modeled in the visual checkerboard environment, since they are binomial problems. However, building a 20 X 20 or a 100 x 100 checkerboard is a little inconvenient. On the other hand, if students have previously worked in the checkerboard environment, they can visualize it in this situation).

I'd like to follow up more on the notion of starting our investigations into students' thinking on data and chance by looking at what our students *can* do rather than concentrating on what they stumble on.

Situation 3--Unveiling what our students* can *do

At the moment there seems to be bit of a boom in research into the teaching and learning of probability and statistics. At least one gets that impression when browsing the proceedings of a number of recent international meetings on research in mathematics and statistics education (Garfield, 1994; Puig and Gutiérrez, 1996; Phillips, 1996; Batanero, 1996). The proceedings of these meetings depict an ever growing body of evidence that deals with students conceptions and beliefs about data and chance. (In fact, there is so much research reported now in the Newsletter for the International Group for Research in Probability and Statistics— begun by Joan Garfield at the University of Minnesota and currently edited by Carmen Batanero at The University of Granada— that it is becoming very difficult to keep abreast of research developments around the world).

Although most of this research is being conducted by mathematics and statistics educators, a good deal of the theoretical underpinnings for the research into peoples' conceptions of data and chance is based on a methodology from cognitive psychology, particularly from the area of the psychology of decision making (Kahneman & Tversky 1972, 1973; Kahneman, Slovic, & Tversky, 1982; Nisbett & Ross, 1980). There have been several past reviews of the literature which point to a large body of evidence in the area of students' conceptions and beliefs about probability and statistics, and to the psychological roots of this particular line of research (Hawkins & Kapadia, 1984; Garfield & Ahlgren, 1988; Shaughnessy, 1992; Borovcnik & Peard, 1996; Shaughnessy, Garfield, & Greer, 1996). Recently a group of psychologists at the University of Chicago has been challenging some of the claims of the earlier psychological research, and finding better "success" with students when questions are posed in terms of frequencies (Gigerenzer, 1994, 1996; Gigerenzer & Hoffrage, 1995). However, the basic methodology has remained the same: pose cognitive tasks to students, record successes and failures, but do little, if any, follow-up on their thinking, and rarely any following of the development of students' thinking over an extended period of time.

In adopting the research tasks and methodology of our brethren in psychology, we have been able to document that our students have certain beliefs and conceptions about probability and statistics that can lead them to incorrect decisions or erroneous analyses of problems. We have been able to characterize some of these erroneous conceptions. We have been able to do an excellent job of documenting what our students *cannot* do in the areas of chance and data. We have not done a very good job documenting what our students *can* do. We, like our compatriots in psychology, tend to ask research questions that will almost certainly expose the pitfalls in our students' thinking. We do not often put students in situations that will give us an opportunity to document potential growth or possible change in their thinking.

I suggest we adopt a somewhat different methodology, that we put students into open ended situations, that we ask them to try and make sense out of data, to generate their own questions about data, to design their own graphical representations of data, to in fact even create their own "measures" on data. This is a somewhat different paradigm for research on the teaching and learning of data and chance than we have taken in the past, and may even have some aspects similar to the Soviet teaching experiments.

I would like to share a couple of tasks that I have used with some middle school students in the past several years which give them an opportunity to explore data. My job

is to watch what they do, and to listen, and occasionally to ask a question. The group of middle school students I worked with is a combined grade 6-7 class in a school located on the outskirts of Portland, Oregon. The class is an experimental class that has been working through the *Visual Mathematics* curriculum as it is being developed (Foreman & Bennett, 1995, 1996).

The students are each given the entire weather page for one day taken from the local newspaper. (In many cities in the U.S., there is a special multicolored page for the weather, complete with many types data sets, graphs and charts, and containing both a national and a regional weather map). The students actually have a set of about 25-30 consecutive days of this weather map among them, with each student having one day's weather summary. The maps are passed out so that students have consecutive days in their small groups of 4-5 students. The task begins with two writing stems, "We notice that...." and "We wonder about....", and asks students to make lists under each stem. After a while, the groups are asked to share and discuss what they have written in their lists. Here are some of the responses I obtained from this group of students.

We notice that:
* the weather page tells weather from all over the world
* a hurricane died down on the day we investigated
* pictures and maps explains things better (than words)
* they predicted rain 3/5 of the time on the 5 day forecast on our groups pages
* the weather pages show information from the past for Portland

We wonder:
*why Medford has the highest air pollution
*what the zeros indicate in the white column
* if the east coast always has more severe weather
* where the maps of Alaska and Hawaii are
* how it can be that the high temperature is in the morning

Of course, not all the "notices" or "wonders" that the students come up with are worthy of an effort at exploration. But several of the "wonders" that the students came up with proved quite intriguing and were investigated by groups of students in great detail. Two of these were subsequently written into the materials in the program as possibilities for teachers to use (Foreman & Bennett, 1996, p. 265).

1) Is the coldest time of day more likely to occur before or after sunrise?

2) How well do the weather folk do at predicting Hi and Lo temperatures five days in advance?

To collect data on the first of these questions, students needed to extract information for the time of day for the lowest temperature from the time-temp graphs given for each day on the weather pages (Figure 7). They also needed to check on the time of sunrise, which they decided was around 7:00 A.M. for the days (in October) they were investigating.

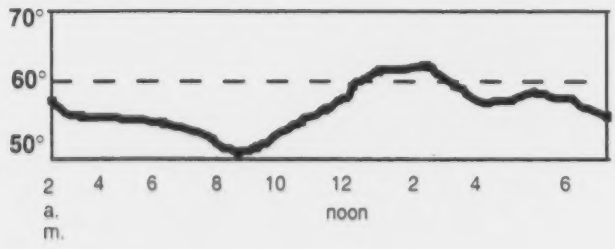


Figure 7

Some of the students represented the information for the time of day for the low temperature in a type of line plot (Figure 8).

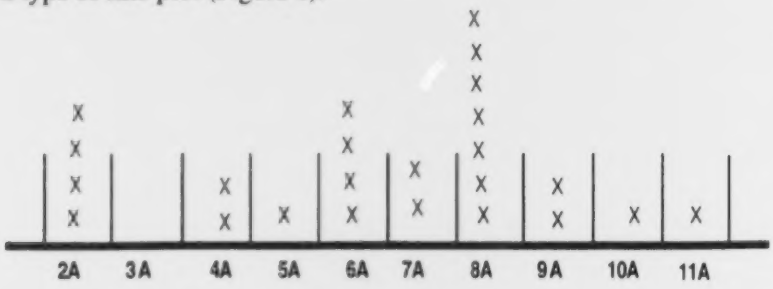


Figure 8. Time of day for the low temperature

These students were surprised that the time of the low temperature for the day occurred *after* sunrise nearly as frequently as it did *before* sunrise. Quite a discussion ensued as to why there was such variability for the coldest time of day.

The second of these questions above led students to gathering and comparing the real and the predicted temperature data. On each day, predictions are given in the weather pages for the next five consecutive days in graphics that include predicted Hi's and Lo's, and predicted weather conditions (e.g. rain, clouds, sun, etc.). The students in this sample are quite visually oriented, since the mathematics curriculum they have been working on emphasizes visual models and visual representations of ideas. So, it was no surprise that they represented the information for this problem in visual displays. Here are some examples of how students represented the information on predicted temperatures vs. actual temperatures (Figures 9, 10).

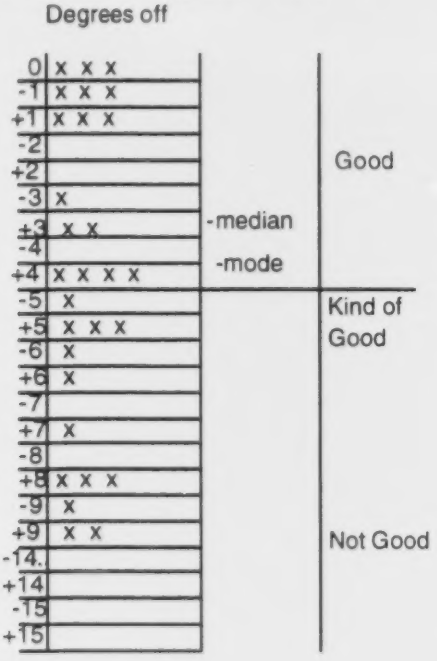


Figure 9

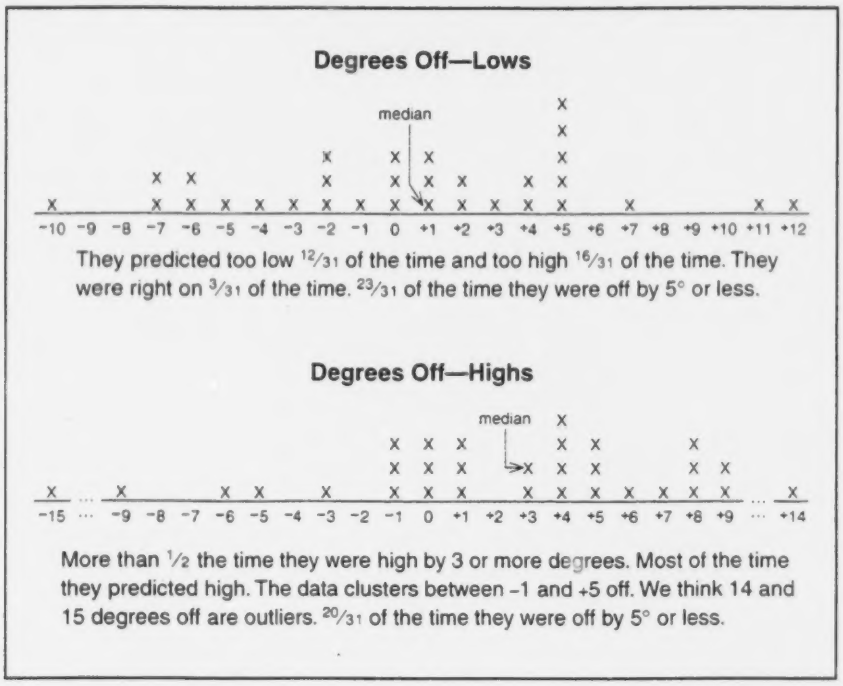

Degrees Off—Lows
median
−10 −9 −8 −7 −6 −5 −4 −3 −2 −1 0 +1 +2 +3 +4 +5 +6 +7 +8 +9 +10 +11 +12
They predicted too low 12/31 of the time and too high 16/31 of the time. They were right on 3/31 of the time. 23/31 of the time they were off by 5° or less.
Degrees Off—Highs
median
−15 ··· −9 −8 −7 −6 −5 −4 −3 −2 −1 0 +1 +2 +3 +4 +5 +6 +7 +8 +9 ··· +14
More than ½ the time they were high by 3 or more degrees. Most of the time they predicted high. The data clusters between −1 and +5 off. We think 14 and 15 degrees off are outliers. 20/31 of the time they were off by 5° or less.

Figure 10

After the students had generated their own representations of the data they were shown how to represent data in a scatter plot. Although the students had some experience plotting points on co-ordinate graphs, this was their first experience representing data in a scatter plot. While they were making this scatter plot, several student-generated graphical measures were created by different groups to "measure" what they meant by a "good prediction" for the five day temperature predictions. First, several students announced that the predictions were exactly correct only when the points landed on the line of equal temperatures--their own creation. This line became known as the "J. Alex" line, after one of its creators (Figure 11).

Student Plot with "J. Alex" line

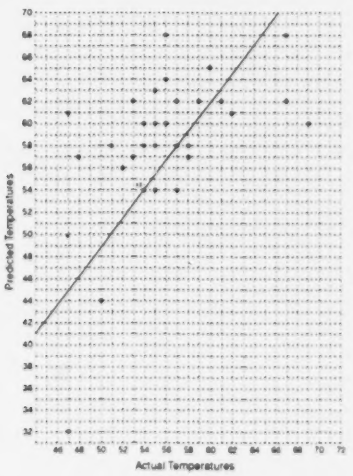


Figure 11

Second, several other students created confidence band lines (they didn't call them this, of course) around the J. Alex line and decided that for this data, a "good prediction" meant that the points had to fall within these ±5 degree bands. These lines became known as the "Morgan" lines, naturally after one of their creators (Figure 12).

Student Plot with "Morgan" lines

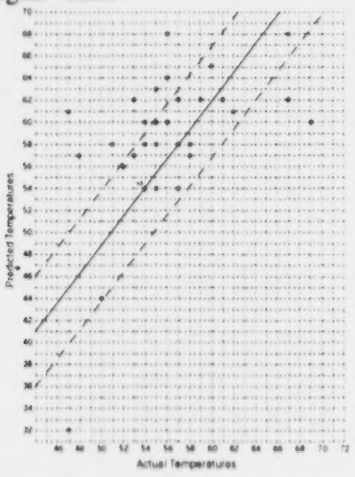


Figure 12

What was fascinating about this entire process were the many things that these students *could* do, on their own, while investigating this weather data. Given some time, the resources of their fellow students, and an interesting (to them) set of data, they were quite capable of organizing, analyzing, and representing the data their own way. Furthermore, they came up with several of their own user-constructed measures for making decisions about this data. If we, as researchers, give students an opportunity to show us what they *can* do, rather than what they cannot do, we may discover some encouraging conceptions and abilities that normally go unnoticed.

Situation 4--What next?-- Investigate students' conceptual development in variability, over time, starting with what they* can *do.

I'd like to conclude with a situation that I am beginning to work on. In pursuing this situation, I have been trying to listen to my own suggestions about some of these missed opportunities in research that we have discussed: 1) follow-up on student's thinking and reasoning by documenting growth and change over time; 2) begin to investigate student's conceptions of variability; and 3) try some research approaches that uncover what our students *can* do in problem solving in chance and data, rather than merely documenting what they are unable to do.

The data for this situation involve time lapses between blasts of one of our U.S. national treasures, the geyser Old Faithful located in Yellowstone National Park. The data were taken from the book *A Handbook of Small Data Sets* (Hand et. al., 1994), which contains data for the time intervals between blasts of Old Faithful for 15 consecutive days. The plan is to put students into a context in which variability runs rampart, in several forms, but in which there is still the possibility of detecting patterns. Students are presented a table of the data taken from one day's blasts of Old Faithful (Table 1). Then they are asked, "What do you predict for the next day's data set, and why?"

Table 1

Time intervals in minutes between Old Faithful's blasts

First Day Data

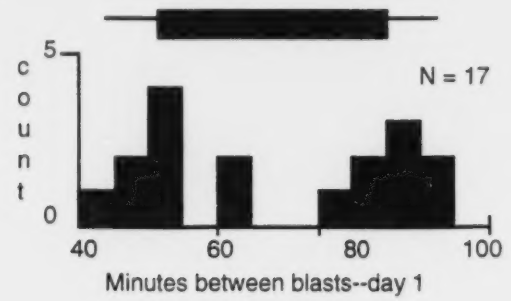
50	87	48	93	54	86	53	78	52	83	60
87	49	80	60	92	43					

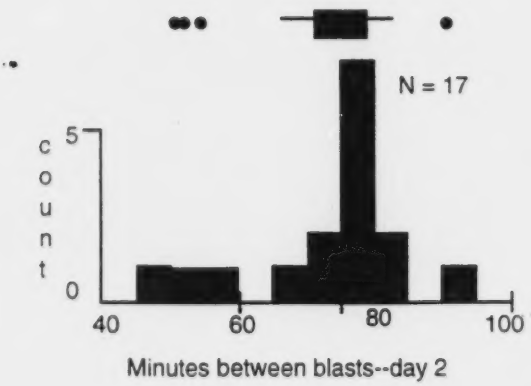
First Day Ordered Data --

43 48 49 50 52 53 54 60 60 78 80 83 86 87 87 92 93

After they make their predictions for the next day's data, students are given data for three consecutive days of Old Faithful blasts, and asked to investigate whether there are differences in the data for these three days, and how they decided. The idea is to create an environment in which students get more and more information, one in which they become detectives while exploring the ultimate question "Just how faithful is Old Faithful, anyway?"

Some visual representations from blasts on three consecutive days for Old Faithful are given in Figure 13. This data set has the potential to highlight the concept of variability, and perhaps to help counter Landwehr's concern that people believe there is "no variability in the real world." First, the time intervals between blasts given in Table 1, indicate that Old Faithful appears to alternate between long time - short time throughout the day. Thus, there is variation within a day. Second, although the three ranges for the time intervals are comparable for all three days (comparative boxplots in Figure 13), the inter-quartile ranges are very different for the three days. They tend to cluster around 70-80 minutes for day 2, for example, but are spread from the mid 50's to the mid 80's for day 1, and from the mid 60's to the mid 80's for day 2. So, there is also variation across days.





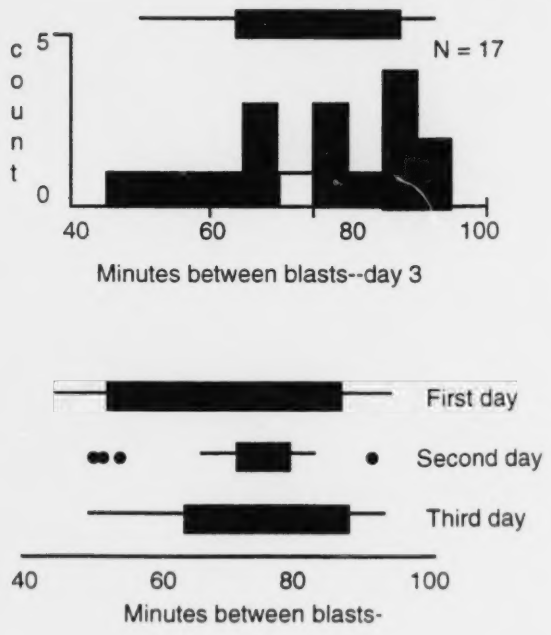


Figure 13

Clearly centers alone are not adequate to describe the Old Faithful's data set from one day to the next day, as there is tremendous variability in the range of times for the blasts, even though there are identical medians for days 2 and 3.

Since middle school students love to play detective with complex data sets, they will likely be interested in exploring questions like: Does this long-short alternation hold up over many days? Do differences between days, like days 1 & 2, continue to occur or is this an unusual phenomenon? They will also come up with questions of their own, and with ways of representing the data that might not occur to us. I believe the Old Faithful environment is an example that will prove to be a fruitful context for investigating students' development of the concept of variability.

In conclusion I would like to recommend that we investigate students thinking about data and chance starting with data sets and tasks that elicit conceptions of variability and difference, rather than tasks that focus on centers and sameness. I would also suggest that a more fruitful avenue for our research might be to begin with rather complex *multivariate* data sets, like the weather pages, rather than simple univariate data sets that can seduce us and our students into looking too much just at averages. And, as we go about our research, let's remember to follow-up on our students' thinking, and to include tasks that show us what they *are* capable of doing.

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Do disadvantaged sudents fail mathematics or does mathematics fail disadvantaged students?

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The persistent differential outcomes in mathematics eduation have been cause for considerable research and concern for over 40 years. This paper considers the social context of contemporary mathematics and its implication in the outcomes for disadvanatged students. It is argued that reliance on individualistic models of theory and research need to be considered in conjunction with the social context.

The focus of this year's conference "People, People, People" is both timely and important. For over 40 years, there has been a surmounting body of evidence which documents the differential outcomes for students based on their background – including gender, social class, ethnicity, race, language background, geographic location and other demographic characteristics (Secada, 1992). This differential access and success has been found in mathematics from early childhood through to tertiary education. The differences are explained, theorised and researched from a range of perspectives including arguments of biological determinism through to systemic injustice embedded in the very structures of schooling. Each perspective has different strategies and foci for social justice initiatives. As we move into more conservative times, consternation needs to be expressed with the re-emergence of the New Right and the increasing and tightening control over teachers' work through systemic mechanisms (Apple, 1995) and the lack of focus on social justice.

As we move rapidly towards the new millennium, inordinate changes are occurring with society and schools to the extent where the social fabric of what we have come to know as "our society" is under threat. This has ramifications for schools and education because of its impact on the people integral to these sites. The impact of economic rationalist ideology and the corporatist State has been profound - government undertaken by the managerial elite, technology occupying prime space, privatisation of state authorities, and a focus on efficiency, debts and cuts. This new style of management and government has seen a shift away from "the public good" to a fixation on self interest. This has resulted in high unemployment and increasing poverty - both of which are at unprecedented levels in the post-war era. Working within increasingly conservative regimes and tighter fiscal management practices, issues of social justice and equity are no longer touchstones for reform. However, the increasing gap between rich and poor in conjunction with the diminishing profile of equity and social justice reforms indicates that the economic rationalist ideology has little or no place for real consideration of the educational disadvantage of students excluded from school success.

The demographic characteristics mentioned by Secada (1992) indicate significant portions of our communities differentially affected through the school system. These groups are not singly disadvantaged by schooling, but often endure multiple disadvantage as in the case of young, indigenous women. A factor common to most of these categories is access to money, or more specifically, the lack of money. Huston (1995) goes so far as to argue that income is the best predictor of academic success.

This paper is concerned with the outcomes of students living in poverty as they relate to education and more particularly to mathematics. The rationale for this interest is the increasing numbers of people forced to live in poverty due to the changing economic climate and workplace restructuring brought about through the new style of management. It has been long recognised that students living in poverty are at greater educational risk than their peers from more affluent backgrounds. When there are increasingly more

people forced to live in poverty, then there are important implications and ramifications for educators and for us as mathematics educators given the high status of mathematics within the formal school curriculum and the wider society, along with its role as a social filter. When the corporatist ideology, in conjunction with the economics rationalist ideology, govern reform agendas then there should be even more concern for the education of disadvantaged groups since their voices are marginalised in the debate for access to scarce resources. This is even more so for mathematics educators where numeracy is recognised as a key indicator of school success. This paper is thus an empassioned plea for mathematics educators to consider the present and the future of the *people, people, people* who undertake the (often forced) study of mathematics. It seeks to move beyond psychological approaches to theorising the dialectic of poverty, disadvantage and mathematics education. It seeks to make explicit the social context within mathematics education is intrinsically linked and in so doing, make explicit the constraints within which teaching and learning occur. In so doing, it proposes that there are systemic aspects of mathematics education which must be considered in the future if mathematics education is to take serious the issue of *people, people, people* .

In taking this line of debate, two aspects of poverty must be considered. First is the immediate effect on the individual and educational outcomes. This is the most dominant approach to the issue and is most commonly couched in psychological discourses. Fine (1990, cited in Apple, 1995, p. 343) is critical of such approaches as they have "benevolent consequences...[in which] there are isolatable and identifiable groups of students, who by virtue of some identifiable characteristic are not likely to graduate." These approaches seek to find solutions that focus on the individual students or their families. This approach denies the wider social, or macro, context which supports and maintains structural inequality. These approaches implicitly support and maintain the status quo rather than offer systemic change.

The second approach is what Apple (1995) calls for – the provision of the "best educational experiences we can provide for our children in every curriculum area" alongside "relentless attention to *systemic* power and critique" (Fine, 1993 cited in Apple, 1995, p. 344, emphasis in original). If systemic critique is not undertaken, then the processes through which already disadvantaged students are systematically excluded and marginalised will continue to be masked.

Poverty, Social Disadvantage and Educational Outcomes

This section is concerned with developing an overview of the issues associated with poverty and education, in particular defining poverty and social disadvantage; drawing on current trends with Australasia and the world; and the effect of poverty in education.

Within the Australian context, "living in poverty" is identified by the economic status of people living below the "Henderson poverty line"[1]. The restrictive lifestyles associated with being poor is not confined to economic resources but also associated with social, psychological and educational disadvantages. There is often a conflation of the terms "poverty" and "disadvantage". For the purposes of this paper, poverty refers to the restricted access to economic resources whereas "disadvantage" encompasses a wider range of indices including fiscal ones.

Australian statistics indicate that there is an increasing number of people (and children) living in poverty. Briggs (1994) argues that one in eight Australian children are living in poverty. These figures are repeated in other parts of the Western the world (Hamburg, 1993; Huston, 1995; Morra, 1994). Hill and Veale (1995, p.9), using information from the Brotherhood of St Lawrence, state that "Australia [and] the United States had the highest poverty rate when measured against other comparable OECD countries." Such statistics indicate that the issue of poverty and social disadvantage cannot and should not be ignored. Significant numbers of students are forced to live in

[1]This is an arbitrary line brought about as a consequence of the Henderson Commission of Enquiry into Poverty, 1975-76. At the time of writing this paper, it was around $12,000.

conditions which are implicated in their positioning as learners of mathematics. Of those most affected by poverty are sole parent families (McInnes, 1996). Within the Australian context, indigenous Australians are disproportionately represented within any measure of educational disadvantage. For example, outcomes from the Year 2 Diagnostic Net and the Year 6 tests (Education Queensland, 1997) have shown that indigenous Australian students are identified as performing significantly poorer than their peers in all areas of numeracy and literacy measured.

The effects of poverty on educational disadvantage is well documented. Poverty effects the educational achievements of students so much so that Connell, White and Johnston (1990) have argued that the increasing levels of poverty are reflected in the increasing levels of educational disadvantage.

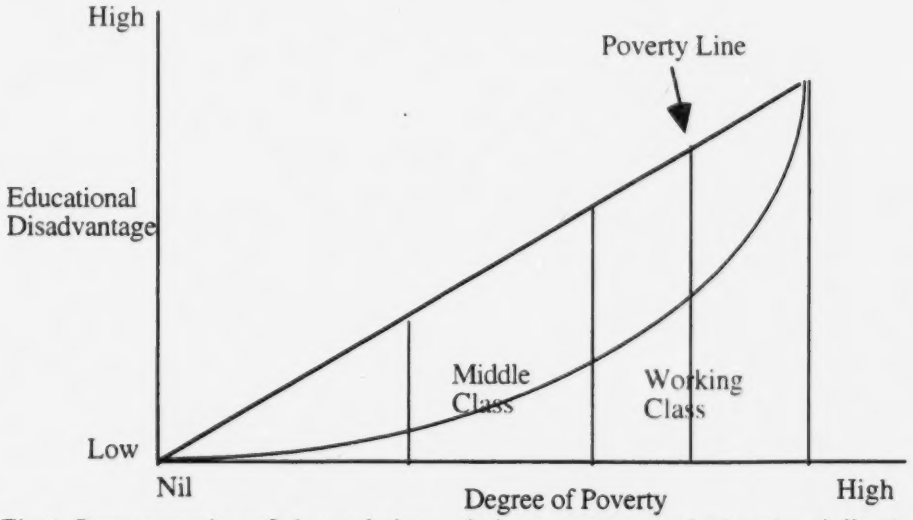

Fig 1: Representation of class relations relative to poverty and educational disadvantage
Source: (Smith, 1993)

Knapp, Adelmann, Needels, Zucker, McCollum, Turnbull, et al., (1991) argue that students at the greatest risk of educational failure are those from low-income backgrounds who disproportionately represent ethnic minorities and NESB/ESL families.
The ways in which poverty manifests itself in educational disadvantage includes:

- spasmodic attention spans due to lack of food (Smith, 1993)
- psychological health - eg depression (Smith, 1993)
- low levels of literacy and numeracy (Smith, 1993)
- teachers having lower expectation of these students (Knapp, et al., 1991; State Board of Education, 1992)
- truancy and poor attendance (Smith, 1993)
- preschool participation in much lower than for their more affluent peers - 35% of poor children in comparison with 60% of high income families (Morra, 1994)
- low retention and completion rates (Smith, 1993; Lamb, 1992)(Teese, Davies, Charlton, & Ploesel, 1995)
- students from lower-status backgrounds were only half as likely to enrol in HSC [senior] group 1 mathematics, physics and chemistry than other students (State Board of Education, 1992; (Sturman, Sharpley, & Polesel, 1992).
- students from lower-social status backgrounds had greater enrolments in technical and applied courses along with economic and business courses than their peers.(State Board of Education, 1992)(Teese, et al., 1995)(Sturman, et al., 1992)
- Poverty is compounded by other factors - such as immigrant families, rural families, linguistically isolated (NESB) families (Knapp, et al., 1991, Morra, 1994)) (Lamb, 1991)

- under-representation of parents in school activities, (Connell, Ashendon, Kessler, & Dowsett, 1982, Smith , 1993)

The Dialectic of Poverty and Education

There is an inter-relationship between poverty and education whereby poverty has some effect on education and education has some effect on poverty (Petrie, 1990). If, this is the case, then schools and mathematics have a role in the production and maintenance of poverty, but also can be involved in the solution of poverty.

Connell, et al. (1982) argue that the core of the problem of poverty and education is the schooling system rather than the character or culture of the poor themselves. Through practices such as public and state schooling; high and low status knowledge; pervasive and competitive assessment systems; streaming; and so forth, the schooling system helps to perpetuate inequalities through the ideologies embedded within those practices. Simultaneously, there are clearly practices which have been implemented which aim to transform schooling so that the role in maintaining social disadvantages are minimised. Bourdieu (with Passeron, 1977) in his earlier work has noted that the schooling system favours and valorises particular forms and ways of knowing through various subtle and coercive means. Through these processes and practices, particular cultural and social systems have more legitimacy than others to the end whereby students from the middle- and upper-classes are more likely to be successful in school than their working-class peers.

Within the contemporary context the issue of poverty is difficult to research because of the ethical dilemma of accessing the background of students. However, through processes which use national statistics, the Ross Index provides an indicator of combined disadvantage This statistic is used to identify schools which are serving a disadvantaged community and support for that school is made available through the Disadvantaged Schools Program[2]. When comparisons are made between the outcomes of statewide testing practices (Education Queensland, 1997) it is quite clear that as a group, students attending DSP schools did not perform as well as their peers from non-DSP schools in the areas of numeracy and literacy. These figures are indicative of group performance rather than individual so that it is not possible to extrapolate about individuals.

A Review of the Literature

An extensive review of the literature in the domain indicates three main trends. The first is concerned with the identification of variables of "at-risk" students. This literature tends to draw on extensive databases of statistics and identifies a number of factors which are seen as significant in the educational outcomes of disadvantaged students. This literature identifies the family structure as significant, and of particular reference is the status of the mother. Most often the causal links drawn from such studies are the single parent family (most frequently the deficit person is identified as the mother), poor educational levels of the primary caregiver (again the mother), and other factors such as drugs, early parenthood (again single mothers) or stereotypical views of "poor" families/communities. For example, Morra (1994) argues that the environments in which poor children grow up involve more homelessness, street violence, illegal drugs, and young, single-parent families. She states that in these families there had been a 46% increase in single-parent families and a 20% increase in families where neither parent had completed (US) high school standards. Within the Australian context , Hill and Veale (1995, p. 9-10) argue a similar position. These family characteristics at often described of as "at-risk" indicators. The more that a student has of these indicators then the more "at-risk" the student is of failing school. Such identifies engender the description of certain families characterised by single parent, welfare, drug or substance abuse, early parenting, where the mother has

[2] The Disadvantaged Schools Program (DSP) was the touchstone of the educationla reform of te Whitlam government. It is the longest running social justice program in the world and has srvived many incoming governements in spite of the numeraous changes to structure and funding.

little formal education, the family is living below the poverty line, the first language spoken in the home is not English, unmarried mother at time of the child's birth, the family is a single-parent structure (Zill, Collins, West, & Hausken, 1995). This general claim is supported by Tunstall (1995) who also raises the issue of race, particularly in the case of the Black Americans and their over-representation in poverty.

The second main theme in the literature are intervention programs. These programs vary from school (or even classroom) to national initiatives. Within this literature, the key initiatives are the Disadvantaged School Program (DSP) in Australia, and the Head Start Program in the United States. The latter has currently been revitalised in recent years due to the increasing issues associated with educational outcomes of significant proportions of the American population. In contrast the DSP has been one of the longest serving compensatory programs in the world and has been in operation since the 1970s. Both programs aim to provide additional educative experiences for students in the expectation that this will have long terms effects for the participating students.

The third theme which needs to be discussed in the very limited, but very specific, literature on mathematics and poverty found in the mainstream mathematics education literature. One of the more pervasive myths in education is that students from disadvantaged backgrounds have (presumed) deficiencies in basic and relevant skills due to their impoverished backgrounds. Until they make up the differences with their more affluent peers, then advanced skills should be postponed until the students are "ready". Typically mathematics programs in these schools are focused on the basics and adopt a rote-and-drill pedagogy (Silver & Lane, 1991). Knapp, Shields and Turnbull (1992) similarly note the curriculum for children living in poverty is guided by principles which emphasise basic skills, sequential curricula and rigid control over delivery. Knapp, et al. (1991) found that in poor classrooms, classroom practice tends to conform with general practices across the US where there is an emphasis on arithmetic computation, pedagogy usually involves teacher presentation followed by seated work and the curriculum is defined by the textbook. These rigid approaches support and foster a myth that poor students should be exposed to curricula where academically challenging work is delayed until a mastery of basic facts has been achieved.

Limitations of the Dominant Trends in Research

The trends noted above are seen to be restricted in their potential for success for students living in poverty. The literature which identifies "at-risk" factors focuses on individual characteristics and intervention is of a compensatory mode and founded on deficits models. The entering behaviour, skills, knowledge of the students are seen to be restricted and in need of some form/s of compensation. Models of these intervention strategies are programs such as the Head Start Program (US) and The Disadvantaged Schools Program (Aust). These programs, while having different foci, have as their central tenets programs and practices which seek to redress the deficits that students have when they enter the formal school context. As has been noted with the Head Start Program, the success of these initiatives are limited. The limitations are at the level of accessing students who are in need of intervention along with the limited long term effects. These programs focus on the individual and do not address wider social and cultural impediments to the participation and success of students living in poverty. Where intervention is at the level of the individual - whether this is student, classroom, or school, long term success is restricted as the wider structural and systemic aspects of schools and society.

In the remainder of this paper, I draw on some of the key factors which impact on the teaching of mathematics in primary schools. These factors similarly impact on secondary and tertiary levels of teaching. The comments are drawn from teacher interviews which were open-ended interviews focused on the teaching of mathematics in schools and classrooms in identified "disadvantaged" schools. The data draw on comments which focus on the individual and social aspects of teaching in these schools. It will be argued that teaching in these contexts is flawed and often compounds the disadvantage of already

disadvantaged students. It is possible that some educators may see these social context as divorced from mathematics. However, teaching mathematics does not occur in a vacuum and this social context is inextricably bound to mathematics. The central thesis that is developed in the following sections is that schools and mathematics contribute to the disproportionate outcomes of education. By considering the social context within which teaching of mathematics occurs, then we are in a better position to implement systemic change which may have a more profound effect on the social justice outcomes of education.

Systemic Approaches to Mathematical Disadvantage

Cultural Constraints

Students come to school with identifiable cultural attributes. When there is a congruency between the patterns of school and home, then students are more likely to be successful in the formal school context. Conversely, when there are differences, then success (and even participation) is more elusive. In the most obvious examples, Maori and Indigenous Australians enter the formal school context having been raised in a family and social setting which values and embodies particular ways of knowing and seeing the world. When they enter the formal school context, their cultural knowledge and behaviours are dissimilar to those of the school context. Similar arguments can be made for working-class students and any other social or cultural groups whose culture is dissimilar to that of the school. For students whose culture is different from that of the school, then significant more effort will be necessary if that student is to be constructed as "successful" within formal schooling. There is a significant body of research in mathematics education which documents the cultural biases in the mathematics curriculum. These biases effectively work to marginalise students in their study of mathematics, that is, they are cultural impediments to learning. Kemmis, working from Habermas, uses three key identifiers which are useful for observing and theorising cultural conditions - the patterns of language, work and power.

Bourdieu's notions of cultural capital are similarly useful in theorising the lack of participation and success of students from poor and disadvantaged families. Within this theoretical framework, students enter the school context with certain knowledges, practices and an ethos towards school. For those students whose knowledge, language, experiences are similar to those valorised within the school context, they are more likely to be successful the participation and outcomes of schooling. Students who come from poor and impoverished backgrounds are less likely to be familiar with the patterns of language, work and power within the classroom and the school. This has been shown on numerous occasions - for example, Walkerdine's analysis of binary opposition terms (such as more and less) indicates that students from more affluent, middle-class families have a familiarity with both signifiers whereas students from poor, working-class families have familiarity with "more". This very subtle difference in background experiences will clearly have some effect on the positioning and learning of students in the mathematics classroom. For those students for whom the patterns of language work and power of school are congruent with familial backgrounds, then success is more likely, thus making culture a form of capital which effects educational outcomes.

Teachers are often aware of the cultural differences in entering students. They recognise that there is a significant amount of work to be done with poor and disadvantaged students to bring them to a level which is similar to that of their more affluent middle-class peers.

> *These children come to school unable to read or even knowing their colours. The parents don't even have the money to buy books. Even if they did, they would not know the difference between a "good" book and a "bad" book.*

This comment indicates not only an awareness of the differences in the knowledge and skills that students bring to the formal school context, but also an implicit recognition of cultural capital effecting the life chances of students. In this context, Marg recognises that not only do the students lack the literacy and pre-numeracy skills of their middle-class peers, but the problem is more than just a deficit. It is that the parents do not have the cultural awareness of what are seen as valuable cultural icons for improving educational outcomes in their students. She also comments

> *The parents are keen for their children to do well at school, but they have no idea on how to help them. They would not know how to help them with homework or even what to do with helping them with reading.*

This comment indicates that the problem is more than just lack of knowledge and basic skills, but more endemic. The students and their families have little idea of what formal schooling values and hence have little idea of how to improve the educational outcomes of their children.

The lack of knowledge that students have upon entering the school context necessitates that teachers will spend significant time teaching fundamental skills. In contrast, their more affluent peers will be encountering curriculum beyond this level and therefore be more "advanced" than their poor peers. While some of this teaching and learning may be focuses on fundamental knowledge associated with the mathematics curriculum, a significant amount of time is also dedicated to the teaching of social skills.

> *There is a lot of work that has to get done with the children just to bring them to a level which works in the classroom. I have to teach basic things like colours and shapes before we can even start sorting activities. Their social skills also need a lot of work. They don't know how to behave in a classroom so I have to spend a lot of frustrating hours just working on these sorts of things.*

The social skills which are valued in the classroom occupy a significant amount of teacher time in the lower primary years. This is, in part, due to the perceived need that students assume the valued patterns of interaction and work of formal schooling - the hidden curriculum. This can be seen to be the imposition of a middle-class cultural system. In raw terms, it essentially means that the teaching of such skills takes even more time away from the teaching of (basic) mathematics and therefore further advancing the educational experiences (and outcomes) of their affluent, middle-class peers.

Structural Constraints

Resources - Technological, Practical and Human

The current infatuation with post modernity and education has brought to the fore recognition of the power of technology and the subsequent impact on classrooms (see for example, (Green & Bigum, 1992)). Some support for the information technology is well documented while others raise concerns that technology is fast becoming the new social filter - the ever increasing gap between rich and poor, and the subsequent differential outcomes.

A survey of the literature alerts us to the impact of technology on the teaching of mathematics. At the primary school, it is quite apparent that computers and CD-ROMs are having a substantive effect on teaching and learning practices. Similarly, at the secondary level, the impact of graphic calculators is very apparent. As I have argued elsewhere (Zevenbergen, 1996) the use of graphic calculators at the senior level need to reconcile the differences between those who have access to them and those who do not. Assessment at

this level must ensure equitable access to such technology before it is can be reified through assessment practices (see Jones & McCrae, 1996).

Technology is not creating the master plan for schools and education insofar as advancement for all, but rather, reifying already apparent differences. Students who have access to technology will be more advantaged than their peers who do not have similar access. This applies to whatever level of technology is considered. Whether this is computers, the World Wide Web, software or even at the level of calculators, the difference between those who have access and those who do not, is largely determined by economic capital. Students who are able to access such are resources are clearly advantaged over those who do not. One only has to consider the implications of access to graphic calculators at the senior levels or even the familiarity of calculators at the primary level. When this is considered within the context of the World Wide Web where students can advertise or access sites which will help them with open-ended tasks, it is clear that students who have the financial resources to enable them to purchase hard- and software designed for these tasks will be far more likely to succeed in mathematical assessment than their not-so-financially-endowed peers.

"Non-electronic" Resources

At a different level, access to technology, where technology is consider within a broader framework to include both electronic and non-electronic equipment, there are considerable aspects of differential treatment again. For example, it is quite common for the more affluent schools to have access to considerably more resources than their less-affluent peers. The difference in access to resources impacts on the programs, and by implication their quality, that can be offered within a school context. Students who do not have access to basic mathematics equipment, such as MAB block, unifix, pop stick, and so forth, are disadvantaged in the modes of learning and working valued in contemporary mathematics education.

> *This new school was a real shock, there is so little equipment around. I thought it must have been locked away in a central store, but my teaching partner told me that this school did not have much in the way of resources. My last school had more resources in one classroom than this school has across the whole school. It is clear that this is because of the socio-economic status of the parents. At my last school we could charge a levy of $40 per student. Where there are 60 students in the double classroom, then we had a lot to spend on resources. At this school, the parents have trouble giving their kids lunch, let alone $40 for a levy. We have to do with the bit that the Dept gives us. It means that it is even hard to find money for photocopying.*

The comments made by this teacher are most applicable at the classroom level of implementing mathematics. In contrast, schools with healthy budgets, particularly elite independent schools, are more likely to have access to greater resources and more flexibility in gaining resources depending on the immediate programs being implemented. This is in stark contrast to schools where there is a lack of access to finances and resources. Denise was commenting on the chronic lack of resources. She was planning a lesson on measurement, but the few tape measures that the school had were being used in the upper school.

> *When I found out I could not have tape measures, I had to make my own using blackline masters. The students will be able to do some measuring, but the measures were not as good as tape measures. They would break easily but it was the best I could do. I had to spend almost a lesson teaching the kids to be careful with them so that they would last longer.*

The constraints placed on teachers in providing for quality teaching in mathematics is strongly linked in the economic resources available to the schools and teachers. While teachers are very resourceful, the extra work created in attempts to make resources intensifies the workload of these teachers. Even when teachers are resourceful, there are great pressures on teachers insofar as basic resources including photocopying, art supplies and so forth, all of which impact on their capacity to create effective learning environments for their students. This compounds the disadvantages already faced by disadvantaged students.

Where schools are able to access special funds, such as DSP funding, there is potential for building up substantive resources. However, the need for mathematics resources must be identified in the budget submission - which in itself takes a substantial amount of extra teacher time. Where schools have allocated money for the purchase of mathematics resources, then without adequate professional development (some) teachers may not know how to use them appropriately and their value is diminished or even lost. One of the major criticisms of DSP was that many of the programs were the initiatives of particular staff members and when these staff member/s left the school, the programs were lost.

A further issue impacting on the potential to deliver qulaity mathematics programs within the primary school is the issue of human resources. Most schools are staffed on a per capita basis whereby standard formulae are used. Some flexibility is possible within the school so that schools have some degree of autonomy on staffing ratios and specialist teachers. For example, it is often possible for higher classroom ratios to allow for specialist teachers. Alternatively, programs can be established within special projects, such as DSP funding arrangements:

> *We are hoping to get DSP subsidiary funding. If that comes through then we have decided that we will employ some extra teachers for intervention. They will take the children identified as having learning difficulties to a specialist area and work on them intensively.*

This process will allow students who have been identified as needing extra time and effort in the areas of numeracy (and literacy) in an attempt to redress their deficits. Such programs, while in rhetoric represent an altruism aimed at improving the educational outcomes for disadvantaged students, they are heavily reliant on extra funding sources. As we move into tighter economic times, tighter fiscal control as noted at the beginning of this paper, then the long term survival of such programs is dubious.

Of further note is the relationship that disadvantaged schools have with their parents and community. As has been noted by numerous researchers and alluded to in an early section of this paper, parents from economically disadvantaged, working-class families are less likely to work with their schools in spite of recognising the value of schooling. This links with notions of cultural capital and cultural incongruency noted previously. What this means for delivering quality mathematics programs is a restricted pool of adequate (human) resources from which teacher s can draw. Not only are parents less likely to participate in school activities, when they do, they are less likely to have the skills and knowledge to help the teachers and students than their middle-class, more affluent peers. Many teachers commented on their lack of contact with parents and the consequent impact on the programs which can be developed.

> *I will spend these holidays making maths games for the kids to take home with their readers. They will be able to play these with their families. It might make them do something mathematical and I hope it will get the parents to see things as being maths, not just the homework sheets with lots of sums to do.*

> *[Researcher asks: Why don't you get a group of parents to come in help you make them? That way they might take more care of them as well as make it easier for you.]*
> *They would not come in. I have got lots of parents who just drop the kids a the gate and you never see them. Also, if they did help, I would have to spend a lot of time explaining to them what has to be done. I think it would be quicker to do it myself.*

This interaction indicates the difficulty experienced by teachers working in this environment and the impact it has on the development and implementation of quality mathematics programs such as those described by Silver and Lane (1996).

Assessment

There has been widespread implementation of standardised testing in most Western countries. Within the Australian context, tests have been developed and implemented by the various state authorities which vary in form and years of implementation across the states and territories. Recently (March 1997) State Ministers of Education agreed to hold standard tests across the nation making Australia comparable with other Western countries. The principle of such tests have been widely contested (see for example, (Ellerton & Clements, 1994; Apple, 1992). For teachers working with students from disadvantaged backgrounds, this has substantive implications. Where funding for intervention is attached to such testing, there is pressure on teachers to ensure that they make "accurate" assessments of their students.

Of particular importance for teachers working in areas of social and economic disadvantage, is the recognition that if a significant amount of time is being devoted in testing. In the first instance, teachers must devote some time to preparation for the tests. Second, and perhaps more significantly, it has been shown from the outcomes of statewide assessment, that students from poor and disadvantaged backgrounds are disproportionately represented in the cohorts identified as needing intervention. This finding suggests that, particularly in the case of diagnostic tests, it is reasonable to assume that teachers working in schools serving this client group will be spending significantly more time testing than teaching. This process even further disadvantages this group of students.

> *I'm so glad that all of this Year 2 validation is over with. I have spent the last four weeks guessing where my kids are, then working with activities so that they can be "tested" with the Net[3]. I have had to do activities which orientate the kids to the work on the Net. Now that I have gone through validating where they are against the continua, I can now teach again.*

Others have complained that they spent an inordinate amount of time preparing their students with the background knowledge needed to do the tests, but the content of the tests was not seen to be in line with syllabus documents so that for these teachers, the implementation of the tests actually meant that their students were being denied access to perceived important knowledge.

Tests, such as the Net, are mechanisms by which the State is able to control teachers' work. Teachers felt compelled to be 100% certain that where they mapped a student was accurate. This was, in part, due to the potential litigation which may occur if they "mis-diagnosed" a student. However, it is worthy to note, that interviewed teachers also commented that their original "hunches" or diagnoses of students were accurate when

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[3]The "Net" is a term used in Queensland to refer to a state-wide intitiative aimed at the early years of schooling Its intention is to identify children at risk in the areas of numeravy and literacy. Students "cuaght int eh Net" will recieve extra fudnign for intervention at the school level. Its full title is "The Diagnotics Net" and is structured around student outcome statements. Teachers must map individual studetns against stated outcomes. Students who are not able to perform scertain tasks will receive this "extra" funding.

compared against the final reports submitted. Ball (1996), using case studies, tracked the implementation of the Net and showed not only the proletarianisation of teachers work, but also the intensification of their work. In this light, Apple's concerns about the proletarianisation of teachers' work are accurate. Teachers are concerned about their professional "intuitive" judgements of students' mathematical performance - their professional judgements have been subjugated to more technocratic aims and processes.

Changing Nature of Teachers' Work

There is considerable recognition of the students in contemporary society. Some authors, such as Green and Bigum (1993) and Smith and Curtin (1995) have gone so far as to argue that students are "aliens" in today's classroom. Their focus is in the impact of technology on the way in which students think and behave. The emphasis on behaviour management programs suggest a crisis in student behaviour. These changes are also apparent in the high applications for stress-related leave. The romantic notion of "good old days" where students would "sit up and shut up and get on with their work" are no longer the environment where teachers' work. The issues associated with "problem children" are seen to be more common in disadvantaged schools. Many explanations are offered for this ranging from the poor background of the children and their families, to the imposition of middle-class culture (of teachers and the schools) onto working-class students.

The issue of behaviour management is particularly evident in mathematics classrooms where there has been an emphasis - rightly or wrongly - on structured, individualistic learning. I would suspect that very few mathematics educators would support this pedagogy, however, the reality is that it is a very common approach in primary and even more so, in secondary classrooms.

The documented evidence which shows that demographics of the typical classroom teacher is one who is approaching mid-40s. There are very few young teachers in education. For most teachers, this would mean that their initial teacher training was undertaken in the 70s where mathematics teacher education emphasised behaviourist modes of teaching and learning. Furthermore, it is also evident that a significant amount of teachers do not undertake professional development, suggesting that these teachers are not up-to-date with contemporary modes of teaching mathematics.

For teachers working in classrooms where there is a cultural difference between the teacher/school and the students, where the teacher still works within a traditional mode of teaching, then the teaching of mathematics is unlikely to be catering for the needs and interests of the client students.

> *The school is changing in its demographics. Once we were known as a very good school but that is changing now. There are a lot of people who still come to the school because of its past reputation, but they don't know how bad things are going. We are now a B1[4] rating for DSP funding. As a new teacher, it is really frustrating to come out of Uni with all these great ideas on how to deal with classrooms. We learned all this stuff about group work, activities , problem solving, constructivism and that at uni, but when you get into the classroom teachers just laugh at you - particularly those who have been out for a long time. You look at their teaching, and you see the children sitting in rows, doing maths sheets. They are quiet and the teachers think they're learning. A lot of the teachers are just really cruel to kids to keep them in their places. They have no idea of what the needs of these kids are, but they are seen as good teachers because the kids are quiet and they are sitting doing their work. I think it is really a case that the kids are scared of*

[4] Schools are rated for funding based on the Ross Index as noted previously. Schools with an "A" rating are those which will receive finding. B ratings do not automatically receive fnding, but will put into a pool and any remaining dfunding will be distributed accoridng to submision evaluaions. Within this context, a B1 rating is the next "best" rating for funding.

> *their teachers. How can they learn maths like that. The teachers have no idea of how to teach these kids. It was not an issue 20 years ago, but it is now.*

New graduates may be more aware of the changing needs of schools and students, more up to date with contemporary methods of teachings mathematics, but their representation in schools is comparatively limited. There is a recognition that the rote-and-drill methods which focus on basic skills is a dominant approach adopted by teachers working with students from disadvantaged backgrounds. Silver and Lane (1991, p.3) argue that:

> a program of repetitive drill and practice on basic computation which has characterized middle-school mathematics education for many American students and which has relegated disproportionate numbers of poor students to the remedial track, thereby blocking their access to most socially acceptable paths to status and success. (Silver & Lane, 1991) pp.3-4)

Knapp, et al.'s (1992) work indicates that quality practice with poor students needs to focus on meaning and understanding where skills are embedded in a meaningful context and where there are explicit links made between the school and outside-school contexts. Silver and Lane (1991) also note that in the context of teaching students from disadvantaged backgrounds, some consideration must be made of the reading comprehension abilities, writing abilities and general familiarity of the task contexts in order to develop an appropriate teaching and learning environment. It would appear from Amanda's comments that such practices are not common in classrooms, particularly among the cohorts of teachers who have been in the teaching profession for some time.

Intervention Practices

As has been noted earlier, the clientele attending schools is changing. The literature indicates more students come from non-nuclear families, more children live in poverty; and more students come from dysfunctional families. Such labels are problematic and should not be accepted without some question. What can be seen as "acceptable" is that there are issues confronting teachers which are different from those of the past. The changing society is evident in the emergence of labels such as at-risk students, behaviour adjustment, behaviour management, effective teaching environments, ADD students, ADHD students, and so forth. The immensity of such labels indicates that the (supposed) emergence of a range of behaviours not evident in earlier times. Again, such labels should be treated with caution since they focus on the individual rather than systemic issues. However, what they do indicate is the need for teachers to apply labels to behaviours (and in some cases, children) which previously were not apparent in society. The state's attempts to address these problems is evident in the range of programs and services aimed at supporting teachers and students.

There is a considerable correspondence between behavioural issues and the familial background of the students where there is a strong link between the perceived "bad" behaviour of students from disadvantaged backgrounds. As previously discussed, this may be due to cultural differences between the school/classroom/teacher and the students rather than some biological basis. What is of importance to this paper is the impact this has on the learning outcomes in mathematics. Because of the perceived "problems" in classrooms, the state implements particular programs and offers various forms of support to teachers to help "deal" with the problems. Intervention at this level can be in the forms of teacher support, special aid teachers or even special programs. Such interventions may be classroom based or involve withdrawal of the student/s from the classroom.

In conjuction with the school-based initiatives, students identified as "at-risk" are often subject to other interventions from the state. Where students come from familial circumstances which have been identified through jurisdictional processes, particular external bodies also may be involved in some forms of intervention. Such initiatives

involve state authorities such as family services or police. In many cases, the implementation of state authority programs may be based at the school so as to cause minimal disruption to the students' lives. This may appear to be a worthwhile practice, but given the immensity of the number of students involved in such interventions, then this can cause considerable disruption to classroom life and effectively the educative experiences of the students.

> *I have been asked to work with teachers to help them with their maths teaching, but I have gone into classrooms and I am amazed at what goes on. I was once in a classroom where I thought I'd spend a couple of hours with the teacher to work on her maths planning. She's planned a maths lesson so that I could see what she was doing and that we'd have time to look, see and evaluate. What ended up happening was that I had to see a procession "support" come through that room so that in the end she only had about 30 mins of maths. Some of the people included police, family services, teacher supports, and so on. It really made me think that this is one of the reasons why she is having trouble delivering her maths, and also why the kids were having difficulties learning the stuff. Until that is changed then there is not much hope for her or the children.*

The disruption to teaching mathematics is noted in this comment and is further discussed by Ball (1996). Quite clearly, the amount and range of services needed to cater for the emerging 'problems' of contemporary classrooms restricts the capacity to cover content. This reinforces the comments made by teachers of the emerging classroom context.

Social Justice and Equity: Research Implications

In light of the comments raised in and through this paper, it is clear that there are many issues confronting mathematics educators. Many of the big issues can not be addressed by considering the individual. Psychological discourses, with their focus on the individual, assume that the problem with exclusion and marginalisation is an issue which can be addressed by focussing attention at the level of the individual. Such initiatives are only "bandaid" solutions. No longer can we keep patching up problems. It is time to address to systemic inequality. Clements and Ellerton, in numerous writings have been urging the mathematics education community to rise to the challenge. The national agendas of state (as in the case of Australia) and federal (as in the case of New Zealand) politics needs to be challenged. In considering how poverty and social disadvantage can be addressed, it is essential to consider the context in which teaching and learning mathematics is occurring and how this impacts on the cognitive aspects of learning. Learning and the act of teaching do not occur in a social vacuum. Teachers are a product of the society in which they are raised, the teacher education facility which trained them, the schools and sectors in which they work, all of which impact on their thinking and practice of teaching. As teachers' work becomes more intense and proletarianised, they need to be made aware of the conservative agendas impacting on their work and the learning outcomes of their students. A greater understanding of the processes through which issues of equity and social justices are realised in and through teachers' work needs to be undertaken in order for a more complete picture of how mathematics education contributes to the construction of social disadvantage can be developed. The recognition of the social context within which mathematics education is essential to this new emerging era of research. This changing contexts of schools is borne out in the comment below:

> *As teachers today we can longer think about just teaching [mathematics] content we have to think about managing the class. Things like inclusion have changed teaching so much that it is not like it was 20 years ago. The*

> *teachers who have been in since then still think it is OK to teach with the children lined up in rows in doing worksheets. They rule through terror but I wonder how much the children learn. Or even what they learn. We have to change teaching to meet the needs of the children that come to our schools these days. When a lot of these children come from really sad homes, schools have to think about this and work with their students.*

As mathematics educators, we need to begin to consider the changing contexts of schools and the impact this has on teaching and learning of mathematics. This context can vary from the level of the classroom through to more macro issues of systems, policies and politics. These variables invariably influence the work of teachers and researchers.

> *As one teacher I can only make a small change to the lives of my students, but as a collective, teachers can make a substantial difference to the life outcomes of our students. Teachers need to know how the system helps to exclude students and how we as teachers are part of that process. Until we know and can accept that role, then we will not be able to change much. At the moment it is just finger-in-the-dyke stuff.*

Returning to comments earlier in this paper, it was noted that there has been more than 40 years of recognition, research and reforms in differential outcomes in mathematics education. In spite of this body of work, there still remains differential outcomes in this area. As one of the most hegemonic subjects offered within the school curriculum I urge the askance of the question which is the title of this paper - do disadvantaged students fail mathematics or does mathematics fail disadvantaged students? By considering the latter question we may begin to reconceptualise our work and make headway into the differential access and success in mathematics. Within a broader conceptualisation of mathematics education where the social context is mutually constitutive and inextricably bound to the learning of mathematics, then a critique of the systemic constraints to differential outcomes can be undertaken.

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Seventh-grade students problem posing from open-ended situations

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The present study was the final phase of a three-year project in which problem-posing programs were developed for the third, fifth, and seventh grades. The aims of the study were as follows:

1. to trace the development of seventh-grade students' problem posing across a range of mathematical situations;

2. to trace the developments of individual children as they participate in a 3-month classroom problem-posing program;

3. to monitor changes in children's perceptions of, and attitudes towards, problem posing and problem solving;

4. to identify links between students' problem-posing and problem-solving abilities.

Background

Problem posing is recognized as a significant component of the mathematics curriculum and is considered to lie at the heart of mathematical activity (e.g., Brown & Walter, 1993; Moses, Bjork, & Goldenberg, 1990; Silver & Cai, 1996). The inclusion of activities in which students generate their own problems, in addition to solving pre-formulated examples, has been strongly recommended by several national bodies (e.g., Australian Association of Mathematics Teachers, 1996; National Council of Teachers of Mathematics, USA, 1989; Streefland, 1993). Despite its significance, problem posing has not received the attention it warrants from mathematics education researchers. We know comparatively little about children's abilities to create their own problems in different mathematical contexts, about the processes they use, and about the extent to which these abilities are linked to their competence in problem solving. There is also insufficient information on how children respond to programs designed to develop their problem posing (Silver, 1994). This situation needs to be redressed, given that problem-posing activities in the curriculum can foster more diverse and flexible thinking, enhance students' problem-solving skills, broaden their perceptions of mathematics, and enrich and consolidate basic concepts. Student-generated problems can also provide us with important insights into children's understanding of mathematical concepts and processes, as well as their perceptions of, and attitudes towards, problem solving and mathematics in general (Brown & Walter, 1993; English, in press c; English, Cudmore, & Tilley, in press; Silver, 1994; Van den Heuvel-Panhuizen, Middleton, & Streefland, 1995).

The lack of research on curricula designed to foster children's problem posing means there are few theoretical and practical frameworks to guide program development (Silver, 1994). Building on my previous studies (English, in press a, in press b) and on related literature (e.g., Brown & Walter, 1993; Cobb & Bauersfeld, 1995; Schoenfeld, 1992; Silver & Cai, 1996) I developed the theoretical framework outlined in Figure 1.

This framework guided the construction of the problem-posing program, described in a subsequent section. As it is not possible to address the entire framework, I consider only those elements that are fundamental to the findings examined here. These findings are concerned with children's problem creations from two types of open-ended situations, namely, descriptive problem stories, and open symbolic expressions. Posing problems from these situations primarily requires an understanding of, and facility with, problem structures, and an ability to think diversely and flexibly.

Theoretical Framework

Understanding Problem Structures

One of the fundamental elements of problem posing is understanding just what a problem is (Brown & Walter, 1993). This includes being able to recognise its underlying structure and to detect corresponding structures in related problems. Structure may be defined as "form abstracted from its linguistic expression" (Freudenthal, 1991, p. 20). While not denying the importance of problem context (Freudenthal, 1991), children need

KEY ELEMENTS OF PROBLEM POSING

Knowledge and Reasoning

. Understanding problem structures and recognising related structures;
. Knowing problem design, in particular, recognising critical units of information;
. Being able to model and transform given structures, as well as build new ones;
. Knowing whether and how a posed structure constitutes a solvable problem;
. Being able to think in mathematically diverse ways;
. Knowing how and when to apply processes of analogical reasoning;
. Being able to reason critically in assessing problems and problem experiences.

Metacognitive Processes

. Communicating one's perceptions of, and preferences for, different problem types;
. Reviewing and enhancing self-efficacy expectations;
. Improving one's disposition towards problem posing and problem solving.

Sociological Factors

. Participating in classroom communities of philosophical and mathematical inquiry;
. Engaging in constructive dialogue and debate;
. Sharing and critiquing problem creations.

Figure 1 Key Elements of Problem Posing

to recognise the mathematical structures of problem situations if they are to utilise these to generate new examples and questions; this requires them to place the contextual features in the background and bring the structural elements to the fore. That is, children need to construct meaningful mental models or representations that recognise the important mathematical ideas and how they are related. The importance of such mental models in mathematical learning has been well documented (e.g., English, in press c; English & Halford, 1995; Novick, 1995; Silver, 1981). Indeed, a lack of such models has been shown to be responsible largely for the difficulties children experience with operational word problems, especially those with complex structures. For example, problems in which there is not a clear mapping between the problem situation and the operation required for solution cause particular difficulties (e.g., Sally has saved $24. This is 3 times the amount Peter has saved. How much has Peter saved?; English in press c; Nathan, Kintsch, & Young, 1992; Stern, 1993).

The complexity of problem structure is also determined, in part, by its linguistic or syntactic properties (Mayer, Lewis, & Hegarty, 1992; Silver & Cai, 1996). Mayer et al. found that problem-solving difficulty seemed to be related to linguistic complexity, with problems containing assignment propositions easier than those with relational or conditional propositions. For example, a problem that asks, "How much did the lunch cost?" would be easier than one which asks, "How much more does the bike cost than the scooter?" or, "How much would the drinks cost if each person was only allowed two cartons of juice?" The nature and number of distinct semantic relations embodied in a problem also have a bearing on its complexity (Marshall, 1995; Silver & Cai, 1996). For example, a story problem that involves both multiplication and subtraction would be more complex than a comparable case involving only one of these.

Another important factor in children's facility with problem structure is their awareness of problem design. In generating their own problems, children must

recognise the critical items of information that are required for problem solution (in contrast to the other items such as contextual information). This awareness of design includes recognising the nature and role of the "known" and "unknown" information entailed in their posed problem, as well as any constraints placed on goal attainment (Moses et al., 1993). This knowledge is necessary for determining whether and how a posed problem structure constitutes a solvable problem, a basic element of problem posing (Brown & Walter, 1993).

Thinking in Mathematically Diverse Ways

Being able to perceive mathematical situations in diverse ways is not only fundamental to children's problem-posing development but also to their overall mathematical growth (English & Halford, 1995; NCTM, 1989; Smith & Silver, 1995). Interpreting a mathematical situation in more than one way is particularly important in children's understanding of operational situations and their ability to generate new operational problems. Although there exists substantial literature on children's abilities to *solve* operational problems (e.g., Bergeron & Herscovics, 1990; Carpenter et al., 1993; Fuson, 1992; Greer, 1992), there is little information on children's abilities to *pose* them. We are thus left with an incomplete account of children's numerical facility. Part of this facility requires being able to assign several meanings to the formal symbols ($+$, $-$, $\times$, $\prod$). Unfortunately, children's school experiences rarely provide them the opportunity to interpret these symbols in a variety of ways; the meanings usually assigned to them are those elementary concepts children are taught first (Fischbein, Deri, Nello, & Marino, 1985; Fuson, 1992; Stern, 1995). This appeared to be the case in recent research (English, in press a) where grade 3 children were found to be inflexible in their problem creation, experiencing considerable difficulty in recognising formal symbolism as representing a range of problem situations (e.g., they would continue to pose only a "take-away" problem for 9-6=3). Broadening children's perceptions of mathematical situations thus appears a major area in need of attention in children's problem-posing development.

Program Development and Implementation

As described in this section, a 3-month problem-posing program was developed for several classes of seventh-grade students. Given the confines of the students' existing curricula, it was not possible to implement problem posing as an integral learning process across the entire mathematics curriculum, although this certainly would have been desirable. Nevertheless, as Wittmann (1995) emphasised, research centred on carefully designed and empirically studied teaching units that are based on fundamental theoretical principles, makes a major contribution to mathematics education. Indeed, Wittmann maintained that "such units are the most efficient carriers of innovation and well-suited to bridge the gap between theory and practice" (p. 369).

The present program represents a refinement of the third- and fifth-grade programs (English, in press a, in press b). The development of these programs reflects elements of Freudenthal's (1991) "thought experiment," which involves envisioning how the teaching-learning process will proceed prior to implementing activities in the classroom. After implementation, one tries to find evidence that indicates whether or not the expectations were realised. The feedback of the practical experience into new thought experiments generates a cycle of curriculum development and research. This cycle has continued throughout the three years of the problem-posing project.

Participants and Selection Procedures

Three classes of seventh-grade students from three state schools participated in the present study, which was conducted throughout 1996. Twenty-three students from across the three classes were chosen for in-depth observation and analysis (mean age of 11. 9 years in term 1). The 23 children (along with an additional six children serving as a small control group) were chosen on the basis of their responses to tests of number sense and novel problem solving; these were administered during the first term of the school year. The tests were modelled on examples that had been used successfully in the

previous studies (English, in press a, in press b). The number sense test focused on facility with number and routine computational problem solving, while the novel problem-solving test included examples requiring a range of reasoning processes (e.g., deductive, combinatorial, spatial reasoning), as well as general strategies.

The selected children displayed the following profiles of achievement: (1) strong in number sense but not so in novel problem solving ("SNS" profile; N=6); (2) not strong in number sense but strong in novel problem solving ("SNP;" N=5); (3) strong in both domains ("SB;" N=7); (4) average achievement in both domains ("AB;" N=5). The intention was to include children from the first three profiles only, however difficulty in obtaining sufficient numbers necessitated adding the last category (which, as the results show, proved beneficial).

The 29 children (including the small control group) were individually administered a comprehensive set of problem-posing activities during the second term (prior to the program) and a parallel set towards the end of the fourth term (after the program). The problem-posing program was conducted during the third and fourth terms and comprised 12 weeks of classroom activities (approximately 1.5 hours per week).

Program Design

The program encompassed six main activity types that addressed the problem-posing elements displayed in Fig. 1. Briefly, these activity types are as follows:

Problem exploration and reflection: This involved group and class discussions on the children's perceptions of problem solving and posing, their attitudes towards these, the different approaches they adopt in problem situations, their problem experiences out of class, how problem solving could be made more interesting for them in the classroom, and other related issues. Included here were group debates on issues such as, "You learn more from creating and solving your own problems than from solving ones the teacher makes up."

Problem preferences: Here, the children examined a range of routine and non-routine problems and discussed which ones they would most/least like to solve. They also indicated how they would modify their disliked problems to make them more appealing.

Problem sorting: This involved the children in sorting sets of routine and non-routine problem cards according to similarity in problem structure. The problems were designed such that those with parallel structures had different contexts, and those with different structures were set in the same context.

Modelling new problems on existing structures: The children compared routine and non-routine problems and talked about the important elements in problem design. These included elements such as the nature of the known information, the type of information that is not known, and any constraints placed on goal attainment. After solving some of these problems, the children created their own examples by modelling them on the existing problem structures (analogical reasoning processes play an important role here, as discussed in English, 1997). Diversity of problem context was encouraged in the children's creations.

Creating a new problem from problem components: Open-ended situations: A range of activities was implemented here. These included posing problems when one is given "knowns" only, constraints only, or knowns and constraints (Brown & Walter, 1993). Symbolic examples of these activities included creating problems from open expressions and statements, such as "135-68," "24 x 5," "The answer is 4/5. What is the question?" "A new table costs $567. Matching chairs cost $68 each and arm rests cost an extra $25 per chair," and "Create a division problem where the remainder must be used."

The open-ended descriptive situations included a number of examples that were taken from travel brochures, newspaper items, historical documents etc. One of these is addressed in the next section.

Transforming a given problem into a new problem: In these activities, the children explored ways of creating new problems by modifying the structures of existing problems. This included reversing knowns and unknowns, removing constraints, adding more knowns and/or constraints, and using a "what-if-not strategy" (Brown & Walter, 1993).

An important component of all the children's problem posing activities was their sharing and critiquing of each other's problems. A special critique form was created for this, which the children completed and returned to the author of the problem (English et al., in press). The form asks the problem "reviewer" to consider a number of points including whether the problem is solvable, whether it is challenging and interesting, aspects that the reviewer likes/dislikes, and suggestions on how the problem could be modified (if necessary) and extended.

Throughout the implementation of the program, we tried to establish a community of inquiry involving meaningful dialogue or "connected talking" among the children and teacher (English, in press c; Yackel, 1995). At certain points during the program, and on its conclusion, we asked the children and their teachers to reflect on their experiences and also to comment on specific aspects (e.g., what they liked, did not like, whether they felt their problem-posing and problem-solving skills had improved, how we could improve the program for future classes etc.). All children maintained journals of their problem creations and reflections, and the responses of the selected children were video- and audio-taped.

Children s Responses to One Activity Type: Open-ended Situations

Given the limitations of space, children's responses to one activity type only are reported here. We consider their responses to posing problems from (i) open-ended descriptive situations, as shown in Fig. 2, and (ii) open symbolic expressions (subtraction and multiplication, of the form, 134 - 29 and 35 x 5). The example in Fig. 2 (see page 8) appeared on the post-program activities, with a parallel example included in the pre-program activities. The children were required to construct three different problems from the given information. The open symbolic expressions listed above appeared on the post-program activities, along with other similar examples (parallel cases were presented in the pre-program activities). The children were asked to create two different problems for each symbolic expression. As previously noted, the children explored a number of open-ended situations during the program.

SPOOKY TRAVEL

A 5-day tour of the ghost castles on No Man's Island, departing from Munster Town, costs $1776 per person. A 4-day tour of the bat caves on No Man's Island costs $1400 per person. Departure from Cape Fear to No Man's Island costs $350 less per person. The cost of food for each of the trips is $450 per person if there is just one person travelling, and $400 per person if two or more people are travelling together.

Figure 2 An Open-ended Descriptive Situation

To obtain an idea of the children's progress with these open-ended examples, we consider their pre- and post-program responses and also include a couple of their program responses. The problems they created on the pre- and post-program activities were analysed using the following coding scheme, which draws upon some of the ideas of Silver and Cai, 1996.

Problem creation and solvability

This was concerned with: (i) whether a mathematical problem was created, and (ii) whether the problem was solvable with a unique solution (although problems with more than one solution are important in the curriculum, such problems in the present context reflected a problem design weakness).

Problem complexity (for the descriptive situations)

This focused on: (i) the extent of critical information units included in the problem, (ii) the number of distinct semantic relations, (iii) the number of steps required for solution, and (iv) the type of question posed (assignment, relational, conditional). A critical information unit, as used here, refers to an item of information that is necessary for problem solution. For example, reference to the point of departure in the example in Fig. 2 is a critical information unit, as is a statement on whether food is required. An assignment question addresses one variable, such as, "How much did the trip cost?" while a relational question compares two variables, such as, "How much more does it cost to go on the 5-day tour than the 4-day tour?" (Mayer et al., 1992). A conditional question imposes a constraint, such as, "How much would you have to pay if you wanted to depart from Munster Town and if you wanted to take a friend with you?"

Problem complexity and diversity (for the symbolic expressions)

Children's creations for the symbolic expressions were classified as either basic or complex, with respect to the mathematical structure of the problem (cf. classification system used by Leung, in press). Basic problems were the elementary change and part/part/whole examples for subtraction (e.g., Peter had 9 marbles. He lost 4 marbles. How many marbles does he have now?; Sally has 9 cars. 3 are red and the rest are blue. How many are blue?) and equivalent set problems for the multiplication (e.g., Four children have 3 balloons each. How many do they have altogether?). Complex problems reflected a broadening of the children's thinking and included the more difficult cases such as: (i) comparison and equalise situations for subtraction (e.g., Sally has 6 goldfish. This is 3 more goldfish than Samantha has. How many goldfish does Samantha have?; Sue has 9 marbles. Jenny has 6 marbles. How many more must Jenny win to have as many as Sue?), (ii) scalar or multiplicative comparison problems (e.g., John has 24 cars. Penny has 3 times as many cars as John. How many cars has she?), and (iii) Cartesian product (combinatorial) problems (e.g., Sue has 3 different blouses and 4 different skirts. How many different outfits can she make?).

Children were considered to show diversity if the two problems they created for each expression had different structures, such as one change problem and one comparison problem, or one comparison problem and one equalise problem, or one equivalent set and one multiplicative comparison.

Overview of the Findings for the Descriptive Situations

The children showed a distinct improvement in their abilities to generate problems from open-ended descriptive situations. On the pre-program example, there were two instances of a non-mathematical problem being generated (both from children in the AB profile) and 17 instances of an insolvable problem. Children from the SNP and AB profiles had the greatest difficulty in creating a solvable problem on the pre-program activity, while children from the SB profile were the most competent. On the post-program activity however, every child was able to create a solvable problem, with children in the SNS and SB profiles better able to create problems with unique solutions than children in the remaining profiles. In contrast, the six non-participants ("control group") had difficulty in generating a solvable problem, with 45% of their creations being either non-solvable or a non-mathematical problem.

Developments in the complexity and sophistication of the children's problems between the pre- and post-program activities can be seen in Tables 1 and 2. We examine Table 1 first.

Table 1 Frequencies of Use of Critical Units (CU) and Question Types (QT) by Achievement Profile

Pre-program Problems Generated	First Problem Created							Second Problem Created							Third Problem Created						
	No. of CU			QT				No. of CU			QT				No. of CU				QT		
Profile	1-2	3-4	5-7	A*	R	C	0	1-2	3-4	5-7	A	R	C		0	1-2	3-4	5-7	A	R	C
SNS**	4	2		3	1	2		4	2			1	5				6		1		5
SNP	5			2		2		4	1		1		4			4	1		1		4
SB	1	5	1	4		3		1	5	1	4	2	2			1	5	1	5	1	2
AB	5			2		3	1	4			2		2		1	3	1		1		3
Post-program Problems Generated																					
SNS		4	2	1		5			4	2	2	1	4				3	3		4	2
SNP	1	4			1	4		2	2	1	1	1	3				5				5
SB		3	4	2		5			5	2	4		3			1	4	2		3	4
AB	1	3	1	2		3		2	2	1	3		2				4	1			5

Note. * A: assignment R: relational C: conditional ** SNS: strong in number sense only (N=6); SNP: strong in novel problem solving only (N=5); SB: strong in both (N=7); AB: average in both (N=5). On the pre-program activity, there were 3 cases where a question type could not be assigned and 2 instances in which the question type was both relational and conditional. On the post-program activity, there was one of the latter instances.

Among the more noticeable developments evident in Table 1 was an increase in the number of critical information units the children included in their problems (reflecting an increase in solvable problems). Children from the SNP and AB profiles in particular, showed substantial growth, as was evident in Nathan's (SNP) case. He progressed from being unable to generate a solvable problem prior to the program to creating this problem after the program: *Which would cost more? Being a single person and leaving from Munster Town or having two people leave from Cape Fear to go to No Man s Island?* While children from the SNS profile also showed considerable improvement in their inclusion of critical information units, those from the highest profile (SB) showed little change between the pre- and post-program activity. These children had few difficulties in generating problems prior to the program and were able to create quite sophisticated examples during the program. For example, Adam posed this problem after examining one of the travel brochures: *I ve taken a leap year off work and decided to go on as many holidays as possible. Each time I go on a holiday I have to take the time of the previous holiday to recover for the one coming up. If two holidays go for the same amount of time I ll go on the most expensive one, then the cheapest, then I ll go on another expensive one, then a cheaper one, and so on. What would be the average cost per day whether I m at home recovering or on holiday? P. S. Money is not a concern.*

The program made little difference to the children's posing of relational questions (see Table 1). These were clearly not favored, reflecting the documented difficulties children experience with comparison problems (e.g., Stern, 1993). On the other hand, 59% of all the children's questions were of a conditional type and 35% were assignment questions. This is in contrast to Silver and Cai's (1996) findings where only 5% of their sixth- and seventh-grade students posed conditional questions. Interestingly, it was the SNP children who tended to favour conditional questions on both activities. This reflects one of the findings from the fifth-grade study where children in this category tended to create problems that displayed structural complexity but operational simplicity (English, in press a).

Table 2 shows quite substantial shifts in the children's use of semantic relations and in the complexity of their problem solutions.

Table 2 Frequencies of Use of Semantic Relations (SR) and Solution Steps by Achievement Profile

	Pre-program Problems Created																													
	First Problem Created										Second Problem Created										Third Problem Created									
	No. of SR				No. of Steps						No. of SR				No. of Steps						No. of SR				No. of Steps					
Profile	0	1	2	3	0	1	2	3	4	5>	0	1	2	3	0	1	2	3	4	5>	0	1	2	3	0	1	2	3	4	5>
SNS*	1	4	1		1	4	1				2	1	2	1	2	1	2	1				3	2	1		3	2	1		
SNP	3	1	1		3	1	1				2	2	1		2	2	1				3	2			3	2				
SB		4	3			4	3				1	4	1	1	1	4	1	1				3	4			2	4			1
AB	2	3			2	3					1	4			1	4					3	1	1		3	1	1			
Post-program Problems Created																														
SNS		2	3	1		1		1	2	2		4	2			1	1	4				1	4	1		1	1	1	2	1
SNP		2	3			2	2		1			1	4			1	4					1	4			1	2	2		
SB		1	4	2			3	2		2		2	5			2	3	1	1			2	3	2		1	2	2	2	
AB		4		**		4						2		3		2		2		1			5				4			1

Note. SNS: strong in number sense only (N=6); SNP: strong in novel problem solving only (N=5); SB: strong in both (N=7); AB: average in both (N=5). ** One child from the AB profile created a combinatorial problem for her first problem.

As shown in Table 2, the SNP children demonstrated the greatest improvement, especially in their ability to incorporate several semantic relations in their problem; the AB children also showed marked gains. For example, they created problems which incorporated addition, subtraction, and multiplication. Children in the remaining two profiles displayed a noticeable increase in the computational complexity of their problems, with 59% of their post-program problems involving 3 or more steps (in contrast to only 10% previously).

Overview of the Findings for the Symbolic Expressions

The children had little difficulty in actually posing a problem for the subtraction and multiplication expressions. However prior to participating in the program, few children created problems that were structurally complex or diverse, as can be seen in Table 3 (see page 13). It is interesting to note though, that the children who were strong in novel problem solving (SNP and SB profiles) showed the greatest complexity and diversity in their problem creations, prior to the program (this again reflects findings from the previous studies [English, in press a, in press b]).

Children in each profile showed considerable improvement after participating in the program, with their problems displaying greater structural complexity and diversity (in contrast to the control group, where only two could create a complex subtraction or multiplication problem, and only two showed diversity). The AB children in particular, seemed to make substantial progress, just as they did on the previous activity. For example, they created a proportionally greater number of complex subtraction problems than all the other children and also showed greater diversity than the SB children. We would have expected the SB children to show greater development in their problem posing with symbolic expressions. Although they did pose a proportionally greater number of complex multiplication problems than the remaining children, this was not the case with their subtraction problems where they also displayed limited diversity.

It is also worth noting the observed improvement in the children's use of context in their problems. Prior to the program, the children used simple contexts such as, *A farmer had 134 cows. 29 died. How many were left*? After the program, more diverse contexts appeared such as, *At the Channel 7 TV station, 134 fan letters were received every day. Channel 10 received 29 fan letters less than Channel 7 each day. How many fan letters does Channel 10 receive each day?*

Table 3 Proportions of Children Who Created (a) 1 or 2 Complex Problems (b) Diverse Problems for the Symbolic Expressions, by Achievement Profile

Profile		No. of Complex Problems Created: Subtraction 1	Subtraction 2	Multiplication 1	Multiplication 2	Created Diverse Problems: Subtraction	Multiplication
SNS*	Pre-program	.17	.17			.33	
	Post-program	.5	.17	.33	.33	.83	.5
SNP	Pre-program	.4		.2	.2	.4	.2
	Post-program	.4	.2	.4	.2	.8	.4
SB	Pre-program	.14	.29	.14	.14	.43	.14
	Post-program	.14	.43	.43	.29	.43	.58
AB	Pre-program		.2				.2
	Post-program	.4	.4	.2	.4	.6	.6

Note.* SNS: strong in number sense only (N=6); SNP: strong in novel problem solving only (N=5); SB: strong in both (N=7); AB: average in both (N=5).

Conclusions and Implications

The findings from this study and the previous two studies suggest a number of important implications for implementing problem posing in the primary school. First, it seems that the program described here was a successful learning experience for the seventh-grade students and their teachers. For example, a typical comment from the children's reflections on the program was, *I understand problems heaps more now... I actually got involved and didn t sit back and be lazy, it s more fun and challenging than when I started.* Similarly, one of the grade 7 teachers commented: *Personally I really enjoyed it. I thought it was a way that I had never ever considered approaching problem solving, and I personally did learn a lot from it. I think it freed up the entire class and they really enjoyed the activities. . . they were actually asked what their opinions were; they were asked their thoughts, their feelings, and I think they gave those quite freely.* Although the present program was refined from the previous studies, there is nevertheless considerable room for improvement both in its design and in its implementation. For example, one area in need of attention is children's conceptions of operational situations. The present program needed to devote more time to enriching and broadening the children's understandings; the elementary operational concepts were still very much entrenched (cf. Fischbein et al., 1985).

Although the present sample is small and the data are limited, the responses of children in the different profiles suggest further implications for classroom practice. The SNP and AB children, both of whom were not strong in the number domain, made substantial gains from participating in the program. In particular, the SNP children showed a good deal of divergence in their thinking and in their problem creations, as was found in the previous studies. Their weaknesses in the number domain did not appear to hinder them in their problem creations. In fact, the classroom observations showed that they felt a sense of empowerment in handling mathematical problem situations and delighted in creating challenging and "unusual" problems for their peers to

try. Because such children lack numerical proficiency, it is easy to overlook their other mathematical abilities in a number-dominated curriculum. We need to broaden children's mathematical experiences so that all children have the opportunity to reach their maximum development in different mathematical domains (cf., Vygotsky, 1962). We might then see substantial improvement in children's overall mathematical growth.

There is also the need to explore further the links between children's problem-posing and problem-solving abilities and to consider these in designing classroom experiences. The three studies collectively suggest that competence in solving routine computational problems is associated with the posing of computationally complex, but not necessarily structurally complex, problems. Competence in nonroutine problem solving appears associated with the posing of more divergent and structurally complex problems. We need to engage students in a range of problem-posing activities that draw their attention to both the computational and structural complexities of the problems they create and solve. At the same time, they need to be aware of the contextual components and how these can both hinder and enhance problem creation and solution.

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Teachers' Reported Use of Problem Solving Teaching Strategies in Primary Mathematics Classrooms

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The implementation of mathematical problem solving in New South Wales (NSW) primary classrooms is being explored. The results of part of a survey that has been administered to a sample of primary school teachers regarding their beliefs and practices in relation to problem solving is explored. Teachers report that they regularly use strategies such as whole class discussion including a focus on suitable problem solving strategies, concrete materials and teacher modelling. They rarely have calculators available for students, allow students to choose problems or spend much time on one problem.

There has been substantial advice to teachers to teach problem solving and to use problems as a focus of learning in mathematics (Cobb, Wood & Yackel, 1990; Schoenfeld, 1992). This advice has been provided in papers in research and professional journals as well as in curriculum documentation. Such advice has been accompanied by considerable efforts through preservice and inservice programs to change teaching practices from more traditional approaches to contemporary or reformed methods. In the reformed classroom, non-routine problems become the focus of learning (Clarke, 1993) rather than drill-and-practice exercises forming the basis of classroom activity.

If the success of advice and programs which focus on mathematical problem solving are to be ascertained, then research that investigates what teachers believe about teaching mathematics and what actually happens in teachers' classrooms is necessary. Teachers' espoused beliefs are influenced by their actual beliefs (Thompson, 1992), by their knowledge and interpretation of advice about teaching problem solving (Fennema, Carpenter & Peterson, 1989) and by their use and understanding of curriculum documents. Reported classroom practices are influenced by espoused beliefs, by actions in the classroom as well as by constraints such as parent and student beliefs and school culture. Figure 1 is a proposed model of the relationship between these factors.

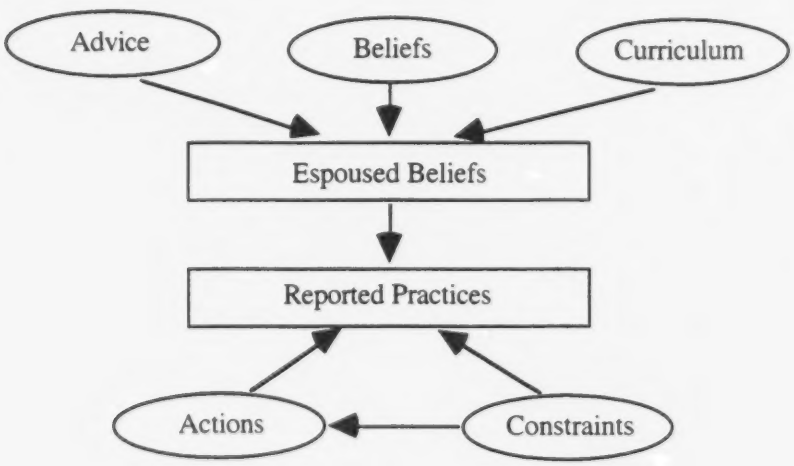

Figure 1. Factors which impact on espoused beliefs and reported practices.

Advice About Problem Solving

Problem solving can be a valuable vehicle for learning mathematics (Cobb et al, 1990). As students are challenged by problem solving tasks, it becomes necessary for them to think about what they already know; to consider strategies that may help them solve the problem; to engage in discussions with other students and the teacher in an effort to resolve conflicts; and to ultimately reorganise existing knowledge to accommodate new ideas.

Reviews of teaching practices (Koehler & Grouws, 1992) and strategies that promote mathematical thinking (Schoenfeld, 1992) reveal a variety of approaches to the teaching of mathematics. Koehler and Grouws (1992) outline five paradigms or different research programs which address approaches to teaching and learning mathematics. While these are varied they all "...accept the premise that students are not passive absorbers of information...[and]...view the teacher as an informed and reflective decision maker" (p. 123). Given these commonalities, the research findings recommend a variety of teaching strategies. In summary, the teacher's role is to plan for instruction, to develop appropriate tasks or problems, to encourage discussion and listen to responses and to modify instruction to meet the needs of individual students.

Establishing and maintaining an appropriate learning environment is another key role for teachers. This environment encourages mathematical thinking (Schoenfeld, 1992) and promotes students' mathematical communication (Weissglass, Mumme & Cronin, 1990). Social interaction is supported as students discuss, speculate and defend their views about mathematical ideas and problems. Teachers need to ensure that students feel comfortable enough to talk about their mathematical understanding without fear of ridicule (Lampert, 1990). In this environment, the teacher manages discussion but students take responsibility and share authority.

Problem solving tasks can be used in classrooms in many different ways. Wright, (1992) describes two approaches which teachers adapt according to their beliefs about student learning. One approach to problem solving is to present students with problems at the end of the topic after skills and procedures have been rehearsed. This approach is referred to as an 'ends' approach and is based on a belief that students need mathematical content and procedures before they can solve unfamiliar problems. Textbooks which place problems at the end of chapters reinforce this approach. Exercises and application problems are typically used in such classrooms.

Another approach, referred to as a 'means' approach, uses problems as the focus of learning. Problems are used to provide a stimulus for student thinking. Good problems create conflict in the minds of students and restructuring of understanding may be needed in order to resolve such conflicts. Unfamiliar and open-ended problems are favoured as the basis of mathematical activity. If this approach is adopted then problems would be used more frequently but fewer problems would be considered.

Beliefs About Problem Solving

It has been argued that teachers' decisions about how to teach mathematics are made on the basis of their knowledge and beliefs about how students learn mathematics (Koehler & Grouws, 1992; Thompson, 1992). Teacher thinking before, during and after lessons, plays an important part in instruction (Clark & Peterson, 1986) and belief systems influence perceptions, planning and actions in classrooms (Ernest, 1989). Teaching is a cognitively demanding activity in that teachers are constantly making decisions based on their beliefs, knowledge, judgements and thoughts and belief systems are important determinants of how individuals organise tasks and goals (Nespor, 1987; Cobb, 1986). The view of mathematics which teachers adopt and the models of teaching

and learning which they support will have a significant impact on classroom practice (Thompson, 1984; Ernest, 1989).

Studies have attempted to improve teachers' knowledge of student thinking in an effort to change beliefs and practice (Cobb et al, 1990; Fennema et al, 1989). In the research conducted by Fennema et al (1989) experimental teachers were given Cognitively Guided Instruction about childrens' thinking in addition and subtraction. As a result, these teachers spent less time on drill activities and more time on problem solving than a control group of teachers. The experimental teachers spent more time listening to student explanations and they accepted a wider variety of problem-solving strategies.

Curriculum Advice About Problem Solving

In NSW, it is mandatory to implement the current syllabus document that provides advice about content and approaches to teaching. The *Mathematics K-6 Syllabus* (NSW Department of Education, 1989) describes teaching and learning units under the topic strands of number, space and measurement. Developing processes and problem solving competence is addressed in the introductory section where there is a brief discussion about the role of problem solving. Advise is provided which suggests that problem solving can form the basis of mathematical activity in classrooms and that the role of teachers is to challenge student thinking and to develop problem solving competence in their students. The *Syllabus* suggests that in mathematics classrooms, problem solving tasks provide opportunities for students to engage actively in processes such as planning, estimating, looking for patterns, conjecturing, generalising and evaluating.

Teaching strategies are also described in the *Syllabus*. These include encouraging discussion in small groups, allowing students to develop their own problem solving strategies and recording procedures, providing concrete materials, designing tasks and choosing situations that relate to students' interests and experiences. Mention is also made of listening to student responses, supporting thinking when necessary, encouraging student reflection and guiding student's invention of their own problems. The document clearly supports much of the advice about problem solving that is contained in the literature.

This study is part of a larger investigation that aims to explore the level of implementation of mathematical problem solving in primary classrooms in NSW. Teachers' beliefs about the role of problem solving in learning mathematics as well as their classroom practices is being investigated. Data collection focuses on teachers' knowledge and beliefs about the discipline of mathematics, teaching mathematics and students' cognition since all of these factors impacts on what occurs in teachers' classrooms (Ernest, 1989).

Method

A survey has been used to gather data about teachers' espoused beliefs and classroom practices, use of particular mathematical tasks and resource materials as well as their concerns and perceived needs in relation to the implementation of problem solving approaches. Part of the survey was designed to ascertain the frequency of use of a list of possible teaching strategies. Teachers were also asked to record how frequently they use particular types of problems. The meanings of each of the terms 'exercise', 'application problem', 'open-ended problem' and 'unfamiliar problem' was outlined in the introductory section of the survey so that a common understanding could be achieved. This approach was taken in an effort to avoid the disparate meanings that are attributed to the term 'problem'.

There were twenty items that related to problem solving teaching strategies. These items were chosen on the basis of strategies mentioned in the literature (Clarke, 1995;

Koehler & Grouws, 1992; Van Zoest, Jones & Thornton, 1994) as well as in curriculum documentation (NSW Department of Education, 1989). Teachers were requested to rate these items under the headings of 'hardly ever', 'sometimes', 'often' and 'almost always'.

The survey was trialed with 25 teachers, five of whom were interviewed to check meaning of items and to clarify interpretation of instructions. Each of these teachers was clear about the definitions of the types of problems. The original survey used the categories 'hardly ever', 'about once a month', 'about once a week' and 'almost always' but these were considered too restrictive by the respondents. Changes in wording and lay-out were also made on the basis of feedback from trialing.

The survey was then administered to a voluntary group of 78 teachers in twelve schools in NSW. The researcher addressed each school staff at a meeting and invited them to complete the survey. The staff sizes of the schools ranged from 4 to 30 in both metropolitan and rural settings. To compare these responses to those from an informed community, tertiary mathematics educators, or lecturers, in NSW and Victoria were invited to complete the survey items relating to the frequency of use of teaching strategies. The prompt asked them to complete it on the basis of their reading of the problem solving literature. Twenty-one lecturers responded to this request.

Results

The responses for both groups were scored from zero for 'hardly ever' to three for 'almost always'. This method was employed so that means could be calculated for each item and the responses of each group compared. It is acknowledged that the differences between categories are not equivalent and therefore this process is used as a gross measure for comparison purposes only.

Less Frequently Used Teaching Strategies

Items that had means less than one, or *low* means, for the teachers included:

- you ensure that students work alone (alone);
- you have calculators available for students to use (calculators);
- you present unfamiliar and open-ended problems to the class with very little indication of how to solve them (little help);
- you provide a set of problems and the students are allowed to choose a problem they would like to work on (choose problem); and
- you allow the class or individual students to spend several lessons on the same problem (spend more time).

Items that had means less than one for the lecturers included:

- teachers ensure that the students work alone (alone);
- teachers explain in detail what the students have to do to solve problems (explain); and
- teachers set exercises to allow the students to practise their skills (exercises).

The first item is the only one in common.

To compare the distributions of each of the seven distinct items listed above, box plots have been drawn in Figure 2. There is a large difference in the distributions for the use of calculators. Giving students problems and allowing them to explore possibilities with little help from the teacher is certainly not a frequently used strategy for the teachers who were surveyed. Teachers reported that they set exercises much more frequently than lecturers believe the literature recommends.

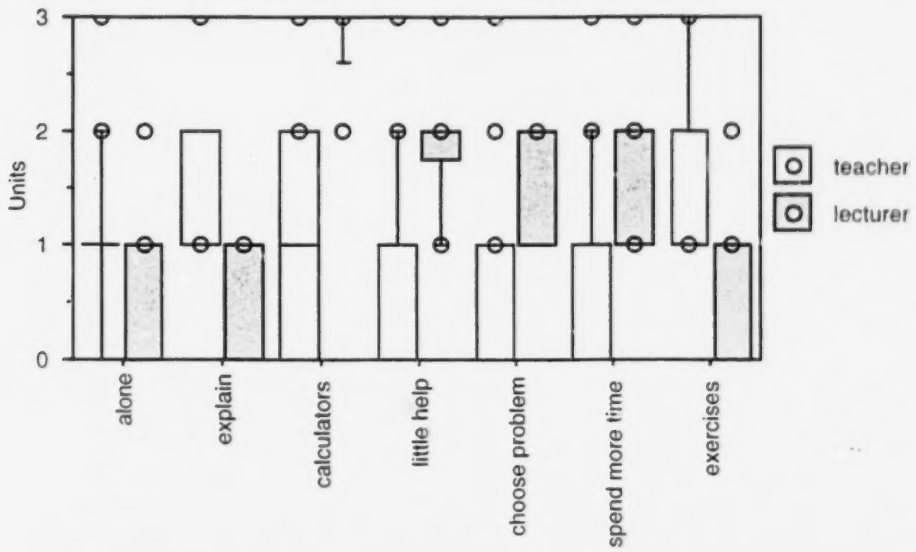

Figure 2. Box plots of items with *low* means for teachers and lecturers.

More Frequently Used Teaching Strategies

Items that had means greater than two, or *high* means, for both teachers and lecturers included:

- at the end of a problem solving lesson you lead a whole class discussion so that students can share solutions and strategies (discussion);
- you provide concrete materials for those students who need them (concrete materials); and
- you discuss useful problem solving strategies (strategies).

The item that had a *high* mean for the teachers alone was:

- you model the problem solving process to the class (model).

Items that had a *high* mean for the lecturers alone were:

- teachers have calculators available for students to use (calculators);
- teachers encourage the students to work in small, cooperative groups (groups);
- teachers encourage students to record their own procedures and methods of solving problems (record methods);
- teachers encourage students to pose their own problems (pose problems); and
- teachers use problems which arise from the school context or which relate to the students' experiences (experiences).

To compare the distributions of each of the nine distinct items listed above, for both teachers and lecturers, box plots have been drawn in Figure 3. Again there are differences in the distributions. Teachers reported a high frequency of use of modelling the problem solving process to the class. It is possible that this item may have been interpreted as teacher demonstration rather than specifically modelling problem solving. Encouraging small, cooperative groups and using problems that relate to students' experiences were used quite frequently by teachers although the means for these items were not greater than two. Encouraging students to record their own methods was used less frequently than lecturers believe to be desirable in mathematics classrooms. Encouraging students to pose their own problems was reported as a rarely used strategy by teachers.

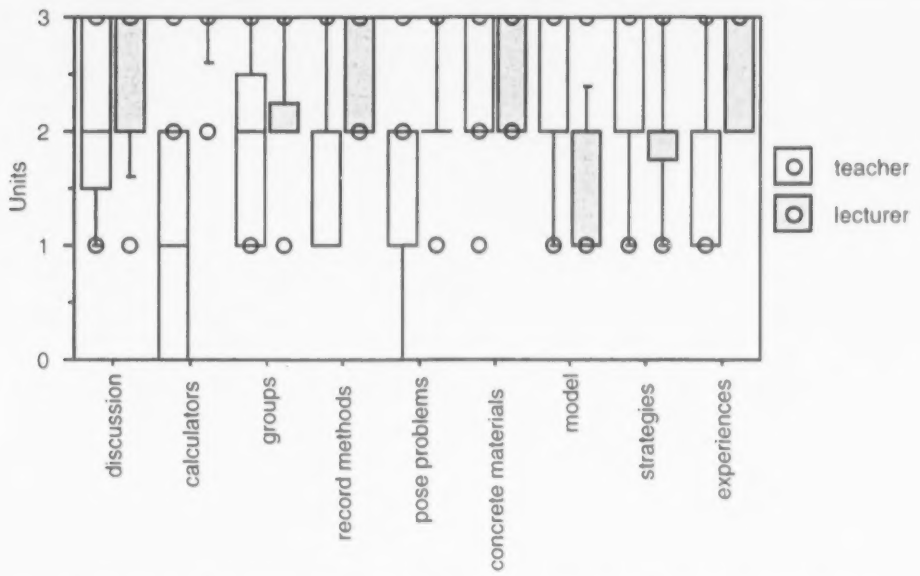

Figure 3. Box plots of items with *high* means for teachers and lecturers.

Frequency of Use of Different Types of Problems

The items that explored frequency of use of different types of problems revealed a wide range of responses from teachers. Table 1 lists the four items and the frequencies for each category. Overall, open-ended problems and unfamiliar problems are less frequently presented to students than application problems and exercises. However, it should be noted that there are clearly individual differences in frequency of use since a total of 24% of teachers pose open-ended problems 'often' or 'almost always' and 21% 'often' pose unfamiliar problems. Also, 7% of teachers 'hardly ever' and 28% 'sometimes' set exercises. These results demonstrate that some teachers appear to use open-ended and unfamiliar problems on a regular basis. This implies that they may use problem solving as a 'means' rather than as an 'end' in the learning of mathematics.

Table 1.
Frequencies of the teaching strategies for teachers which relate to problem types.

Teaching Strategy	Hardly Ever	Sometimes	Often	Almost Always
you present *application problems* which allow students to practise the skills they have just learnt	0	18	49	9
you pose *open-ended problems* to allow students to explore mathematical situations for themselves	8	50	16	2
you set *exercises* to allow the students to practise their skills	5	21	39	10
you pose *unfamiliar problems*	14	44	15	0

In response to these four items, lecturers recommended a higher frequency of use of open-ended and unfamiliar problems than the teachers reported. They also recommended a lower frequency of use of application problems and a much lower frequency of use of exercises. These differences between teachers and lecturers raise several questions about why all teachers have not responded to advice from the problem solving literature as well as from the *Syllabus*. It may be the case that implementing such

approaches is quite difficult given the constraints of classrooms or that lecturers are not in touch with the practical aspects of classroom and school life.

Implications

The strategies that lecturers have rated as being appropriate for regular use are recommended in advice given to teachers about problem solving in the reformed classroom as well as in curriculum documentation. These include whole-class discussion for sharing solutions and strategies; small, cooperative group discussion; provision of concrete materials and calculators; recognition of the need to encourage individual student recording of methods and procedures; and encouragement of student posed problems as well as using problems that relate to student interests.

Teachers still seem to be reluctant to implement some of these. It is clear from the results that most teachers see the value of whole-class discussion, teacher modelling and using concrete materials but teachers are not yet convinced that calculators are an integral part of the primary mathematics classroom. Also, there is less frequent use of individual student methods of recording and student created problems. This may suggest the persistence of a reliance by teachers on standard algorithmic procedures since teachers may believe that students need to learn the traditional or textbook method. It is also possible that teachers might feel threatened by unfamiliar methods and believe that it is important to maintain control of what children learn and how they represent their mathematics.

A small number of teachers seem to have responded to advice and do report using many of the teaching strategies that the literature claims promotes learning in mathematics. This does not of course guarantee that this is what is happening in their classrooms. Part of the larger investigation is to observe teachers as they implement problem solving strategies and to compare reported practices with observed practices. If teachers are in fact implementing these strategies then it is of interest to explore why other teachers have not responded to the same advice. Or, is it the case that teachers have different beliefs about teaching and learning mathematics? If teachers hold similar beliefs, it may be possible that constraints in some school cultures are much stronger than in others. Alternatively, particular school cultures may actively support a problem solving approach and encourage teachers to implement the strategies explored in this survey.

The results of the survey completed by these teachers provide valuable information about reported teaching strategies in relation to the use of problem solving in learning mathematics. A larger sample would help to confirm particular strategies that teachers have not adopted as recommended practice. There may be items where interpretation of meaning was unclear. This will be investigated in interviews that will be conducted with a sample of twelve of the respondents. From this small group, two teachers will be observed in an in-depth investigation of problem solving teaching practices. The school environments in which each of these teachers works will be explored so that constraints and support mechanisms can be identified.

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Students' and Lecturers' Perceptions of Factors Influencing Students' Success in First-Year Undergraduate Mathematics Courses.

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Qualitative data from lecturers and students were used to identify factors which were perceived as making the most important contributions to students' academic success or failure in first-year mathematics courses. A questionnaire based on this information highlighted similarities and differences in the perceptions of lecturers and students about influences on students' success and failure. The results confirm the importance of motivation and suggest further research is needed in the areas of active learning and student effort and workload.

Introduction

Most universities offer a range of first-year mathematics courses which cater for students of varying background knowledge and career aspirations. While in the past the assumption was that students entering a standard first-year mathematics course had recently completed a complementary mathematics course at secondary school, it is increasingly evident that students in calculus and algebra courses come from a wider spectrum. On enrolment students are advised of the recommended prerequisite mathematical background - but what emphasis should we place on this, and what should our response be to the mature student who successfully studied mathematics many years ago, or the otherwise successful student who, despite a weak background in mathematics, needs mathematics as a prerequisite for another major?

Models of learning (Anthony, 1996b; Biggs, 1993) all suggest that presage, process and context factors interact to influence outcomes of learning. While student presage factors such as prior content and metacognitive knowledge, and previous learning experiences are acknowledged as significant factors many of the other factors influencing students' success occur after enrolment. Teaching strategies, student motivation, approach to studying, cultural expectations, and numerous other factors are likely to influence students' success in undergraduate mathematics courses.

Research Study

The purpose of this study was to identify specific factors which are seen as having important influences on students' level of success. In support of the view that perceptions are not stable entities within cognitive structures but are dynamic and context dependent (Prosser, Trigwell, & Taylor, 1994) the study is specific to first-year mathematics courses. Moreover, because teaching and learning processes jointly influence learning outcomes the study is based on the perceptions of both lecturers and students. Crawford, Gordon, Nicholas and Prosser's (1994) study involving beginning mathematics students stressed "the need for a shift in attention away from teaching and learning as independent activities towards a more systemic examination of the relationships between the two activities and the context in which they occur" (p. 344). An exploration of potential differences in lecturers' and students' perceptions will provide an opportunity for both teachers and students to assess their respective roles in the learning process.

The study was conducted in three phases involving exploratory open-ended questionaries, Likert-type questionnaires and student interviews. The first two phases, which are reported in this paper, use a similar methodology from an earlier study (Killen, 1994) which involved students and lecturers across all university disciplines.

Phase One

In an attempt to identify factors which have important influences on success 65 students (from a range of first-year mathematics courses) and 22 lecturers and tutors (all experienced in first-year undergraduate teaching) were asked to complete a questionnaire on which they responded freely to two items: "List five factors that you think are most

important in contributing to students' success in first-year mathematics courses" and "List five factors that you think are most likely to lead to student failure in first-year mathematics courses". A covering information letter explained the purpose of the research and indicated that success was defined as completing the course and receiving a passing grade, whereas failure was defined as receiving a fail grade.

The students and lecturers identified 63 and 53 separate factors, respectively, (such as "completing assignments" and "interesting lectures") as contributing to student success, and 60 and 54 factors, respectively, (such as "not attending lectures" and "boring lectures") as contributing to student failure. These responses were analysed separately and placed into the broad categories suggested by Killen's (1994) study: lectures, course, students and other external factors. While both lecturers and student attributed student success and failure to the same four general influences there was a difference emphasis on the factors within each category. For students, 24% of success factors related to lectures, 23% to course material and design, 47% to the student, and 6% to other external factors. By comparison, lecturers' responses included 15% related to lectures, 12% related to the course, 68% related to students and 5% related to other external factors. For factors influencing failure, student responses were apportioned 21% to lectures, 15% to course material and design, 55% to students and 5% to external factors. By comparison, lecturers' responses included 13% of factors related to lectures, 10% related to the course, 67% related to students and 10% related to other external factors. Thus, the trend indicated in this initial data suggested that while both students and lecturers suggested student factors most often, students tended to place more responsibility for their level of success on the lectures and course design that did lecturers.

Phase Two

In order to investigate the trends that were evident in the data from the first phase of the research, the factors that had been identified by students and lecturers as contributing to student success and failure were used to create two parallel questionnaires (one for students and one for lecturers). The questionnaires contained 40 statements related to factors influencing student success and 37 statements related to factors influencing student failure in first-year mathematics. The items were selected on the basis of frequency from the initial surveys in phase one of the research. The questionnaires were administered to 92 students (64 male and 28 female) nearing the end of a first-year calculus course and 26 lecturers (19 male and 7 female). The respondents used a five-point Likert-type scale to indicate the extent to which they thought that each factor influenced student success or failure (1= not at all, 5= greatly).

RESULTS

'Success' Questions

On all but two items on the 'success' scale student responses covered the full range from 1 (not at all) to 5 (greatly). The items concerning the availability of help and plenty of worked examples in the lectures received a minimum score of 2. For lecturers the range of responses reflected a more unified opinion, with 13 of the top-ranked responses having a minimum score of 3 or 4. In general, ranking agreement between students and lecturers was high (Spearman's Rho was 0.611, $p < .01$). The item rated most highly (most likely to influence success) by both students and lecturers was "self-motivation". Additionally, students' high ranking items indicated that assessment demands were very important, as was the quality and availability of support, but high ranking items did not include "regular practice of examples".

Data analysis, using the non-parametric Mann-Whitney U test based on ranking, was used to determine if there were any significant differences between lecturers' and students' scores for each item. An analyses of scores and respective ranking of 'success' items is presented in Appendix A. Responses for five items were significantly different at the 1% level, two at the 5% level, and a further two at the 10% level.

Lecturers gave significantly more support for the need for courses to have realistic expectations of prior knowledge and clear requirements, regular practice of examples, and adequate background knowledge: all factors which should support optimal motivation and sustained student involvement (Paris & Turner, 1994).

In contrast, students placed more emphasis on factors related to course design and organisation (availability of help and availability of information booklet in exams) and student behaviours (regular attendance at lectures and tutorials, taking notes during lectures, active attention in lectures and an appropriate balance of social and academic life).

'Failure' Questions

It is apparent, that for the most part, those factors which influence failure most significantly indicated a lack of those that influence success. On all items on the 'failure' scale student responses covered the full range from 1 (not at all) to 5 (greatly). Again, lecturers appeared to be in much more agreement for 'failure' items, with minimum scores of 2 for most items and scores of 4 or 5 for "insufficient work". The rank order correlation (Spearman's Rho) was 0.586 which is significant ($p < .01$), but not as strong as the rank order correlation on the 'success' questionnaire. The items "lack of effort" and "lack of motivation" were jointly ranked highly influential factors for failure by both lecturers and students.

There was considerable overall agreement in scores (Appendix B) with only four items which showed significant differences at the 5% level and a further six items at the 10% level. Lecturers placed more importance on poor study techniques, insufficient work, inadequate mathematics background knowledge, difficulties adapting to university life, and problems with student's personal life, than did students. These factors are all related to student characteristics or behaviours. In contrast, items which were rated significantly higher by students included boring presentations of lectures, non attendance of lectures, and a perceived lack of relevance of paper content. These factors all related to lectures and course design.

Related to lack of effort is the item "insufficient work". This item was ranked first by lecturers but only eighteenth by students. In an additional section in the questionnaire concerning student workload only half of the students reported attending all four mathematics lectures per week and 55% of the students reported spending 4 or fewer hours on individual mathematics study and assignment work per week. The disparity between lecturers' expectation and student work is clear when less than 20% of students reported completing the recommended hours of study and lecture attendance. These figures also reflect students' higher rankings of attendance of lectures and tutorials than regular practice of examples and reverse order rankings for lecturers.

Both students and lecturers rated poor study techniques as a more influential factor in failure than inadequate mathematics background knowledge. For many students poor performance is largely due to ignorance about the study skills required, or the inability to apply these skills appropriately, rather than lack of ability (Manalo, Wong-Toi, & Henning, 1996).

Comparisons by Performance

Comparison of lecturers and students by performance further differentiates differences in perceptions. In response to the 'success' questionnaire A-grade students accorded less importance ($p < .05$) than lecturers to adequate background knowledge, regular practice of examples and reading before lectures, but more importance to regular attendance of lectures, active attention in lectures, taking notes in lectures, assignment completion, and the availability of the information booklet in exams and help in general. Failing students, on the other hand, placed less importance than lecturers on the need for the course to have clear requirements and realistic expectations of prior knowledge, and adequate background knowledge, and more importance than lecturers on the availability of information booklet in exams and note-taking in lectures.

In response to the 'failure' questionnaire A-grade students accorded less importance ($p < .05$) than lecturers to the influence of lecturers with unrealistically high expectations of students, inadequate background knowledge, too much content, insufficient work, difficulties in adapting to the university environment, and financial and persons problems. However, A-grade students ranked lack of relevance and non-attendance at lectures as more important contributors to failure than did lecturers. Failing students showed only three significantly different responses to lecturers on the 'failure' questionnaire. They accorded less importance to insufficient work and rote learning and more importance to too many demands on time.

Comparison of student responses, based on student success (defined as receiving a passing grade) and failure, were also conducted. Statistically significant findings ($p < .05$) indicated that when considering factors influencing success successful students placed more importance on:

- the availability of worked examples in lectures and tutorials;
- clear presentation of lectures;
- the need to make the paper requirements clear;
- assignment completion;
- understanding rather than rote learning;
- the ability to work independently;

and less emphasis on reading before lectures.

When considering factors influencing failure, successful students suggested that rote learning was a more significant influence than did unsuccessful students. Unsuccessful students, however, considered that difficult lectures, inadequate mathematics background and too many demands on time were more influential factors than did successful students.

Gender Comparisons

No statistically significant ($p < .05$) differences between student gender were noted in the responses to the 'success' questionnaire and only two differences were found in the responses to the 'failure' questionnaire. Females placed less importance on the influence of boring lecture presentations, and males placed less importance on assignment completion as a reason for failure. These finding may offer limited support to gender differences for attributions to failure, with males less likely to attribute effort and females less likely to attribute external sources. Furthermore, assuming students' perceptions influence learning approaches (Taylor, 1996), then the overall lack of differentiation by gender offers support to Richardson's (1993) assertion that there is no significant difference between male and female responses to approaches to studying.

Discussion and Implications for Further Research

Motivation is seen by both students and lecturers as the most influential factor related to levels of success. While highlighting the importance of motivation, data of this type does not allow us to explore whether or not motivation is directed towards achievement or performance goals, nor investigate factors in the learning environment which serve to increase or decrease motivation. The fact that students, and A-grade students in particular, placed significantly more importance on the regular attendance of lectures than did lecturers may reveal the indirect motivational values, apart from cognitive content, that lectures offer. Lecturers in this study, aware of the fact that study material prepared for distance teaching is available for students to study independently, may regard lectures as an 'extra'. However, by helping students become aware of conflicting points of view or by challenging ideas that students have previously taken for granted lectures can stimulate further learning. Moreover, the lecturer's own attitudes and enthusiasm, modeling of problem solving and mathematical thinking are added values which are difficult to portray in written study material.

Within the lecture situation students placed more importance than lecturers on active learning and note-taking. Does this mean that lecturers see their role as information providers, rather than providing opportunities for students to be actively engaged in analysing and processing information? Burroughs-Lange (1996) found that lecturers assumed student needs were about mastery of the subject matter rather than the needs that had been articulated by their students. Further research is needed to explore approaches involved in the lectures from both the lecturers' and students' perspective.

How do students and lecturers apportion responsibility for learning? Studies (Anthony, 1996a; Burroughs-Lange, 1996) suggest that in the classroom/lecture situation teachers do most of the cognitive work, leaving students to do the less demanding tasks. Broadly speaking, it appears that lecturers in this study were more inclined than students to attribute student failure either to factors which were related to student entry characteristics or within the control of students. Students, however, placed greater

emphasis than lecturers on those factors related to lectures and course design. Taylor (1996) noted that although students expressed a clear, but individual sense of who was responsible for what in terms of learning, few appreciated that the negotiation and exercise of that responsibility could be a joint rather than individual project.

When comparing successful and unsuccessful student, A-grade students accorded more weight than lecturers to those factors directly within their control, such as lecture attendance, active participation and assignment completion, whereas failing students placed more importance than lecturers on the supply of information, either in the form of lecture notes, or information booklet. A distinguishing factor between successful students and failing students was the relative importance placed on the need to make the paper requirements clear. Other studies (Anthony, 1996b; Jones & Nuich, 1992) found significant variation in how well students are 'cued into' the kind of work that is necessary to achieve examination success. 'Cue deaf' students can put in a lot of hard work, achieve significant understanding, yet because of ineffective study skills achieve little success in terms of grades.

This study provides a starting point for bringing together the voices of students and lecturers. In particular, the comparison between students' and lecturers' perceptions provides a basis for further research into how perceptions are formed, and what influences perceptions have in affecting learning and teaching approaches and subsequent learning outcomes. The results of the initial phases of this study suggest that the exploration of the teaching-learning phenomenon from the viewpoint of all participants will build an environment of open communication and understanding between teacher and student, with the expectation that improvements in the quality of learning and education will follow.

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Appendix A: Comparison of lecturers' and students' responses on 'success' items

Questionnaire Item	Mean Student	Mean Lecturer	p level	Rank Student	Rank Lecturer
Self motivation	4.46	4.73		1	1
Study for tests and exams	4.34	4.58		2	2
Making sure you understand the basics in each topic	4.33	4.35		3	5
Assignment completion	4.20	4.19		4	13
Willingness to seek help when needed	4.18	4.38		5	3
Availability of help	4.17	3.77	.006	6	25
Clearly presented lectures (boardwork and language)	4.15	4.23		7	9
Plenty of worked examples in the lectures and tutorials	4.15	4.15		8	18
Desire to understand the material rather than rote learn	4.13	4.27		9	8
Assignments are related to lecture content	4.12	4.00		10	16
Supportive and approachable lecturers/tutors	4.11	4.19		11	11
Consistent effort	4.07	3.96		12	17
Paper material is well designed with lots of examples	4.07	4.31		13	15
Well structured lecture presentations	4.05	4.15		14	14
Regular attendance at lectures and tutorials (where appropriate)	3.99	3.50	.025	15	31
Taking notes during lectures	3.96	2.85	.000	16	39
Revising past test and exam questions	3.95	3.73		17	27
Ability to work independently	3.91	3.81		18	24
Suitable study environment free of distractions	3.90	3.62		19	29
Regular practice of examples	3.89	4.31	.097	20	6
Effective study skills	3.87	4.19		21	12
Student interest in the paper	3.86	3.92		22	19
Active attention in lectures	3.84	3.38	.049	23	34
Ability to cope with stress	3.83	3.58		24	30
The paper requirements are made clear	3.82	3.88		25	20
Overall workload for paper is appropriate	3.81	3.81		26	25
Lecturer/tutor is enthusiastic	3.80	3.85		27	22
Availability of 'information booklet' in tests and exams	3.79	2.76	.000	28	40
Interesting lectures	3.77	3.85		29	23
Paper has realistic expectations of prior knowledge	3.75	4.38	.003	30	4
An appropriate balance of social and academic life	3.72	3.35	.090	31	35
Ability to think mathematically	3.71	3.68		32	28
Lecturer who can inspire students	3.71	3.88		33	21
Time management	3.69	3.92		34	18
Appropriate level of internal assessment	3.68	3.42		35	33
Adequate background knowledge	3.58	4.23	.005	36	10
Orderly and controlled lecture environment	3.47	3.42		37	32
Relevance of paper to other subjects	2.87	2.88		38	38
Support from friends	2.62	2.92		39	37
Reading material before each lecture	2.61	3.00		40	36

Appendix B: Comparison of lecturers' and students' responses on 'failure' items

Questionnaire Item	Mean Student	Mean Lecturer	p level	Rank Student	Rank Lecturer
Lack of effort	4.13	4.55	.098	1	2
Lack of self-motivation	4.11	4.36		2	3
Lectures are difficult to understand	4.09	4.16		3	6
Poor quality lecturing	4.00	4.12		4	8
Not keeping up with paper progress	3.97	4.32		5	4
Failure to seek help when needed	3.94	4.16		6	7
Lack of interest in the paper	3.87	3.64		7	22
Boring presentations by lecturers	3.86	3.48	.046	8	30
Lecturer/tutor is unsupportive or unapproachable	3.85	3.72		9	19
Not completing assignments	3.83	4.08		10	9
Overall workload too heavy	3.81	3.79		11	17
Becoming discouraged	3.78	3.92		12	13
Not paying enough attention in lectures	3.77	3.52		13	27
Not enough worked examples in paper material	3.76	3.92		14	14
Poor study techniques	3.74	4.20	.012	15	5
Not attending lectures or tutorials (where appropriate)	3.74	3.24	.065	16	34
Under-estimation of the work required	3.69	3.92		17	12
Insufficient work	3.68	4.60	.001	18	1
Too many demands on students' time	3.68	3.52		19	28
Inability to balance study and social commitments	3.68	3.50		20	29
Heavy paper workload	3.65	3.40		21	33
Emphasis on rote learning rather than understanding	3.61	3.72		22	18
Lack of a regular study routine	3.60	3.64		23	23
Inadequate mathematics background knowledge	3.56	3.96	.065	24	11
Too much content in lectures	3.55	3.80		25	16
Lack of mathematical ability	3.54	3.54		26	26
Suitable help difficult to find	3.51	3.84		27	13
Noisy or crowded lectures	3.49	3.60		28	24
Lack of confidence	3.43	3.68		29	21
A perceived lack of relevance of the paper content	3.37	3.00	.066	30	36
Expectations for the paper are not clear	3.36	3.40		31	32
Complacency or over-confidence	3.34	3.56		32	25
Lecturers/tutors who have unrealistically high expectations of students	3.30	4.04	.003	33	10
Problems with personal life	3.15	3.71	.051	34	20
Difficulties adapting to the university environment	3.03	3.48	.083	35	31
Not enough internal assessment	2.85	2.76		36	37
Financial problems	2.67	3.00		37	35

The Construction of Personal Theory on Gender and Mathematics: Nine Case Studies of Women Primary Teacher Trainees

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This paper discusses the experiences of nine women in their high school mathematics and their personal theories on gender and mathematics. Interviews with the women showed that while their performance was above average up until Year 10, only one student completed higher mathematics. In general, support from mathematics teachers was seen as very negative. Further, their personal theories on gender and mathematics are the result of interaction between their own experiences and the research knowledge they encountered at university during their completion of a teacher training course.

There are encouraging signs that gender differences in achievement in mathematics are now less significant than earlier research has shown (see Burton, 1986, 1990; Fennema & Leder, 1990; Leder, 1992; Leder & Forgasz, 1992). There does remain however, some evidence for differences in attitudes toward mathematics as expressed by females and males beyond the primary schools. For instance, females tend to underestimate their ability in mathematics and attribute it to hard work rather than to natural ability. Even in cases where females attitudes toward the subject is not lower than males, they tend to opt out of higher level mathematics sooner. Also the impact of early childhood and primary schooling on how girls and women develop negative attitudes toward mathematics has been well documented (Clark, 1990; Kenway & Willis, 1993; Large, 1993). Although several factors may contribute to the development of attitudes toward school subjects, the role of the teacher is quite significant. Recent reports (Cobbin, 1995) have argued that many trainee teachers remain weak in mathematics and science and have little interest in teaching the subjects. This is especially significant for women as they continue to dominate the ranks of early childhood and primary training and teaching (Aspinal & Drummond, 1989; Cobbin, 1995). Furthermore, Southwell and Khamis (1992) hypothesised that "if pre-service primary teachers hold beliefs to the effect that mathematics is a very difficult subject, that they are not good mathematicians and that mathematics has limited appeal and use, their teaching will reflect these beliefs and each new generation of learners will be locked into similar beliefs and negative attitudes" (p. 497). Clark (1990) reported that many Australian women went into primary teaching because they saw themselves as "not good at mathematics." Other reasons women cited for taking up primary teaching included family pressure on them to do so, the preference they have in working with younger children, lack of other options and a fear that they would fail in non-traditional fields of work and study.

Previous research had not examined primary teachers' views on gender issues in mathematics education. This paper investigates the construction of personal theories on gender issues of a group of graduating women teacher trainees as a function of the women's experiences in school mathematics. Firstly, the achievement, confidence and participation of these women teachers in high school mathematics will be examined. Secondly, the opinions of these women on the support that they obtained in their school studies will be discussed. Thirdly, the paper outlines the personal theories of these women on gender issues. Fourthly, the relationship of the personal theories and the women's experiences in mathematics will be examined.

The Study

The present study is the first stage of a longitudinal action research project (Atweh, Kym, Burnett, 1996) conducted with a group of women teachers in transition from a four year university course into the first year of teaching practice. The project investigates the needs of these women to make primary mathematics teaching more inclusive. The study commenced in 1996 with a survey of final year women students from a Bachelor of Education course at a university and was followed through with an interview with a

selected sample. Atweh, Kym and Burnett have reported on the survey results and this paper reports on the interviews conducted with the women.

In 1996 about 100 women students undertaking a final year mathematics education unit were given a questionnaire on their attitudes and beliefs about mathematics and mathematics education. The women were invited to participate in a longitudinal study. About 13 women volunteered, only 9 made themselves available for the interviews. Semi-structured interviews were conducted to investigate in depth the previous experiences of this group of women in both school and university mathematics classes, their expectations and concerns about teaching mathematics and their views on gender issues. These interviews were audio taped and transcribed. They were conducted at the university and were between 1-2 hours each.

In Queensland, most high schools begin steaming mathematics in Year 10 where students choose from Ordinary Mathematics - the lower non academic stream - and Advanced Mathematics. At Years 11 and 12, students may take Mathematics and Society, Maths I or Maths I and II.

Findings

Achievement, Confidence and Participation

Four of the nine women interviewed described their mathematics performance up to Year 10 as "top of the class" (Fran, Mary, Sandra, and Nellie). However none of them did higher level mathematics in Years 11 and 12. Two of the women attempted such a course but then dropped after 1 or 2 semesters. Mary, received High Achievement in Advanced Mathematics in Year 10, yet she always felt that she lacked the understanding and meaning behind the content learnt. She saw no point in learning mathematics. She attempted Maths I to keep her options open for university entry. When she realised that mathematics was not required for entry into her chosen course, she dropped mathematics all together rather than doing the lower stream mathematics ("vegie maths" she call it) or risk failing Maths I. However, she indicated that she feels quite confident in doing and using mathematics in everyday life situations.

Similar experiences were related by Sandra. She was at the highest level of performance throughout primary school and lower secondary school. In Year 11 she choose Maths I. She reported that she "was always asking the teachers 'why are we doing this and where would we use it'" She stated that none of the teachers could answer these questions - "I just didn't see the point in doing it, I just couldn't understand it and it was just too difficult - so [I dropped it]".

Nellie did not attempt Maths I. Although she topped the class in Mathematics and Society, she described herself as "not a mathematical person". She indicated that she feels more confident in low level mathematics and feels quite confident in using mathematics in everyday life. Similarly, Fran belonged to a group of three top achieving students (herself and 2 male students) that always finished class work early and were given additional advanced work by the teacher. Algebra "scared" her because it took her longer to understand, it did not have any apparent application and was a source of failure for many of her friends. Considering her career aspirations, she decided she did not wish to enrol in higher mathematics in Years 11 and 12. Her mathematics teacher in Year 9 made several attempts to encouraged her to change her mind. She considered his views but decided against the advice. Failing to convinced her, he rang her parents and requested their support. Fran argued that she did not need higher mathematics, she enjoyed Mathematics and Society and she was a "little scared" of the higher mathematics.

Two other students had achieved above average up to Year 10 (Jane and Jackie). Jane was the only women in the sample to have completed Maths I in Years 11 and 12. She reported that she enjoyed mathematics when its relevance or application was obvious. She "switched off mathematics" at Year 11 when she encountered higher algebra (this she calls "scientific mathematics" e.g. trigonometry). She realised she could have achieved better with extra work and rote learnt the content, but she did not feel that this would have given her the understanding and meaning she lacked. She enjoyed mathematics in accounting where she could see its logical structure and obvious use. She had a positive self image of her ability up until Year 10. When she joined the higher course of Maths I, her confidence and self image declined when she compared her achievement with that of

the more able students. Similar experience was related by Jackie who felt that she "was above average up to Year 9". Being placed with the top stream of advanced mathematics at Year 10 she felt "dumb" compared to other students. Jackie also suffered from exam failure anxiety. She declared: "In Year 10 I failed one test, I cried - I could not handle it. I felt so sick of not being able to do it. I thought I put a lot of work into it". In Years 11 and 12 she did Mathematics and Society.

Louise also described herself above average up to Year 10. She felt confident in using mathematics and problem solving in everyday life yet she felt anxious if someone asked her to do something in her head. (Public anxiety about lack of mathematics ability identified by a few of the respondents.) For the same reason she felt low confidence in teaching mathematics - "you can not afford making a mistake with the children, because "once they learnt it wrongly, is very difficult to unlearn it." She developed low expectations of her ability for success in mathematics - "I just want to pass". She did Mathematics in Society because she "was not good enough to go into higher mathematics - I did it because it was compulsory."

Pam and Kym had difficulties in mathematics that can be traced to the primary school. Pam had an eye problem in the early primary school that delayed her school achievement. She developed low self esteem in her primary and secondary years and a self image of "lower than average in mathematics" and "not very competitive". Having studied ordinary mathematical at Year 10, she was locked into the lower stream mathematics. She is very confident now that she could have handled Maths I at Years 11 and 12. Sometimes she wished she "could do it all over again" to show that she could pass Maths I! She indicated that she was seriously considering it. Similarly, Kym's problems in learning mathematics started early. She recalls the difficulty she had in learning number facts. To cope with her lack of understanding, she often resorted to "learning thing by heart without understanding". She developed a good sense of what was important on the exam and "cramming it" and "somewhat manage to get through". Because she was good in other subjects her difficulty in mathematics was not noticed, or if noticed not taken as a serious problem by teachers nor family. Similar pattern occurred at university; for instance, because she was getting 7's in her subjects, when she complained to a lecturer that she was not really good in mathematics, he simply affirmed that she was "a good student - a really good student".

Support Received

Kym grew up in a rural area where little help was available. She realised she had difficulties in mathematics in the primary school. While at some time she may have felt "dumb" about her lack of ability she now says that she does not blame herself. She harbours some anger against the teachers who "did not really pick on the fact that I was having trouble". With her father being away most of the time, and request for help form her mother often lead to "screaming matches", Kym ended up in a situation of not seeking help at all. She often tried hard to conceal her lack of ability and understanding. "I always had really neat work, ... so that it looked like I was understanding it". She stated that patience from her parents would have been beneficial for her to overcome her difficulties. She tried to ask for help from teachers, only to be given more examples to do on her own.

This pattern of concealing difficulties in learning mathematics from others is not isolated in this group. Mary had decided she did not want to do lower level mathematics at Years 11 and 12. She felt she did not receive any support from her parents. She described her father as someone who "really does not know what is going on in my subjects". Her mother supported her decision to keep her options open by attempting the higher level mathematics. Her parents did provide a computer tutoring package, Promath, and a private tutor. In school, Mary did not want her teachers to know about her problems. She tried to hide her difficulties - "did not want any attention". She described her anxiety about "not knowing what to ask for" when she did not understand mathematics. Talking to a mathematics teacher about her low performance during the first term in Year 11, and about her plans for the future, he said "well you can keep going [in the subject] if you want to" but he made it clear that "he didn't really see the point" of her doing so. This was not seen as negative by Mary. She described her feeling as: "I was like, Hallelujah!" It was the ticket to her freedom from mathematics!

Louise and Sandra had generally supportive parents. Yet they both reported regrettable lack of support from teachers in schools. Sandra described her mathematics teachers as people who "do not care and they were distant". Louise said they "just give you a lesson and its up to you to understand it or not". They both mentioned other teachers, such as English and humanities, who were normally much more supportive than any of their mathematics teachers.

Two more students complained about negative effects of lack of support from mathematics teachers. Pam indicated that she tried to enrol in a higher mathematics in Year 10. She did receive some support from her parents on that decision. Her parents "did not push", but then she added "I am the type of a person who if I was pushed I would have done it". However, support from the teacher at school was minimal. One teacher in particular "was negative all the time and her attitudes was 'if you can not handle it you should drop back [to a lower mathematics class]'". Pam was quick to add that "I hold no grudges. That is why I am interested in this project - when I become a teacher, I want to do the best and encourage [everybody to do well in the subject]". During her high school, one teacher showed special interest in Pam and sought the help of a science teacher. Her performance improved considerably, until the assisting teacher left the school and things went "back to the same old problem".

Jane also received little support from her teachers. When asked whether anyone in high school encouraged her to do well in mathematics she said:

> *No, I wouldn't think so. The teachers probably discouraged me more than anything because it was the one's who were always getting it right they were getting the attention and the one's that were falling behind it just didn't matter. Happened more in senior grades, like it was your fault, if you weren't doing well you hadn't done your homework so it's out of their hands.*

Jackie attributed the lack of teacher support towards her as a gender issue. The first secondary school she attended was an all girls' school where she felt she received considerable support. Moving to a mixed, and larger school in Year 11 and 12, made it difficult for the "teachers to know you - I felt in the background." Perhaps what made it more difficult was that the new school was all boys until Year 10 and then mixed at 11 and 12.

Only two of the interviewees indicated positive support from teachers in high school. Nellie was grateful for her teachers efforts to correct errors and look closely at her work. Fran talked about having a good supportive, yet not pushy family. At one stage of the interview she said "it is definitely the family influence that's where I am". Likewise, the support of the teacher in Year 9 reported above, was also given by Fran as an example of teachers encouraging girls to do well in and study more mathematics.

Personal Theories of Gender and Mathematics

Three of the women interviewed indicated that their experience did not match what they read at university about gender differences in mathematics. Sandra disagreed with emphasis from research findings that, "more attention is given to boys" in mathematics classes. In her experience she "had not observed this at all". She asserted that mathematics is compulsory up to grade 10 and un-streamed until grade 9. Although she did not do higher mathematics she thought that the gender distribution was about equal. She added "from my experience the people who got extra help were the ones who asked for it; I found that it was mostly girls who asked for help and they got it." Sandra does not refute all research findings. In fact when she was asked about evidence that even successful girls tend to move away from mathematics she said, "I mean it's been proven so I accept it, but I really don't know why they have moved away from it". In her view, to avoid this drainage away from mathematics, teachers can encourage girls more and commence career awareness at the end of primary school.

Mary also questioned common assumptions that are derived from research, "[I]n all my classes there was a balance, so I really didn't see that there was a gender inequity happening - except in English where the teacher related differently to the female and male students". When asked about the reasons why girls dropped out of mathematics more than boys, she presented various hypotheses: "Maybe its still that stereotypical thing that there are the jobs that women don't need a mathematics basis for them; ... maybe they

find it too mentally demanding and they cant handle it or something; ... [or maybe] they do not see why they need it". Her answer to the question of how to encourage girls to stay in mathematics longer, brought a lengthy, and very illuminative response that is worth quoting at length. The underlined phrases will be used in the analysis below:

> *I don't know I really think ... [dropping out of mathematics is] a personal thing. If you don't, I don't know, maybe I'm not looking at the wider picture, it was like an individual thing with me because even though I knew that I was doing the stereotypical thing of going into teaching like "good girls" do, [i.e.] it's a female oriented thing, and doing the humanity thing at uni, but I liked them. Maybe, you know, you have to say "but what influenced you to like them and not the other", and it's like I, I didn't see myself as being worked against in the feminist stereotypical thing. But then again, that's just because I'm not aware of it doesn't mean it's not happening, but I never sort of felt any pressure that I should do it or I shouldn't do it. It was sort of just never there with me. I don't know whether it's there with other people. I don't know, sometimes I think it's being over rated that just girls are not going into maths. ... But then again I was at an all girl's school from grade eight to ten anyway, so I didn't see the difference in numbers going into maths. Like they were all girls there anyway so that's probably just based on my own personal experience. ... Maybe girls don't want to be seen as being bright or especially girls in the more low socio-economic areas that want to be seen, because they've been urged into the direction that their life is going, to be like their parent's life....*

It is interesting to note the following tensions and forces in this response. First, there was the tension between the personal experiences ("it was an individual thing with me") and the generalisations from research findings ("maybe I'm not looking at the big picture"). Similarly, there was an obvious tension between personal freedom in choices ("I liked them") and societal conditioning in making selections ("I was doing the stereotype thing"). There is also a tension between the obvious blatant coercion ("I did not see myself as being worked against") and the more subtle influencing ("Because I'm not aware of it does not mean its not happening"). There was also the tendency of making research findings apply to the "other" ("especially girls in the more low socio-economic areas").

Similarly there was some kind of ambivalence in Fran personal theory on gender and mathematics. She asserted that gender is not a factor of success in mathematics. Success in mathematics is a personal decision. She accepted the possibility that there may be cases where boys were encouraged more to do higher mathematics, but "I have never came across it". Likewise she has "never seen teacher favouring boys" in mathematics classes. Her experience with the male teacher who encouraged her to study higher mathematics at grade 11 related above was presented as a counter-example to what she has "been taught at the university". She exaggerated the research findings by portraying a picture in which the girls are "just being scared off it [i.e. mathematics] altogether". However, Fran conceded that some high school students may miss out upon role models and hence have no incentive to do mathematics as a result of male domination of the numbers of mathematics teachers. Further, she believed that being aware of the issues of gender and mathematics is important for primary school teachers in order to avoid falling into practices that may alienate or discriminate against the girls.

Other students did not see a conflict between their experience and research findings portrayed to them during their university course. Nellie indicated that she went to a girls school. Although conceding the school may differ in their assessment, she did not think that the school had a high focus on mathematics. It was "rather acceptable for the girls just to go to Mathematics and Society". She related the experiences of brother who attended a mixed school where "the attitude there is to just do the mathematics and see where it lead you". As for the reasons why girls drop out of mathematics, she present two possibilities: boys' domination in the classroom or that "girls find mathematics too rigid to express their emotions so they choose language subjects".

Louise asserted that gender had a limited effect on achievement in mathematics. She held the view that "it is proven that men are better in spatial ability than women. I do

not know if it is the educational system, men and women are totally different and they think differently. It could be biological". However, teachers can do something about success and participation of girls by making mathematics more "fun or interesting" and do more to encourage girls to succeed.

Girls' participation in higher mathematics, according to Jane, depended upon whether the girls could see mathematics as an important subject which in turn depended upon their career aspirations and values in life. Girls self confidence is important for their success and participation which in turn is influenced by attitudes of others in the classroom.

Pam also went to an all girls' school which placed special emphasis on mathematics and science. She recalled the example of her brother who has a better understanding in mathematics, hypothesising that his involvement in manual arts may have increased his ability to understand mathematics. She pointed out the "gender bias" in her household in terms of division of labour between the females and males.

Jackie also sighted examples of males within her family who have been able to achieve good results in mathematics. She asserted that:

> *as a girl in classroom you tend to be forgotten unless you are outspoken. You have to demand assistance and attention. Behaviour problems form the boys imply that they do not get much attention. ... When I was in a single sex school I felt much more confident - even though we had bad behaving girls, it was different - they cared more. The mixed school had a better reputation so I went to it. I would probably have done much better if I went to single sex school at 11 and 12.*

Finally, Kym points out the "stereotype [sic] view that boys should be better at maths and science and girls should be better at English and arts and that sort of thing. If the teacher holds these kinds of views then it can become a self fulfilling prophecy". Female teachers are not immuned from such stereotypes, according to Kym. Further, this stereotyping can also come from the home. She recalls her "mum always used to say "oh don't worry I wasn't very good at maths either".

Conclusion

Undoubtedly, the case studies reported above show a wide range of experiences, attitudes, beliefs and expectations of this particular group of women as they approach their first year of teaching. In the midst of this diversity, the following patters may be identified. The majority of the women saw themselves as high achievers, or above average, up until Year 10. Yet, almost none of them attempted higher mathematics at the senior school. There was also strong evidence that the majority found teacher support to be minimal, if not of a negative effect.

For someone who is familiar with the research on the experiences of women in mathematics, these patterns are not surprising. What is of interest here are the views of the women on gender issues. The three women (Sandra, Mary and Fran) who questioned research evidence based on their own experiences were high achievers in school mathematics. The choices that they made to drop mathematics seemed to them to be based on individual choice. Arguably, they failed to see the pattern of dropping out of mathematics as more relevant to women in general. There was also evidence that the other women who have seen their experiences as consistent with research findings also were influenced by their own experiences. For example, Kym has developed difficulties in mathematics early in her schooling. As an adult she had some anger at her parents and teachers for not taking action about these difficulties. One reason she raised for not identifying the problem, was that she "was good in other subjects". This lack of action on her difficulties by the others is consistent with a possible belief on their part that girls are not supposed to be good in mathematics and/or do not need as much mathematics. This is consistent with Kym's belief that teachers stereotypes may become self-fulfilling prophecies. These findings have implications for teacher preparation courses. It may be necessary to discuss research findings on gender in the light of the students' own experiences in studying mathematics. The sharing of personal stories and aspirations may be as beneficial in constructing more effective personal theories of gender and mathematics.

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A Different System: Notions of Education and Ways of Mathematising in Fijian Society

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Ethnomathematics considers the parallels between traditional indigenous mathematics perspectives and western ones. Generally shared truths are revealed, but notable differences also emerge. This paper, drawn from research in progress, examines Fijian notions of education and in particular ways of mathematising which are embedded in our cultural and everyday activities. The data show that Fijian systems for identifying, naming and classifying are coherent and adaptable. They are distinct from, but complementary to, western methods.

Introduction

Our people have a rich foundation of cultural values and experiences that are well adapted for their purposes and have been persistent and stable over generations. From their experience living amongst the people, missionaries Williams (1982) and Hocart (1929) write of, among other things, the Fijians' remarkable attention to agriculture, their taste for the fine arts and their exceptional mental capacity. While over time culture has evolved, and language has developed and transformed its meanings, our people are fortunate to have retained a language and stable traditional base. Today, Fijians make up about 50% of Fiji's population, 61% of whom live a communal existence in rural areas.

Contemporary Fijian society is a traditional hierachy, that is, "a rigid model, composed of universally defined groups uniting together to form other groups of increasing inclusiveness, and ultimately a social structure of pyramidal form" (Routledge, 1985). *I tokatoka* (extended family), *mataqali* (clan), and *yavusa* (tribe) are the essential social strata to which every Fijian belongs, and are also the basic landowning units under which we are registered at birth. The units are identified and named by specific ceremonial functions that members perform for the equilibrium and viability of the whole organisation. So there are *mataisau* (traditional carpenter) and *gonedau* (traditional fishermen) clans whose knowledge and skill is unmatched by any outside it, just as there are chiefs, priests, and others. It is the specialist knowledge and skills of functional clans that I analyse for Fijian notions of education and ways of mathematising.

Research Questions

In my study, the following research questions are addressed:

- What is the knowledge base of the ritual and ceremonial function of clans ?
- What mathematical ideas exist in the cultural everyday experiences of people ?
- What are the Fijian concepts for the ideas identified ?

Methodology

Fifteen weeks field work was carried out in Fiji. For the historical and cultural information needed, I used a qualitative approach wherein I *talked with* and *watched* people, and *reviewed* archival material. For people's data, there were two main groups of informants: elders and members of chiefs, fishermen, and carpenters clans, and an urban group of senior educators, retired administrators, and church leaders. With the former I used group interaction, with the latter, individual representations. There were eight groups altogether: two of fisherman (F), four of carpenters (C), and one each of women at matweaving (W1) and tapa making (W2). The only individuals from the village group

were two chiefs (T). The urban group comprised ten men and six women. It included five educationalists (E), nine retired administrators (R), and two church leaders (CH) of denominations that most Fijians belong to. All interaction was in Fijian. The different locations and experiences of the two groups of informants necessitated two forms of asking. These were:

- Oral Testimonies (McMillan & Schumacher, 1993), for the village group, and
- Unstructured Interviews (Drew, Hartman & Hart, 1996), for the urban group.

Analysis and Discussion

The data from both the informants and records in government archives are equally important and used concurrently. In this paper, the Fijian people's data is an English translation that is my interpretation of the Fijian originals. I take responsibility for any misrepresentation. Analysis is in two parts: first I examine Fijian educational ideas, and then I identify mathematical ideas in cultural and everyday socialisation.

Fijian Notions of Education

> *The participation of the young in various tribal activities constituted what we would call schooling in Fijian society. Most of these activities concerned the immediate needs of, or duties of the social unit, hence the training was direct, realistic and purposeful as well as exacting. Each social function, each activity, was an opprtunity for the uninitiated to learn and acquire the skills and knowledge of the federation.*
>
> (Lewis-Jones, 1957, p.110)

This is a reasonable summary of traditional learning. It suggests a learning that is a preparation for life. Its primary aim, as suggested, is to teach an understanding of, and conformity to, customs and traditions. There are no new ideas, only old ones of history. As one of the carpenters said,

> *Everything that we know and practice, our forefathers gave to us. Their carving technique is the same we follow today. We have not brought much else ourselves.*

In an oral tradition like ours, the significance of history and the past gives status to elders who represent the past in the present. Lewis-Jones's statement recognises other facets of traditional learning: I attend to some of these.

Oral Tradition

In society, authority is still in the spoken word and there appears no need for the written. In all levels of society there are tellers and listeners, the roles defined by age and experience. While elders are duty-bound to pass on tradition to the young, the latter know their education is gained by keeping alongside people of quality. They listen and do, knowing that in their time they will become transmitters of culture and perpetuators of the system. An elder of a renowned carpenter clan heard about plans by educated members to document their tribal heritage, the argument being that documenting will preserve knowledge, prevent erosion, and be beneficial to urban children with little access to the village-context. The plan did not impress the elder; he was ambivalent about the exercise and said,

> *It's impossible to condense everything into a book; there is so much. Things will be left out, and that's not right. It would kill tradition. I will have no purpose to my clan.*

Cooperative Effort and Interdependence

Dependence upon chiefs and elders plays a large part in the Fijian character. Chiefs are believed to be "of God" (Fijian Bible translation, Rom 13:1) and should be obeyed. They are gifted to lead and are looked up to for appraisement, judgement and direction. As head of the hierarchy, they administer to hold the commune together. For their part, people esteem a strong hand and are implicitly submissive to the word of elders and chiefs. As Lewis-Jones (1957) observed,

> *Living in a commune, related to everyone, cared for and under the nurturing care of almost every adult, teaches the young the values of relationships and obligations. By participating in group activities, they learn the value of team-work and cooperation. His interests and sentiments are reconciled to those of his society, and he thrives to maintain, foremost, the honour and interests of his community.*

Everyone of my informants, including those interviewed individually, talked of *keitou / keimami* (we) and *neitou / neimami* (ours) more than the singular *au* (I) or *noqu* (mine). This is suggestive of a collective partaking both in decision-making and action, as well as in property ownership (to which further reference is made later). A Fijian set to work on his own easily loses heart and becomes lackadaisical and without interest in the task. He becomes immediately enthusiastic and energetic when with company.

Significance of the Past

The continual reference to *i liu / i mada* (before, the past) and the tendency to introduce stories with phrases of the form:*our forefathers used to say ...* is significant. In one recording alone, there were at least twelve references to *matua i mada* (the elders of the past) and *tubui keitou* (our grandfather). It seems that elders and their achievements *then* are the point of reference for today, and the instruction now is *to do as it was done then.* When I enquired whether these achievements could be equalled or bettered, the response was affirmative *but* quickly followed by excuses suggesting the contrary.

Spiritualism

Our people acknowledge many gods since we are both spiritual and superstitious. The idea of a deity and supernatural powers have featured prominently even before christianity. Today, there are ancestral gods of clans and tribes, there are spirits of the dead, Jehovah the christian god, and others. In people's lives, there seems an indefinable merging of religious beliefs of traditional and christian gods. People attribute all phenomena either to gods and spirits, or to withcraft. This gives a reason for everything, answers all questions, and leaves nothing to chance. A fisherman's comments reflect this perspective,

> *Our fishing task is not dependent on weather. Whatever the conditions, as long as protocol is observed, there will be fish aplenty. Fish is 'given' to us. We go to collect.*

Learning Actively in Situ

Not one of my informants said they were taught by anyone even though they learnt things by listening, watching and imitating. Young people know there is no formal schooling for things-Fijian but that they actively grow into knowledge in situ. As one of the fishermen explained,

> *My first job was as bale-out boy, then I was steering when I could hold a steering pole, then I was helping to hold the net. But I loved to stand at the*

helm with my father, looking out and finding fish as they came up with the swells.

The point is further illustrated by one of the tapa-weaving women,

While collecting tapa stalks, we may cut some big ones for my mother and small ones for me. Or using only big stalks, my mother would cut off the end tips for me. Then together we'd make masi, both of us looking and listening to the other for direction.

Fijian Concepts of Education

As culture has evolved so has language developed and meanings transformed through social changes. An analysis of the morphology of Fijian words and concepts of education indicate what traditional society attends to as important at any point in time. A case in point about Fijian sentence structure is that the order of elements is different from English, with the verb phrase coming before the noun phrase.

Child

There is no word for baby in Fijian, only *gone* for child. To differentiate small *gone* from the big, the descriptors *tawa vakayalo* (without spirit) for 0-5 year olds and *yalowai* (of watery spirit) for 5-9 year olds are common terms used to excuse children's mistakes and justify the firm hand of adults'. They are also the reason for Fijian parents' indifference to kindergarten and pre-schools, the belief being that it is useless to teach children anything until their spirit has become stable. Yet, as one of the carpenters said,

I was always around my father's workplace from when I was little. I had my own little hammer that he made for me and I did whatever he did. Today, my son is eight months old, and he is already at my side where I work. I'm doing as my father did.

If Fijian children, like this carpenter and his son, are learning at home from their parents (girls from mothers and grandmothers, and boys from fathers and grandfathers) and relatives very early, even as they are still *watery-souled*, then the descriptors cannot be about traditional learning. They can only be related to the perceived stress and anxiety that people believe come with school learning, that is *away from living and reality*, and in a new environment. All the same, readiness even then is still a measure of *yalo* or spirit.

'Vuli'

This is the root word which can be verb, noun and adjective. From *vuli*, words for 'learn', 'teach', and 'school' have developed. It is a difficult word that attends to the total education process. Its derivative *vuli-ca* is a verb, and is freely used for both 'learn' and 'teach'. The dual meaning in 'teach' and 'learn' in the words *vuli* and *vulica* is indicative of the traditional learning context which is not separate from, but part of, everyday living. There are no specific instructions as there is no designation of teacher or student roles. There are only mothers, fathers, children and family living their culture and demonstrating as a matter of behaviour and performance. Today though, *vuli* has developed to take account of defined roles and actions in formal schooling.

'Yalomatua'

The concept of attaining and becoming in Fijian society is described in *yalo-matua* (mature spirit). It is suggestive of an intense spiritual being that "was required and necessary to help people apply the various skills, knowledge, and attitudes that they had learnt from the elders through the various activities" (Baba, 1993, p.4). It is differentiated from *kila* (know) which is a mental capacity, and normally associated with school. A related term is *mana* described by one of the chiefs as "a thing which was from the beginning". *Mana* is like the stamp of approval bestowed by the gods on one who has

attained. It is *mana* when something succeeds or comes to pass.

Fijian Way of Mathematising

Defining mathematics as a social construct, Bishop (1988) identifies six activities - counting, measuring, locating, designing, playing, explaining - as those that conceptualise and define the field of mathematics developed in any culture. That framework provided a starting point for my search and analysis. I have since modified it to accentuate those features our people attend to more. For this paper I draw upon and discuss three elements that are closely related in Fijian society: counting, traditional economy and exchange, and kinship.

Counting

Our people quantify, but differently from western societies. Preoccupation with counting in the sense 'to calculate' or 'to be precise' is not part of the fabric of Fijian society, and the use of numbers is limited to small values. This does not mean an inability to count, only that we count as desired. Our engagement in mathematical and other activities has to be culturally appropriate. To this end, preference is given to qualitative evaluations that result in a rich register of qualitative evaluative terms in the language. For counting objects, pronouns show the difference between *dua* (one), *rua* (two), *tolu* (three), and *ira* (more than three) items. While in English 'they' and 'their' can be used for two, three, and more than three, in Fijian (meaning the lingua franca) the terms *rau* (them two), *iratou* (them three), and *ira* (them more than three) are distinct. Counting with these pronouns is used for objects, whereas people are addressed using polite and respectful conversation forms. For this, the prefix *kemu,* makes *kemu-drau* (you two persons), *kemu-dou* (you three persons), and *kemu-ni* (you all). Of these three, the greater plural *kemu-ni* is the most preferred, while *iko* (you) is impolite. Living in a commune where people share and are interrelated, there is no reason to count. Counting suggests a calculating inclination and precision that we do not need. Perhaps to outsiders, what we bring to counting tasks seems beyond logic and reason, but to Fijians it makes good sense. The following comments come from a chief, an educationalist and a retired administrator respectively.

> *In society people are identified with their clans, tribes and many other groupings, rarely as individuals. Every task, or allocation and distribution, is done using one of many group types, never individually. We avoid calling individual by their names.*
>
> *If 8 people are required for a job, we see no problem with sending 10 or 12. After all that's only a few more, and shouldn't be any problem.*
>
> *In counting one normally points with the index finger. We do not do that; we take exception to pointing at a person's forehead, and it's taboo. The five digits of the hand have Fijian names according to the functions they perform. The index finger is "the chief's pointing finger' and they alone can point..*

There is, however, a *Fijian Counting System* that counts goods normally used in traditional presentations, including foodstuffs and traditional prized goods. These are counted in units of term, whereby a special term is given to tens of the goods counted. The more abundant goods like coconut and taro, which are normally presented in large quantities, are counted also in groups of a hundred and thousand. Some of these are shown in Figure 1.

The practice of counting in tens, as one of the retired administrators explained, is qualified in the following way:

It makes counting easier. Presentations are usually big and it's difficult to count the items individually. It's easier to use smaller numbers.

10 pigs = 1 *rara* (village green) pigs	10 turtles = 1 *bi* turtles
10 mats = 1 *sasa* (underlay) mats	100 coconuts = 1*koro* (village)
64 mats = 6 *sasa* and 4 mats	1000 coconuts = 1 *selavo*
10 salt baskets = 1 *wai*	Taro: not counted but piled high - makes 1 *duludulumata* (tired eye)

Figure 1: Fijian counting system

One of the chiefs also commented,

> *Ceremonies are not meant to be competitive. They are about strengthening ties and relationships. It is the custom to underrate and depreciate one's presentation no matter how big it really is. It's being modest about yourself and respectful of the other party.*

Three modes of ***tallying*** were given to me, namely of the widow in mourning, of planting clumps of yams, and that of the traditional messenger.

> *The smallest finger called the "mourning finger for the chief" is often that cut off a widow's hand at the death of her husband. Mourning lasts 100 nights. To keep tally, the widow goes through a"knotting" exercise where a string is hung around her neck and she successively ties a knot each night until the 100th. The knots, normally using maybe 10 coloured strings successively, are easier to count, because we only look at colours.*
>
> *When our ancestors planted yam or cassava, they dig and plant in individual clumps. On a morning's work they might plant a hillside of several clumps. To keep a rough count for comparing notes later, they would plant with one hand, while on the other they hold a small branch with leaves that he plucks off and places on each clump planted, as he moves on. It's the branch of a local plant that everybody uses. In the end they deal only with branches, usually less than five.*
>
> *In his mission as a traditional messenger, the ambassador carries in his pouch several sticks of varying length, each mnemonic to a thought. Arriving at his destination, he lays each stick in front of his kneeling posture, then delivers the messages exactly, and in the exact order.*

Traditional Economy and Exchange

As Ravuvu (1985) suggests, the Fijian way of life "emphasises the expending of material things for social and communal goals" (p. 17). Wealth to the Fijian is not in terms of a bank-balance; nor does it have the same value as it does to others. If 'wealth' is taken to mean 'an abundance in possession', Fijian *i yau,* measured in terms of goods such as mats, tapa, and whales teeth, is different, because they are not kept as possession. A retired administrator explained,

> *For us, 'i yau' is kept for in individual homes, or for clans and tribes by its head, and every member has a certain lien upon it. But it is only kept until the next presentation.*

Redistribution reveals the Fijians' regard for the world. *Reciprocity* is the rule. There seems no need to amass wealth because the natural world surrounding us also sustains us.

There is no accumulation, but always movement of goods across people and groups. Movement is either through the channels of ceremonial presentations between groups, or as direct gifting between people in the tradition of *kerekere*. Either way it is direct and controlled, and of a different character to a monetary economy.

Solevu: This is a public ceremonial exchange of goods between groups, normally pre-planned months ahead. Traditionally the networks existed across regions on a needs basis. The exchange would be of what group A has in abundance that group B needs, for something group A needs that group B can supply. An example would be an exchange between people of the highlands with those from the coast. Today the ceremonies are extended to numerous forms. Whatever the reason behind the exchange, Derrick (1957) is quite clear that,

> *it was never the object of trade to make a profit in the commercial sense. The important thing is not that the exchange is trade, but that the framework within which it takes place is primarily social, not economic.*

Kerekere: This term has been misunderstood and misrepresented as borrowing, requesting, begging, and soliciting. None of these defines what *kerekere* means to the Fijian. There is no shame or stigma attached to the act because, as a church leader and fisherman explained respectively,

> *We do not 'kerekere' haphazardly but only to special kinsfolk. Both parties know the rights to' kerekere'.*
>
> *I cannot refuse anything of my sisters and their children because they're sacred to me. Whatever they want I get for them, and always the best.*

One's right to *kerekere* of the other is demonstrated in the latter giving *more than* what is asked. The prefix *vei* meaning 'with' implies that the transaction is not a right exercised only by one party on the other, but is dual and the roles interchange. However, people do not keep records of 'I owe you', as it is a social obligation.

Locating: Kinship

The concept of relations, concerned with specific properties that link pairs of objects, is basic to many ideas in mathematics. Just as 'greater than' (>), and 'square' (x^2) are relations that will link sets of numbers, the relation 'capital city of' will link pairs like Suva and Fiji, and Wellington and New Zealand. In the same way, a family tree diagram or geneological table with standard symbols, Δ, o, and =, denoting sexes and marriage pairs, can create several relation patterns. Some others may be "sister of", "aunt", and "cousin". The interesting aspect of kinship is that they are different for different groups of people. Ascher (1991) quite rightly makes the point that, "...to understand the kinship system of a culture is to understand much about the culture" (p. 69). There is a variety of kinship systems to be found throughout Fiji and kinship terminology varies considerably with locality (Nayacakalou, 1955). Although the strong dichotomy of cultures and languages would suggest two main groups, the basic principles underlying the kinship structure is the same (ibid). What has become generally accepted as Fijian culture and language is that of the eastern tribes, in particular of the leading Bau tribe who were first contacted by Europeans. Nayacakalou identifies "common descent" that binds together members of a lineage group and "common parentage" which binds together members of a sibling group, as the two unifying principles within Fijian totaki kinship structure. However, "distinction of sex within sibling group" is the most fundamental differentiation. Hocart (1929, p.33) defines the system in this way:

> *If a man is related in any way to another man, he will be related in exactly the same way to that other man's brother, but not to his sister. If he is related in any way to a woman, he is related in the same way to that woman's sister, but not to her brother.*

In addition, the *classificatory principle* of kinship terminology (Radcliffe-Brown, 1950) extends the principle of common parentage and therefore of a sibling group to include, "..all members of each generation within the lineage group to be classified together as siblings" (Nayacakalou, 1955, p.46). Take the simple case of 'father':

> *I have many fathers, whom I may address as small father, big father, or just father. There is my biological father, but all his male siblings are also 'my fathers'. In addition, in the same way that my biological father gains 'many fathers' (in his biological father's male siblings), he will gain more 'brothers', and I, more fathers. The children of all my fathers and all my mothers are my brothers and sisters.*

To complicate matters further and bring in still more fathers, the principle of *unity of the sibling group* allows unity in a sibling group "for a person outside it and connected with it by specific relation to one of its members" (p. 47). Suffice to say that, having numerous parents, brothers and sisters, and progeny, is a special trait of Fijian kinship. *Cousins* in Fijian kinship terms, are the siblings of cross relatives. In other words, my cousins are my father's sisters' children and my mother's brothers' children. For us, the distinction of *lineal* and *cross* relations creates a special level of relationship in *familiar* and *taboo* relations respectively. Taboo relations, like brother and sister, do not talk, and generally avoid each other, while familiar relations are the 'marriageable' partners. Identifying one's *taboo* or *familiar* relations is crucial for proper conduct, and respectability.

Conclusion

It is not my intention to take anything away from Fijian culture or the practices therein, only to identify the 'hidden' or implicit mathematical heritage that can be seen to be part of the socialisation processes of all cultural groups. It has to be recorded for two reasons: firstly to make the case that we do mathematise, and secondly, that we do so differently from western ways. The work gives our children pride in themselves and their culture, and allows them a broader implementation of the knowledge.

The work has important implications for classroom practice. The mathematical ideas may be able to be used as stepping stones and as critical pathways to making sense of the more formal and difficult concepts of school mathematics. Further, knowing that Fijian children come from a society where quantification is normally in qualitative terms, and that they are likely to have a less developed number sense than normally assumed, should enable teachers to structure learning to address these differences With respect to Fijian knowledge, however, it must be emphasised that the ideas are meaningful in context, and that decontextualising them would deprive us of the *mana* and holistic nature of that knowledge.

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Student Error Patterns in Fraction and Decimal Concepts

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This paper uses data from an international study of number sense in Australia, the United States, Sweden and Taiwan to investigate students' misconceptions and error patterns in fraction and decimal concepts. Selected items that were administered to two or three age levels were analysed. It was found that misconceptions about basic concepts were prevalent, and these persisted even among 14-year-olds. Results suggest the need for more meaningful treatment of fraction and decimal concepts, and some relocation of these topics in the curriculum.

The Main Study

Mental computation tests were developed and administered to students aged 8, 10, 12, and 14 in Australia, Japan, and the United States (McIntosh, Bana & Farrell, 1995a; McIntosh, Bana & Farrell, 1995b). In order to investigate number sense further, more extensive number test instruments were developed and administered to these same students in Australia and the USA. Many of these items were also given to Swedish and Taiwanese students. This international follow-up study of number sense is reported by McIntosh, Reys, Reys, Bana and Farrell (1997).

The items developed for the number sense tests were based on the number sense framework outlined by McIntosh, Reys & Reys (1992). Group tests ranging in length from 30 to 45 items were administered to sample sizes ranging from 110 to 160 students in each of the 8-, 10-, 12-, and 14-year-old student groups in Perth, Australia and Columbia, Missouri in the USA. Some of the items were modified for US students. Many of these items, often modified slightly, were administered to 10-year-olds and 14-year-olds in Gotheborg, Sweden. Some of the items, again with modifications in certain instances, were also given to 12- and 14-year-olds in Taiwan.

The major purpose of this study was to attempt to measure number sense over a large age range and in a number of countries. It was not the intention to compare countries' performances since the samples were not truly representative nor were they matched except in fairly crude terms. However, there was some matching of age groups as indicated above. In order to observe the development of number sense across age levels, many of the items were common across two or three different age groups.

Purpose of this Study

The particular purpose of the study reported here was to analyse a representative sample of the items from the main study of number sense relating to fractions and decimals to determine error patterns and prevalent misconceptions in these topics. The items used in the main study focussed on understanding rather than on factual knowledge or routine computation. This analysis should provide some guidance for teachers in effecting remedial strategies, and provide evidence as to whether or not some relocation of these topics in the curriculum may be necessary.

In order to undertake such an investigation the analysis below concentrates on items that were common across a number of age levels; thus providing a good overview of the development of fraction and decimal concepts through the students' schooling, and signalling any desirable re-locations of these topics in the school mathematics curriculum.

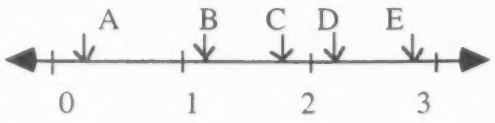
Results

Selected items with their results are presented below. Table 1 shows an item asking students to identify a point on the number line corresponding to 2.19. All items were presented in 'boxes' in this format, so that students wrote or selected responses alongside

the item itself on the test paper. In this case a high percentage, especially among 10- and 12-year-olds, selected the point closer to three rather than to two. It seems that there is a lack of understanding of basic decimal numeration concepts here. Yet these same students who still have difficulty locating a decimal on a number line are expected to undertake computations using decimals. It would be interesting to ask these students whether \$2.19 is closer to \$2 or to \$3. Presenting the item in context is likely to bring about a different result.

Table 1
Decimal locations on the number line

Item:



Which letter on the number line above best represents 2.19? _____

Response % for age groups	Aus 10	Aus 12	Aus 14	USA 10	USA 12	USA 14	Swed 10	Swed 14
A, B or C	11	2	2	7	3	2	20	3
D	47	79	97	20	34	73	35	84
E	40	18	2	64	62	25	44	12
Misc Incorrect	0	0	0	3	1	0	0	0
No Response	1	0	0	6	1	0	2	1

The item in Table 2 asked students to identify the fraction representing the largest number. It is clear that many students selected the fraction with the greatest denominator (9) as representing the greatest number. This misconception was quite pronounced, even among the 12-year-olds, and persisted for 14-year-olds in Sweden and the USA. Again,

Table 2
Identifying the fraction for the largest number

Item:
Circle the fraction which represents the largest amount.

A $\frac{5}{6}$ B $\frac{5}{7}$ C $\frac{5}{8}$ D $\frac{5}{9}$

Response %	Aus 10	Aus 12	Aus 14	USA 10	USA 12	USA 14	Swed 14
A	42	79	98	29	74	85	92
B or C	2	5	1	1	4	3	1
D	57	17	2	69	22	11	6
No Response	0	0	0	1	0	1	1

as is the case for decimals, it is clear that many students are having to carry out computations using fractions while still having misconceptions about the fractions themselves.

The Table 3 item tests the notion of an infinite set of numbers between any two given numbers expressed as decimals - in this case, 1.52 and 1.53. The results show that the majority of students have the misconception that there are no, one, or a few numbers between 1.52 and 1.53. Even among 14-year-olds there was a lack of understanding. However, Taiwanese students in this age group performed quite well, although they were not required to support their responses with examples.

Table 3
Awareness of an infinite set of numbers between any two given numbers (decimal form)

Item:	
How many different decimals are there between 1.52 and 1.53? Circle your answer and then fill in the blank.	A None. Why? _____ B One. What is it? _____ C A few. Give two: _____ and _____ D Lots. Give two: _____ and _____

Response %	Aus 12	Aus 14	USA 12	USA 14	Swed 14	Taiw 12	Taiw 14
A	42	23	55	45	23	36	5
B	33	11	23	8	14	10	5
C	6	2	9	0	2	5	3
D & examples	16	62	3	40	51	*	*
D no examples	3	2	2	4	6	32*	78*
No Response	0	0	8	3	4	17	9

*Taiwanese students were not asked to give examples

Students were asked the same question as in Table 3 regarding the fractions $^2/_5$ and $^3/_5$. The results of this item are presented in Table 4. Approximately half the 12- and 14-year-old students said there is no fraction between $^2/_5$ and $^3/_5$. Many others said there is one,

Table 4.
Awareness of an infinite set of numbers between $^2/_5$ and $^3/_5$

Response %	Aus 12	Aus 14	USA 12	USA 14	Swed 14	Taiw 12	Taiw 14
A	57	29	57	53	47	60	40
B	29	21	14	9	14	8	4
C	4	7	8	5	5	6	3
D & examples	7	40	1	7	12	*	*
D no examples	3	3	11	15	13	11*	35*
No Response	0	0	10	11	8	15	18

*Taiwanese students were not asked to give examples

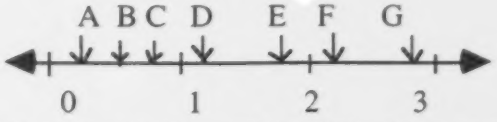
or a few fractions. Performance on this item was considerably lower than for the decimals item in Table 3. It is very likely that students focussed on the numerals or fractions rather than on the numbers they represented, and this would help explain the results. The form of the question might have led to students considering only fifths as possible answers. That is, many students probably concentrated on the fact that there are no fifths between the two given fractions, ignoring the fact that there are many numbers represented by other fractions.

Table 5 gives results for an item that required students to identify a point on the number line representing a fraction whose numerator would be slightly greater than the denominator. Overall, only about a quarter of students were successful in this item. Even among 14-year-olds in all three countries, only a third of them showed understanding. Significant numbers of students in all age groups chose points F or G, which represent numbers greater than two. There is no obvious explanation for this misconception. Once again, we must question whether these students who have difficulty with basic fraction concepts can be expected to undertake meaningful computations with fractions.

Table 5
Fraction locations on the number line

Item:

In the fraction $\frac{5}{8}$, 5 is the numerator and 8 is the denominator.



Which letter on the number line above names a fraction where the numerator is slightly more than the denominator?

_____

Response %	Aus 12	Aus 14	USA 12	USA 14	Swed 14*	Taiw 12*	Taiw 14*
A	6	7	15	10	6		
B	2	5	7	3	1		
C	11	6	11	10	7		
D	14	37	15	30	19	14	32
E	10	7	6	11	17		
F	34	20	13	10	16		
G	13	12	14	6	8		
Misc. Incorrect	0	0	8	0	11	25	44
No Response	10	6	11	19	16	61	24

*For Swedish and Taiwanese students the introductory sentence was omitted from the item

The item shown in Table 6 required students to estimate 87 x 0.89 in relation to the number 87, and they were not very successful on this item. The importance of estimation has been well documented (Sowder, 1992). On this item, many students thought that the product 87 x 0.89 would be more than 87, again indicating significant misconceptions about decimals and their use in computation.

Table 6
Estimation of a product involving decimals

Item:	
Without calculating the exact answer, circle the best estimate for: 87 x 0.89	A a lot less than 87 B a little less than 87 C a little more than 87 D a lot more than 87

Response %	Aus 12	Aus 14	USA 12	USA 14	Swed 14
A	51	82	16	50	65
B	12	8	18	17	16
C	30	8	28	20	10
D	7	2	35	8	8
No Response	0	0	2	4	1

The quotient estimation shown in Table 7 is a similar item in that students were asked to relate the result to a benchmark - this time to 29 - given that the quotient to estimate was 29 ÷ 0.8. Over half the students, even among 14-year-olds, thought that the quotient was less than 29. We had previously found that many of these students showed significant misconceptions in mental computation with fractions and decimals (Bana, Farrell & McIntosh, 1995).

Table 7
Estimation of a quotient involving decimals

Item:	
Without calculating the exact answer, circle the best estimate for: 29 ÷ 0.8	A less than 29 B equal to 29 C greater than 29 D Impossible to tell without calculating

Response %	Aus 10	Aus 12	Aus 14	USA 14	Swed 14
A	64	73	49	55	51
B	4	2	1	3	4
C	20	21	49	36	37
D	11	4	1	5	4
No Response	0	0	0	2	5

Understandings of basic properties of the operations were assessed in the number sense tests. For example, the multiplication property of one as it relates to equivalent fractions was tested in the item detailed in Table 8. However, it could be argued that this item is assessing understanding of equivalence rather than the multiplication property of

one, since students are required to recognise that $^1/_2 = {}^3/_6$, so indicating that the missing number is one. The results show that approximately a third of the students across all three countries gave three as the missing number to make the sentence true. To these students, three lots of a half is three sixths, which was possibly reasoned by multiplying both the numerator and denominator by three, which indicates significant misconceptions.

Table 8
The multiplication property of one and equivalent fractions

Item: Circle the number you can put in the box to make this sentence true: $\frac{1}{2} \times \square = \frac{3}{6}$	A $\frac{2}{4}$ B $\frac{2}{3}$ C 1 D 3

Response %	Aus 12	Aus 14	USA 12	USA 14	Swed 14
A	8	10	19	14	16
B	28	13	21	28	22
C	28	46	16	28	21
D	34	31	43	27	33
Misc Incorrect	0	0	0	0	1
No Response	2	0	1	4	7

Discussion

The use of items that probed understandings rather than routine procedures led to the identification of significant misconceptions about fractions and decimals. These misconceptions relate to the basic notions about fractions and decimals, and they persist through the age groups. Thus even among 14-year-olds we found that many students struggled to come to grips with concepts that we consider to be fundamental.

Despite these significant misconceptions, students are still expected to deal with a great deal of mathematics content involving fractions and decimals. For example, in Western Australia, the mathematics curriculum requires students to carry out addition and subtraction with decimals in Stage 4, which corresponds to Year 4 for most students (Ministry of Education of Western Australia, 1989). But it seems clear that students in this age group are unable to handle decimals in a meaningful way.

Thus the curriculum location of topics involving fractions and decimals needs to be carefully reviewed. It seems that all too often students are being required to compute using fractions and decimals before they have any real concept of the numbers and numerals involved. This is likely to result in instrumental rather than relational approaches, simply to cover the given content. We recommend that much more emphasis be placed on meaningful treatment of these topics in the classroom through the provision of relevant experiences for students. We also urge teachers to delay using fractions or decimals in computation until fundamental concepts are well established.

There is still much scope for further research in this area. We could only speculate on possible reasons for students' incorrect responses, as there were no follow-up interviews. Further work is needed through clinical interview procedures to determine the precise nature of these very prevalent misconceptions about fractions and decimals

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The implication of multiplicative structure for students' understanding of decimal-number numeration

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This paper reports on a study that examined the importance of multiplicative structure and whether, after several years of formal instruction in the decimal number system, Year 6 students had acquired an understanding of this structure. To this end, 173 Year 6 students were tested with a pencil-and-paper instrument developed to assess the decimal-number numeration processes that are normally taught in primary school and from which 45 students (32 high performing, 13 low performing) were selected for interviewing on tasks related to multiplicative structure. The interviews revealed that only the most proficient students (≥ 90% for tenths and hundredths in the test) had acquired a structural schema of multiplicativity that enabled access to application tasks

Baturo (in preparation) developed the model shown in Figure 1 to represent the "levels" of knowledge required for an understanding of place value in the decimal number system. Level 1 knowledge comprises the baseline knowledge of *position*, *base* and *order*, without which students cannot hope to function with any understanding in numeration tasks. Level 2 knowledge comprises *unitisation*, namely, the assignment of a numerical value to part/whole relationships (Behr, Harel, Post, & Lesh, 1994; Lamon, 1996) and *equivalence*, both of which require an understanding of base in processing decimal numbers. Level 2 knowledge is seen as the knowledge that connects and provides meaning for Level 1 knowledge. Level 3 knowledge comprises the structural knowledge (*additive structure*, *multiplicative structure* and *reunitisation*) that provides the superstructure for integrating all levels of knowledge.

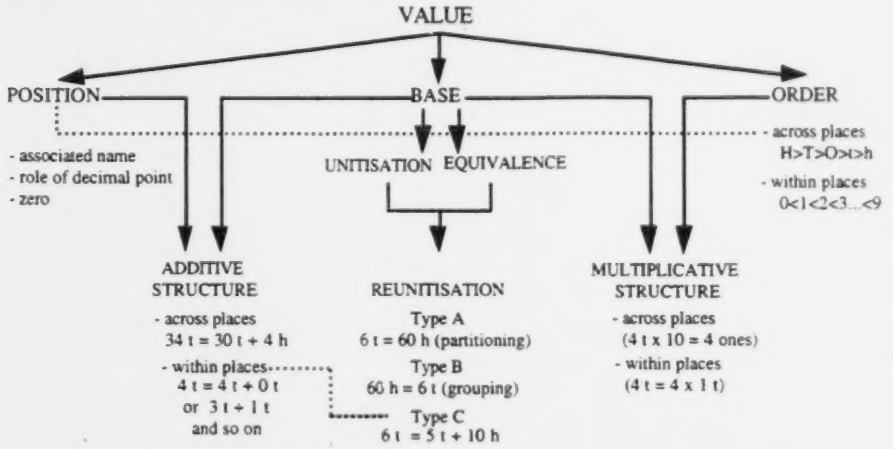

Figure 1. Cognitions and their connections embedded in the decimal number system

This paper reports on a study which explored Year 6 students' knowledge of the continuous, bi-directional and exponential properties of multiplicative structure (Smith & Confrey, 1994). Having an understanding of the multiplicative structure embedded in the decimal number system (see Figure 2) is crucial and, as argued by Baturo (1995), if not explicated for whole numbers, denies students one of the major conceptual underpinnings of decimal numbers.

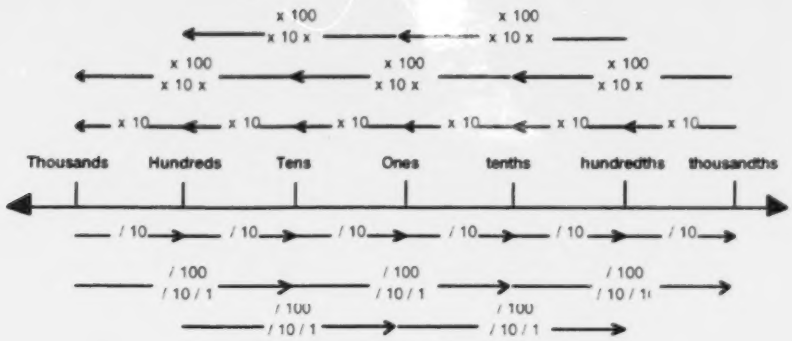

Figure 2. Multiplicative relationships embedded in the decimal number system.

The study is part of a larger project (Baturo, in preparation) which explored students' acquisition and access of the cognitions required to function competently with decimal numbers. To this end, 173 Year 6 students from two schools (different socioeconomic backgrounds) were tested with a pen-and-paper instrument that included items designed to assess number identification, place value, counting, renaming and regrouping, comparing/ordering, and approximating and estimating for: (a) tenths, and (b) hundredths.

The results showed that some students performed equally well on the tenths and hundredths components of the test whilst some performed much better on tenths than hundredths. To distinguish between the regular and irregular distributions, the students were classified as *proficient* and *semiproficient* respectively. The final interview selection comprised 12 high proficient (HP) students (test mean ≥90% for tenths and hundredths), 12 high semiproficient (HSP) students (≥85% tenths; <75% hundredths), 8 medium proficient (MP) students (80-90% tenths and hundredths), 8 medium semiproficient (MSP) students (75-85% tenths; < 65% hundredths) and 5 low proficient (LP) students (40-60% tenths and hundredths).

The 45 students were interviewed individually. All tasks were undertaken in the same order but the interviewer was free to probe responses when required. Each interview took place at the student's school during school hours and took approximately 30 minutes. The interviews were all conducted during the first quarter of the school year and were video-taped for further analysis.

Multiplicative tasks

To investigate the students' understanding of multiplicative structure, Baturo (in preparation) developed the following tasks.

Task 1. Two sets of place name cards (see Figure 3) were used in this task, one set for the interviewer and one set for the students.

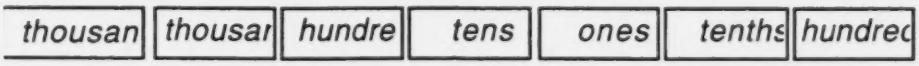

Figure 3. Place name cards used in Task 1.

To assess binary relationships (×/÷ 10), the interviewer put out *hundreds, tens* and asked the students if they could find another pair of places that were related in the same way and, if so, to explain their choice. (This procedure was duplicated with *hundreds, ones* to assess the ternary relationship of 100.) This first task was made as abstract as possible so that the students would be focused on the relationship between the places and not have the potential distraction of the syntactic features of the symbol in order to ascertain what components of multiplicative structure the students had acquired after several years of formal instruction.

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Task 2. This task consisted of the 12 items (6 multiplication, 6 division) shown in Figure 4. In this task, the students were required to explain how they predicted a multiplicative shift (i.e., determined the finishing number when given the starting number and a multiplicative operation).

a	b	c	d	e	f
$0.3 \times 10 =$ __	$0.04 \times 100 =$ __	$6.23 \times 10 =$ __	$0.7 \times 100 =$ __	$2.16 \times 100 =$ __	$0.2 \times 10 \times 10 =$
g	h	I	j	k	k
$4 \div 10 =$ __	$72.5 \div 10 =$ __	$0.9 \div 10 =$ __	$37 \div 100 =$ __	$8 \div 100 =$ __	$14 \div 10 \div 10 =$ _

Figure 4. Task 2 items.

Task 3. The third task required the students to use a calculator to select the operation in examples similar to the following: (a) *change 7 tenths to 7 ones*; and (b) *change 8 ones to 8 hundredths*. Thus the students were provided with the starting and finishing numbers and were required to use their knowledge of the bidirectional nature of multiplicative structure to provide the operation that would make the change. For those students who were unable to do this, teaching intervention was undertaken using a place value chart and a set of digit cards to show the direction and size of the change.

Task 1 was seen as a means of determining the students' available knowledge of multiplicativity whilst Tasks 2 and 3 were seen as a means of determining whether students could access their available knowledge in application tasks.

Students' responses

Task 1

With respect to the binary relationships, 28 students (9 HP, 9 HSP, 8 MP, 2 MSP) selected an appropriate pair of place names. The most common inappropriate responses were to make no response (7 students – 1 HP, 2 MSP, 4 LP) or to select *hundredths, tenths* (7 students – 2 HP, 2 HSP, 3 MSP). Each of these latter students explained that their selection had been made on the basis of the names (a syntactic feature of the task). However, when asked to consider the order of the names, 6 (2 HP, 2 HSP, 2 MSP) of the 7 students changed their selection to *tenths, hundredths*.

When asked to explain the reasoning for their selection, 56.6% of the students (11 HP, 7 HSP, 5 MP, 1 MSP, 1 LP) gave responses which indicated that they had the appropriate knowledge of bi-directionality ($\times$, $\div$) available as well as exponentiality (the relationship of 10). Their explanations included comments such as "if you times tens by 10, you'll get a hundred; divide it by 10 to make it 10 times smaller". A further 6.7% of the students (1 HP, 1 HSP, 1 MP) indicated that they had uni-directional ($\times$, not $\div$) knowledge of the relationship available. All of these students associated the relationship from left to right (larger to smaller) with subtraction. One student (MP) had bi-directionality ($\times$, $\div$) but did not know the relationship of 10. She had selected *thousands, hundreds* and explained that *hundreds* are multiplied by hundreds to produce *thousands* and that *thousands* are divided by thousands to produce *hundreds*. Eight students (4 (HSP, 1 MP, 3 MSP – 17.8%) applied additivity (+, –) to the relationship (e.g., "add 9 tenths to make ones and take away ones to make tenths"). The remaining 8 students (4 MSP, 4 LP – 17.8%) were unable to explain their selection. (See Table 1 for the results of this task in terms of the performance categories.)

With respect to the ternary relationship (100), 71.1% (12 HP, 11 HSP, 7 MP, 2 MSP) of the students selected an appropriate pair. Their spontaneous explanations revealed that they had used the syntactic feature of position to help them select a matching pair [e.g., "the middle one (place) was missing" or "it jumps one (place)"]. However, probing their explanations revealed that 48.9% (11 HP, 6 HSP, 4 MP, 1 MSP) had full multiplicative structure (bi-directional and exponential), 6.7% (1 HP, 1 HSP, 1 MP) had

uni-directional (×) and exponential (100) knowledge, 4.4% (1 HSP, 1 MP) had bi-directional but not exponential knowledge whilst 15.6% (4 HSP, 2 MP, 1 MSP) applied additivity. (See Table 1.)

Task 2

The students performed better for multiplication than for division as shown by the following results: HP (97.2%, 83.3%); HSP (79.2%, 44.4%); MP (43.8%, 10.4%); LP (25.9%, 06.7%). For multiplication, the successful students tended to use two main strategies: (a) *shifting* either the digit/s or the decimal point, and (b) *renaming* (e.g., "3 tenths times 10 is 30 tenths; 30 tenths equals 3 ones"). The unsuccessful students used one or more of the following strategies: (a) inserting a zero (the *whole-number* rule – "10 has one zero so add a zero") either at the end of the given number (e.g., $0.3 \times 10 = 0.30$) or at the end of the whole-number part ($6.23 \times 10 = 60.23$); (b) *guessing* (which produced erratic responses across the tasks); and (c) omitting the task. The *shifting* strategy was successfully transferred to the division items but the latter strategy and would therefore be successful for both multiplication and division.; which would be more difficult to apply to division.

For division, the successful students used the *shift* strategy. (No other successful strategy was revealed.) The unsuccessful students used one or more of the following strategies: (a) *reversed* the operation (e.g., $10 \div 4$ instead of $4 \div 10$); (b) used the *wrong operation* (multiplied instead of divided); (c) inserted *zeros* inappropriately to make the number smaller (e.g., 0.0037); (d) *guessed*; or (e) *omitted* the task.

The interesting feature of this task was the way that students changed appropriate strategies for inappropriate ones. For example, 37 students (12 HP, 12 HSP, 5 MP, 5 MSP, 3 LP) used the shift strategy for at least one multiplication item but only 25 students (12 HP, 9 HSP, 2 MP, 1 MSP, 1 LP) maintained the strategy across the multiplication tasks. Furthermore, very few students transferred the shift strategy to division. One suggestion for these behaviours (failure to maintain and transfer a successful strategy) is that students whose knowledge is not semantic are swayed by the syntactic features of tasks, a phenomenon which suggests that, for many students, their knowledge of multiplicative structure is rule-based and does not include an understanding of the continuous, bi-directional and exponential properties of multiplicativity.

The students' explanations revealed that the shift strategy can be semantically-based (i.e., emanating from an understanding of the multiplicative structure of the decimal number system) or rule-based (i.e., × 10 has one zero so shift all digits one place to the left). Students who maintained the shift strategy across all the items revealed that they understood the principles underlying the strategy.

Task 3

With respect to effecting the change from 7 tenths to 7 ones, 60% only (12 HP, 8 HSP, 6 MP, 1 MSP) selected the appropriate operation (× 10), 6.7% (2 MSP, 1 LP) selected the correct direction (×) but did not know the relationship (10). With respect to effecting the change from 8 ones to 8 hundredths, all but one of the students (HSP) who was successful with the binary relationship was successful with the ternary relationships. Thus 57.8% only (12 HP, 7 HSP, 6 MP, 1 MSP) revealed an understanding of the continuous, bi-directional and exponential properties of multiplicative structure. Of the remaining 19 students, 2 (1 HSP, 1 MP) revealed that they had uni-directional knowledge (×, not ÷), 4 (3 HSP, 1 MP) revealed that they could only access full additivity (+, −) whilst the remaining 13 students were unable to do either task. (See Table 1.)

Summary

Table 1
Responses across the tasks in terms of the performance categories.

	Performance categories				
	HP (n = 12)	HSP (n = 12)	MP (n = 8)	MSP (n = 8)	LP (n = 5)
Task 1 (binary)					
Full MS (×, ÷)	11 (91.7%)	7 (58.3%)	5 (62.5%)	1 (12.5%)	1 (20.0%)
Partial MS	1 (08.3%)	1 (08.3%)	2 (25.0%)	0 (00.0%)	0 (00.0%)
– ×, –	1	1	1	—	—
– Op.$_2$, base ^{a}	—	—	1	—	—
Full AS (+, –)	0 (00.0%)	4 (33.3%)	1 (12.5%)	3 (33.3%)	0 (00.0%)
Task 1 (ternary)					
Full MS (×, ÷)	10 (83.3%)	6 (50.0%)	4 (50.0%)	1 (12.5%)	0 (00.0%)
Partial MS	2 (16.7%)	2 (16.7%)	2 (25.0%)	0 (00.0%)	0 (00.0%)
– ×, –	1	1	1	—	—
– Op.$_2$, base ^{a}	1	1	1	—	—
Full AS (+, –)	0 (00.0%)	4 (33.3%)	2 (25.0%)	1 (12.5%)	0 (00.0%)
Task M2					
Full MS	10 (83.3%)	5 (41.7%)	0 (00.0%)	0 (00.0%)	0 (00.0%)
Partial MS	2 (16.7%)	4 (33.3%)	1 (12.5%)	1 (12.5%)	0 (00.0%)
Task 3 (× 10)					
MS	12 (100%)	8 (66.7%)	6 (75.0%)	1 (12.5%)	0 (00.0%)
Partial MS	0 (00.0%)	0 (00.0%)	0 (00.0%)	2 (25.0%)	1 (20.0%)
– op.$_2$, base ^{a}	—	—	—	2	1
– op.^{2}, base $_a$	—	—	—	—	—
Task 3 (÷ 100)					
MS	12 (100%)	7 (58.3%)	6 (75.0%)	1 (12.5%)	0 (00.0%)
Partial MS	0 (00.0%)	2 (16.7%)	0 (00.0%)	1 (12.5%)	0 (00.0%)
– op.$_2$, base ^{a}	—	—	—	1	—
– op.^{2}, base $_a$	—	2	—	—	—

Note. MS = multiplicative structure; AS = additive structure; op. = operation.

Table 1 shows that, across the tasks: (a) the HP students generally displayed expert knowledge of multiplicative structure; (b) the HSP students had some knowledge of multiplicative structure; (d) the MP students displayed knowledge of multiplicative structure in Tasks 1 and 3 but displayed no knowledge of multiplicative structure in Task 2; and (e) the low-performing students (MSP, LP) revealed that they had no knowledge of multiplicative structure. These results suggest quite strongly that multiplicative structure is a determining factor in differentiating high-performing (HP, HSP, MP) and low-performing (MSP, LP) students and in differentiating HP students from the other high-performing groups.

Knowledge availability and access

In Task 1, the students' syntactic explanations were probed to determine whether they also had the appropriate semantic knowledge. Thus Task 1 was viewed as a means of establishing the students' available knowledge of the continuous, bi-directional and exponential properties of multiplicative structure. Tasks 2 and 3 were designed to reveal whether those students who had the appropriate available knowledge had structured their knowledge in a way that promoted access in application tasks. Table 2 provides the findings of these tasks in terms of availability and access. Note that *continuity* is not included in the table because it was not a factor in prohibiting availability or access.

Table 2
Students' availability and access of multiplicative structure in terms of the performance categories.

	PERFORMANCE CATEGORIES				
	HP (n = 12)	HSP (n = 12)	MP (n = 8)	MSP (n = 8)	LP (n = 5)
Availability (Task 1)					
– Bi-directional (×, ÷)	11	7	5	1	1
– Uni-directional (×)	1	1	1	1	0
– Exponential (B, T)	11	6	4	1	0
– Exponential (B)	1	3	1	1	1
Access (Tasks 2 and 3)					
– Bi-directional (×, ÷)	10	2	0	0	0
– Uni-directional (×)	2	6	0	0	0
– Exponential (B, T)	10	2	0	0	0
– Exponential (B only)	0	2	0	0	0

Note. B = binary; T = ternary.

The results show that the poor performance of the low-performing students (MSP, LP) was due to unavailability of knowledge rather than lack of access. The poor performance of the MP students was shown to be due to access, suggesting that their available knowledge was not connected to form an integrated schema of multiplicative structure. The HSP students generally had available and accessible knowledge of binary relationships only. The HP students alone revealed that they generally had the appropriate knowledge available for both binary and ternary relationships and could access this knowledge, suggesting that they had constructed a schema of multiplicative structure.

Conclusions

The interview tasks revealed that the HP students alone had full multiplicativity of binary relationships, could transfer this knowledge to ternary relationships and could access this knowledge. Thus they were assumed to have constructed *a complete structural schema* of multiplicativity that incorporated the three components of continuity, bi-directionality and exponentiality, and connections between the components.

Figure 4 illustrates the available knowledge of the components and the assumed degree of connections held by each performance category. The black shading indicates full knowledge of that particular component (e.g., continuity across and within domains, bi-directionality, and exponentiality for binary and ternary relationships). The grey shading indicates limited knowledge of that component (e.g., continuity within but not across domains, uni-directionality, and exponentiality for binary but not ternary relationships). No shading represents no available knowledge of the component. The arrows indicate that the knowledge components are connected and therefore the knowledge is accessible. No arrows indicates no connections and therefore no access.

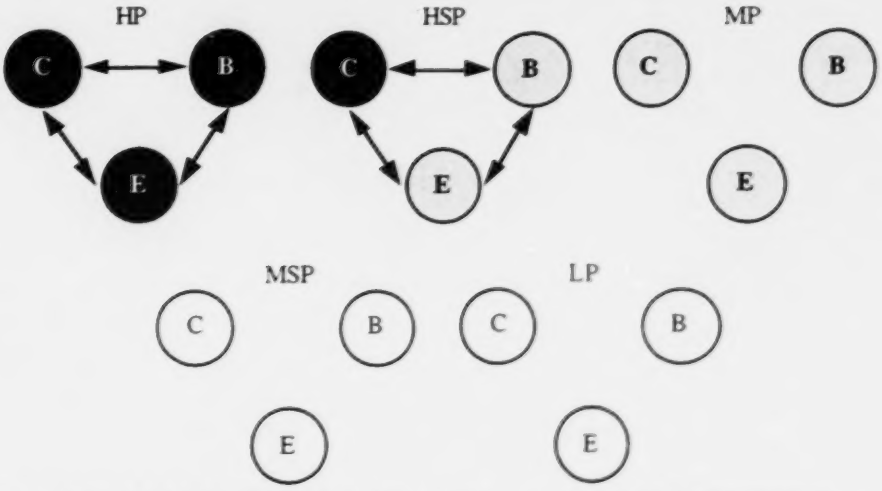

Figure 4. Structural schema of multiplicativity exhibited by the performance categories.

Two points of interest emerge from the models of multiplicative structure. One relates to the complete lack of knowledge held by the low-performing students. In the earlier interview tasks related to position and order, the students in these groups had revealed no knowledge of the decimal-fraction place names or of the value order (i.e., tenths are larger in value than hundredths) of the decimal-fraction places. This lack of knowledge was thought to be due to the absence of the exponential mental model shown in Figure 1, a supposition that is supported by their performance on the multiplicative tasks.

The HP students had scored ≥90% for both the tenths and hundredths items in the test so it seems that a complete structural schema (knowledge and connections) is essential for processing tenths and hundredths with understanding. In the test, the HSP students had scored ≥85% for tenths and <75% for hundredths so it seems that having a structural schema for tenths but not hundredths is not enough to enable processing of both tenths and hundredths with understanding. The MP students had scored 80-90% for tenths and for hundredths so having a schema of multiplicativity that is limited to uni-directional and binary relationships that are unconnected prohibits enables limited processing of tenths and hundredths. The MSP and LP students have no available knowledge so their processing of tenths and hundredths will be ad hoc and limited to prototypic tasks.

The HP students' structural schema of multiplicativity suggests that will be able to extend this knowledge with minimal difficulty to accommodate thousandths and ten-thousandths, etc. However, the HSP and MP students' structural schema indicates that they will be unable to accommodate new-decimal fraction places with any degree of understanding. The lower-performing students' lack of structural schema prohibit them from understanding tenths and hundredths and it is predicted that their problem will be exacerbated if introduced to thousandths without intervention to remediate their present knowledge of multiplicativity.

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Researching cultural issues with NESB secondary mathematics students: "In my country it is more difficult."

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This paper presents data from a two year study on Year 7 and 9 students, who were classified using a new construct of Non-Anglo-Cultural-Background (NACB). This was used to determine results at the individual level which showed for example that the NACB students generally were more confident and less anxious about their school mathematics than were their ACB counterparts. The paper also discusses some of the problems concerned with doing research into cultural issues in mathematics education

Research on cultural aspects of learning mathematics seems to play a minor role currently in the mathematics education scene in Australia. While there has been, and continues to be, a strong interest in gender issues, in contrast there has been relatively little attention paid to the mathematical experiences of either Aboriginal or immigrant students. For example, in MERGA 1995, in Darwin, which was an occasion where there could have been some focussed attention on cultural issues, although two of the plenaries dealt with this theme, only three other papers did (one of them being from an international visitor) (Howard, 1995, du Toit, 1995, and Thomas, 1995).

Earlier research on cultural aspects, summarised and reviewed by Atweh, Cooper and Kanes (1992), and by Ellerton & Clarkson (1992) in the context of language, generated a wide literature. In recent times however the research is limited in quantity but replete with possibility. As examples, Howard (1996) continues his important research with Aboriginal teacher educators, making us aware of the importance of sensitivity in dealing with cultural issues. Leder, Rowley & Brew (1995) present some challenging data about ESL students' performances in the Victorian Certificate of Education, while Thomas (1995) develops her arguments about improving language policies in schools to improve the chances of NESB students succeeding at University mathematics.

The rarity of such research could not be due to any lack of potential benefit. Australia, with its significant Aboriginal population, also has the world's second highest variety of 'home' countries from which its immigrants have come, and would seem to be a 'research site' of huge significance for this theme. As well as having intrinsic and pragmatic interest, there is also potential pay-off for the development of theoretical constructs generally, not to mention the economic importance of improving the quality of the participation of a large segment of the school population, nor the contribution such research could make to the multi-cultural nature of Australian society. However another reason for the paucity of this research could well be due to the inherent challenges of doing it. It is indeed a complex research field.

In this paper we will present some data from a study of year 7 and 9 students, which sheds some light on both the immigrant student's experience of learning mathematics and on the difficulties to be overcome if such research is to increase. The study was concerned with cultural diversity and involved the variables of gender, ethnicity and Socio-Economic Status (SES). It was hypothesised that the typical school mathematics culture could be characterised as predominantly 'Anglo, male, middle-class', and that these three variables would assist in the interpretation of students' experiences with any kinds of conflicts with that culture.

In previous papers we have referred to the gender-related data from this study (Brew et al., 1996), to the differences between high and low achieving students and their parents'attitudes (Leder et al., 1995), and to the role of 'significant others' in affecting peers' attitudes and perceptions (Bishop et al., 1996). At last year's MERGA conference we reported on the 1995 cohort data (Pearn et al, 1996) from schools with a large percentage of Non-English Speaking Background (NESB) students. In this paper we

shall present some of the intriguing data arising from the contrasts between these '95 students and their '94 counterparts who came from predominantly ESB schools.

Selection of sample - procedures and problems

The procedure for the selection of the sample of students was firstly to select eight State secondary schools, using such Government-based data which would maximise the possibility of contrasts between the students. There are two kinds of governmental classification which concern ethnicity - Non-English Speaking Background (NESB) and English as a Second Language (ESL).

An NESB student is commonly defined as one who was born in a country with a non-English first language, or born in Australia to parents either of whom were born in a country with a non-English first language.

An ESL student is defined as one who was born outside Australia less than seven years ago, in a country with a non-English first language, and who also has not studied English as a formal subject for more than seven years.

The SES of the school was determined by considering the Census information of the school catchment/enrolment area, participation in the Disadvantaged Schools Program, which has now ceased, and the proportion of students receiving the Educational Maintenance Allowance. In addition, four of the schools were selected from the SE Victorian region and four from the NW region.

Eight schools were finally selected, and agreed to participate in the research, according to the following plan:

	Low SES	Medium SES
Low NESB	1,2	3,4
High NESB	5,6	7,8

Schools 1, 3, 5, 7 were in the SE region while schools 2, 4, 6, 8 were in the NW region. Within each school, a Year 7 and a Year 9 class was then selected, and within each class the teacher was asked to choose four students whom they considered to be High Achievers and four students whom they considered to be Low Achievers, for follow-up interviews.

This then was the formal planned procedure. However as the research progressed from school level analysis to individual student level analysis, two aspects of the classification became increasingly apparent:

- the SES classification was suspect and uninformative at the individual student level, but it was not ethically possible to obtain individual students' SES levels, even if that could be meaningfully defined;
- the ESL/NESB classification was also suspect at the student level, and we decided to determine and use our own classification.

While the ESL/NESB labels may help funding and examination decisions at the state and national levels, there are too many inconsistencies and overlaps for good research at the individual level. For example, there are confusions about how to classify Hong Kong students, as they are considered NESB, even if their parents are of Anglo background, and Cook Islanders are not classified as NESB even though their first language is not English. Often schools classified pragmatically a student as ESL because of his/her English language problems despite being here for more than seven years. Also some ESL-classified students could have received all their schooling in Australia and in English.

Accordingly, we used the following criteria for our own ethnicity classification of individual students, which we will refer to hereafter as Non-Anglo-Cultural-Background (NACB). These were students:

- who were born into a non-English speaking home, regardless of country,
- whose first language is not English, and which was predominantly used at home,

- whose observable characteristics and language usage in class (if other information was unavailable) suggested that they should be called NACB.

Conversely, ACB students were those who were born either inside or outside the country, into an English-speaking family environment, and who used English as their first language.

Collection of data - procedures and problems

The data for this study came from four sources: questionnaires to all the students in the 16 classes, video-tapes of three lessons in each class, interviews with the selected students and with their parents where possible. The formal description of the data sources have already been explained (see, for example, Pearn et al., 1996).

Regarding the data-gathering, the questionnaires were in English, but they were administered personally by the research team who had ample opportunity to explain any difficulties. Initially teachers were concerned about the effect of video recording their classes, but after the initial phase no difficulties occurred. Gaining interviews with the targeted students was occasionally a problem, particularly with some of the low achieving students. Several visits needed to be made to the classes in order to complete the interviews. Even then, some interviews were not able to be carried out.

The most challenging part of the data gathering process however was gaining the interviews with the parents. These were either carried out at home, at the parents' work-place, or at Monash. There were particular difficulties with gaining interviews with the parents of the NACB-targeted students. The ethical procedures rightly permit any chosen subjects to decline to participate in the research, and unfortunately several of the NACB students' parents took that option. Whether this was due to language worries, being too busy, or for fear of giving information to strangers, is of course not known. Whatever the reason, the difficulties of engaging these parents is one of the problems to be overcome in doing this kind of research.

Results

Students' views about mathematics

In this part of the questionnaire there was not a great deal of difference between the students at the High NESB schools and those at the Low NESB schools. However three attitudinal constructs out of the 13 in the questionnaire showed a significant difference:

Table 1

	High NESB schools			Low NESB schools				
	mean	sd	n	mean	sd	n	F	p
Anxiety	2.27	0.76	178	2.45	0.82	179	5.65	<.05
Confidence	3.81	0.75	179	3.60	0.93	180	6.41	<.05
Maths not a male domain	4.09	0.66	176	4.36	0.54	180	27.21	<.001

Scale: 1 = strongly disagree to 5 = strongly agree. Six questionnaire items contributed to each construct.

The first two findings are consistent with each other, in the sense that the High NESB school students are less anxious and have more confidence than the Low NESB school students. In terms of the earlier expectations of cultural conflict, this was a surprising result. One could well believe that the differences should be round the other way, but it is not the case.

Table 2

	NACB students			ACB students				
	mean	sd	n	mean	sd	n	F	p
Anxiety	2.28	0.76	177	2.44	0.82	178	5.136	<.05
Confidence	3.79	0.79	179	3.63	0.91	178	4.022	<.05
Maths is not a male domain	4.10	0.64	175	4.34	0.57	179	20.21	<.001
Maths is logical	4.01	0.82	182	3.76	1.04	178	7.28	<.01
In maths there should always be one right answer	3.12	1.32	183	2.84	1.21	179	4.546	<.05

Scale: 1 = strongly disagree to 5 = strongly agree.

At this point we have to remind readers that these are school level effects. In order to see what happens at the individual level, we present in Table 2 the significant results from the equivalent analysis done using the NACB/ACB classification described earlier.

As can be seen by comparing the Tables 1 and 2 there is not a lot of difference in the results, but the finer classification has picked up two more significant differences. Hereafter we shall therefore present the data using the NACB/ACB classification.

The data show that the NACB students feel less anxiety and have greater confidence, have a preference for logical and 'one right answer' mathematics, and have a stronger view of mathematics as a male domain.

Attributions of success and failure

This aspect is one which often produces interesting results, and here was no exception. Table 3 below shows the significant differences and the trends:

Table 3

	NACB students			ACB students				
	mean	sd	n	mean	sd	n	F	p
Success due to ability	3.53	0.76	174	3.34	0.84	174	3.442	<.01
Success due to effort	3.64	0.77	176	3.48	0.79	175	3.94	0.077
Failure due to ability	2.90	0.78	172	3.13	0.84	175	2.433	<.01
Failure due to effort	3.06	0.78	173	3.26	0.72	174	5.426	<.01
Failure due to the task	3.40	0.73	174	3.52	0.69	175	0.683	0.055

This pattern of results is interesting, as it shows that the NACB students rate the internal factors (ability and effort) to be more important for their success than the ACB students, while it is the reverse for the failures. Also important for the ACB students are the task reasons for failure.

Classroom environment

On the Fraser classroom environment questionnaire three out of the five aspects were significantly different or showed a trend, see Table 4.

The NACB school students viewed their mathematics classrooms as offering a significantly greater degree of independence, with more possibility for investigation and

participation, than did the ACB students. Possibly this may reflect a different school experience from that in their country of origin.

Table 4

	NACB students			ACB students				
	mean	sd	n	mean	sd	n	F	p
Independence	2.68	0.65	175	2.51	0.60	178	4.29	<.05
Investigation	3.27	0.50	175	3.10	0.44	178	18.11	<.01
Participation	3.54	0.45	175	3.43	0.49	178	2.282	0.063

Student's Perception of their level of achievement.

This part of the data comes from a short questionnaire which asks the students to rate their own level of achievement, and what they perceive their teacher's, their peers' and their parents' opinions are about their level. The ratings go from 1 (weak) to 5 (excellent). Table 5 shows four areas of significant difference:

Table 5

	NACB students			ACB students				
	mean	sd	n	mean	sd	n	F	p
Self rating	3.51	0.92	189	3.27	0.96	180	3.314	<.01
Perceived teacher rating	3.50	0.98	186	3.19	1.01	178	5.811	<.01
Perceived class mate rating	3.66	1.02	186	3.22	1.11	179	8.950	<.001
Student wished-for rating minus self-rating	1.21	0.86	189	1.44	0.91	180	2.808	<.01

In each of the first three cases the NACB students rated themselves significantly higher than the ACB, which confirms the earlier findings about their feelings of greater confidence and less anxiety. They also seemed more 'content' with their performance in that their 'wished-for' rating was not as far ahead of their self rating as it was for the ACB students.

Discussion

Having anticipated that the school mathematics learning experience for the NACB students would be one of difficulty and conflict, we have been surprised by the results. We have found an interestingly coherent set of perceptions and attitudes of the NACB students, when compared to the ACB students. They generally appear to be more confident, less anxious, and more at ease with their perception of their own abilities. They generally attribute their successes to their own abilities and efforts, but not their failures, and they feel that their teachers and their classmates rate them higher than the ACB students. Furthermore, NACB students perceive their classmates rate them at a higher level compared to the rating they give to themselves.

For example, M., a year 9 high achieving student from the Ukraine, in response to the question "How good do you think your friends think you are?" answered:

M. Oh they think I'm good.
Int. Do you think that they think you are better than you are?
M. Yeah.

The NACB students also have a more 'logical' and 'right answer-oriented' view of mathematics, and they believe that maths is more of a male-subject than do their ACB peers. Furthermore the interviews revealed another interesting aspect. This was that several of the students spoke of the difference in standard of mathematical work at schools between their former country and Australia. Two example interview extracts follow. The first extract is from the interview with M., and the second is with a Year 9 Low-achieving male student from Vietnam:

Int. I keep getting a sense that in the Ukraine you worked at a different standard of maths, is that the case?
M. Yeah well I only went up to 3rd grade, so I can't remember. It was harder the stuff I did then, when I came here the stuff I did in grade 5 I did in grade 3.

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Int. What was it like doing maths in your own country?
T. In my country it is more difficult, when in year 7 we had equations and parallelograms or parabolas and bar graphs.
Int. So you did all that in year 7?
T. Yes
Int. And now you are doing it in year 9?
T. Yes.

This is clearly not the whole story, and it would be interesting to speculate on other reasons why the NACB students generally in this study seemed not to be revealing the kinds of difficulties, anxieties and conflicts which we had anticipated we would find. Despite the fact that not all of the NACB students were achieving well, it appears to be the case from this study that being from a non-English culture is not necessarily a handicap when it comes to learning mathematics at school. Moreover one might well have more concerns about the situation of the ACB students!

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Monitoring Change in Attitude of Preservice Teachers Towards Mathematics and Technology: A Longitudinal Study

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This paper reports on the quantitative data gathered during the second and third years of a 5 year longitudinal investigation into the attitudes of preservice primary teachers towards mathematics and toward the teaching of this subject. Findings indicate that teacher education programs can improve the attitudes of preservice teachers towards mathematics, and that this shift in attitude can continue to improve well into the first year of teaching.

The prior experiences of preservice teachers can not only have long lasting repercussion on the way they teach mathematics, they can also impact upon the outcomes of their students. Research findings link teacher attitudes toward mathematics with student attitudes and achievement levels in the subject (Bishop & Nickson, 1983; Schofield, 1981). In addition, work completed by Ernest (1988) suggests that teachers' attitudes toward the subject may affect their attitudes towards the teaching of the subject, which in turn impacts upon the ethos and culture of the mathematics classroom, which in turn impacts upon the attitudes of their students. Hence, a cycle of poor attitudes and poor achievement levels can spiral from generation to generation. Given the evidence indicating that a majority of prospective primary teachers hold negative attitudes toward mathematics (Ernest, 1988; Sullivan, 1987) it is imperative that shifts in attitude to the subject be an affective outcome of teacher education programs with the intention of stopping this spiralling effect. However, this desired outcome begs the question: Does preservice teacher education really make a difference to prospective teachers' attitudes towards mathematics and mathematics self-concepts? If so, how long does this affect last?

Schuck (1996) suggests that prior experiences are responsible for existing beliefs and attitudes that act as "chains" inhibiting movement towards reform in the way mathematics is taught in schools. We are in agreement with her proposition that knowing the nature of these constraints can help in shifting the attitudes of preservice teachers toward more favourable ones. Further to this, it is our premise that monitoring such shifts in attitude can facilitate the movement towards more positive ones as teacher education programs undergo restructuring and take account of the monitoring process in their course designs. The importance of such monitoring is acknowledged by Relich, Way and Martin (1994) in their work to develop a reliable and valid instrument to evaluate student attitudes so as to gauge the success of teacher education programs in this area.

This paper provides an update of a longitudinal research project begun in 1993 which aims to "investigate the attitudes of two groups of preservice primary school teachers toward mathematics and science/technology and toward the teaching of these subjects as they begin their teacher education". Furthermore, it seeks "to monitor these attitudes as the students progress through their respective teacher education programs...and to follow these students into their first years of their teaching careers" (Bobis & Cusworth, 1994a, p. 115). This paper, as have previous papers presented at MERGA conferences (Bobis & Cusworth, 1995; 1994a), focuses solely on the findings relating to mathematics. While the on-going investigation utilises the multiple data gathering techniques of questionnaire, structured and semi-structured interviews and reflective journals, only findings from the questionnaire will be included in this report.

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Method

Participants

Participants were preservice primary-school teachers enrolled in their third year of a Bachelor of Education (BEd) program at the University of Sydney and graduates of the Diploma of Education (DipEd) program with one year's teaching experience. While 138 (83 BEd and 55 DipEd students) responded to round 1 of the questionnaire, 93 students (47 BEd and 46 DipEd) completed the questionnaire in round 2 and 67 students (42 BEd 3rd year students and 25 DipEd graduates) completed it in round 3.

The Questionnaire and Procedure

The questionnaire used in rounds 2 and 3 of the investigation was the same used during round 1, but some items were changed slightly in round 3 to reflect the teaching status (that is, one year of teaching experience) of Diploma of Education graduates. The Attitudes Towards Mathematics questionnaire used in the study has been presented in detail previously (Bobis & Cusworth, 1994b), but will be described briefly here. In short, the instrument was a modification of Relich and Way's (1992) instrument designed to assess attitude toward mathematics and was reformulated to incorporate attitude toward teaching mathematics with the aid of technology (namely, calculators and computers). The final version was a 24 item, Likert-type instrument comprising 3 scales: an attitude to teaching mathematics (ATM) scale (11 items), a mathematics self-concept (MSC) scale (9 items), and an attitude to teaching mathematics with technology (ATMT) scale (4 items). Each student received 3 scores - one for each of the attitudinal dimensions of the questionnaire. These scores were the algebraic sum of the individual items in each of the dimensions. Higher scores represent more positive attitudes.

Table 1 summarises the data gathering procedure for the first 3 years of the investigation. Results of stages 1 and 2 have been reported at previous MERGA gatherings (Bobis & Cusworth, 1994b; 1995) and will only be referred to in this paper so that comparisons can be made with new findings.

Table 1 Summary of data gathering procedure for each group of subjects at each stage of the study.

Date	Stage	DipEd	BEd
February 1993	1	Round 1 of Questionnaire, general background survey and initial interviews.	Round 1 of Questionnaire, general background survey and initial interviews.
November 1993	2	Round 2 of Questionnaire and follow-up interviews.	
February 1994	3		Reflective Journals kept by BEd students.
November 1994	4	Round 3 of Questionnaire completed by 54% of graduates after one year's teaching experience. Follow-up interviews.	Round 2 of Questionnaire and follow-up interviews.
November 1995	5		Round 3 of Questionnaire and follow-up interviews.

For BEd students, round 2 of the questionnaire occurred at the end of 1994. Since no professional education subjects were undertaken in their first year of tertiary education, the questionnaire was considered unwarranted at the end of 1993. By round 3

of the questionnaire BEd students had completed three mathematics education subjects - one of which focussed on the use of computers.

Results and Discussion

In this section results for DipEd and BEd students will be reported separately. While some comparisons between the results of the two groups will be made in the discussion, it is considered inappropriate to make definitive comparisons when BEd and DipEd were at different stages of their careers for rounds 2 and 3 of the questionnaire. Table 1 presents the Diploma of Education group's mean scores and standard deviations for each round of the questionnaire and Table 2 presents the Bachelor of Education group's results.

Diploma of Education

It is clear from Table 2 that scores for each subscale of the questionnaire continued to improve for each of the domains even after the DipEd group had graduated and been teaching for one year. The greatest improvement occurred for the attitude toward teaching mathematics (ATM) domain at both rounds 2 and 3. Two tailed t-tests were used to determine if there were significant differences between the mean scores calculated for rounds 2 and 3 on each subscale. There was a significant difference for ATM at this stage in the study ($t = 3.282$, $t < 0.05$). Previous findings had also revealed a significant difference between the ATM mean scores of rounds 1 and 2 ($t = 3.5$, $p<0.05$). This means that students' attitudes towards the teaching of mathematics improved significantly at each round of the questionnaire, both during the time spent in their teacher education program and during their first year of teaching.

Table 2 Mean scores and standard deviations for each subscale for rounds 1, 2 and 3 of the questionnaire (DipEd only).

Factor		Dip Ed		
		Round 3	Round 2	Round 1
		n =25	n = 46	n = 55
Attitude to teaching mathematics (Max = 55)	mean	43.1	38.5	33.1
	SD	11.2	8.1	8.7
Self-Concept (Max = 45)	mean	32.3	31.1	29.5
	SD	8.5	9.5	8.1
Attitude to teaching with technology (Max = 20)	mean	15.8	15.0	13.6
	SD	4.0	3.1	4.4

While each round of the questionnaire revealed an improvement in math self-concept (MSC), consecutive rounds did not yield significant results ($t = 1.9$, $p>0.05$ and $t = 1.3$, $p>0.05$, respectively). However, the difference between MSC mean scores at the start of the DipEd program (round 1) and those after one year of teaching (round 3) were significant ($t = 2.5$, $p < 0.05$). The difference between the mean scores for each round of the questionnaire were 1.6 and 1.2 respectively. This means that, like the ATM of DipEd students, math self-concept continued to improve (though at a slightly less rate)

after they had been teaching for one year. This finding is of particular interest, since the resilient nature of self-concept as compared to attitude has been noted by other researchers in this field (Relich and Way, 1993).

While there was a significant difference between mean scores for attitude toward teaching mathematics with technology (ATMT) at rounds 1 and 2 ($t = 3.5$, $p<0.01$), the increase in mean score for ATMT between rounds 2 and 3 was not significant ($t = 1.3$, $p>0.05$). This indicates that after their initial improvement in attitude toward teaching with technology there had been little subsequent improvement.

Bachelor of Education

As can be seen from Table 3, the mean scores of Bachelor of Education students on each subscale of the questionnaire also improved at each round. Similar to the DipEd group, greatest improvement occurred for the attitude toward teaching mathematics domain with a mean difference of 11.4 between rounds 1 (the start of the program) and 3 (at the end of their third year in the BEd program). Two tailed t-tests were used to determine if there were significant differences between the mean scores calculated for round 3 and previous rounds on each subscale. While there was a significant difference between the mean scores for ATM at round 1 and round 2 ($t = 7.1$, $p< 0.0001$), there was no significant difference for ATM between rounds 2 and 3 of the questionnaire ($t = 0.9$, $p>0.05$).

Table 3 Mean scores and standard deviations for each subscale for rounds 1, 2 and 3 of the questionnaire (BEd only).

Factor		BEd		
		Round 3	Round 2	Round 1
		n = 42	n = 47	n = 83
Attitude to teaching (Max = 55)	mean	43.2	41.8	31.8
	SD	7.4	9.2	8.1
Self-Concept (Max = 45)	mean	33.9	32.4	30.8
	SD	7.6	7.8	7.8
Attitude to teaching with technology (Max = 20)	mean	16.9	15.7	12.3
	SD	2.5	3.1	4.7

There were significant differences for ATMT between rounds 1 and 2 ($t = 4.0$, $p<0.001$) and between rounds 2 and 3 ($t = 2.6$, $p < 0.01$) indicating that attitudes toward the use of technology in teaching mathematics improved significantly at each round of the questionnaire. Further analysis of these scores was conducted since comments made by DipEd students during previous rounds of the questionnaire and in interviews indicated that the attitudes of many students toward calculators and computers differed quite significantly. Results showed that BEd students' attitudes towards the use of calculators and computers had both improved significantly by round 2 of the questionnaire ($t = 3.6$, $p<0.001$ and $t = 3.6$, $p<0.001$ respectively), but round 3 results indicated that only students' attitudes towards the use of calculators had improved significantly from those found in round 2 ($t = 3.5$, $p<0.05$). This finding was interesting given the fact that a significant amount of practical instructional time was allocated to the use of computers prior to round 3 of the questionnaire for BEd students. It is hoped that analysis of

interviews conducted shortly after the questionnaire was completed may shed some light on this finding.

There were no significant differences between the mean scores for MSC in rounds 1 and 2 ($t = 0.7$, $p>0.05$), in rounds 2 and 3 ($t = 1.5$, $p>0.05$) or in rounds 1 and 3 ($t = 1.6$, $p>0.05$), indicating that the mathematics self-concepts of BEd students had not improved significantly in the three years that they had been monitored by this investigation. Unlike the DipEd finding of round 3, this result is in accordance with the suggestions by Relich and Way (1992) concerning the resilient nature of self-concept and highlights how critical it is to foster positive self-concepts from a very young age. While the mathematics self-concept of DipEd graduates improved significantly when the MSC mean score from round 1 was compared to the MSC mean score achieved after one year of teaching, it remains to be seen if the same will occur for BEd graduates after their first year of teaching.

Summary and Conclusions

Findings indicate that both DipEd and BEd groups improved their mean scores on all three subscales at each round of the questionnaire, though the improvements were not always significant or at the same rate for the two groups. At each round, attitude toward teaching mathematics had the greatest improvement, indicating that it is easier to shift this domain than the others. It is interesting to compare the attitude to teaching mathematics mean scores of the DipEd with those of the BEd at each round. While there was little difference between the mean scores at the start of their education programs, there was greater improvement in BEd students while still undertaking their teacher education and it was not until DipEd graduates had taught for one year (round 3) that the two groups achieved similar mean scores on this domain. Since the BEd students still have another year of teacher education, it is likely that their scores on this domain will continue to rise. It is possible, however, that attitude scores may reach some peak and then plateau. If this is the case, then it will be interesting to determine the career stage that this occurs for each group.

The results so far also suggest that the improvement in attitude toward teaching mathematics and math self-concept of DipEd graduates as a consequence of their initial teacher education is deep enough to not only remain more positive but to continue to improve well into their first year of teaching. The reasons for improvement are not evident from the quantitative data presented here, but it is hoped that analysis of interview data may shed some light on these reasons. Subsequent rounds of the questionnaire will determine if scores on each of these domains actually plateau or whether they start to drop back at some stage of their careers.

Of particular interest is the finding that mathematics self-concept improved significantly for the DipEd group between rounds 1 and 3 of the questionnaire. While this domain has been shown to be particularly resilient to change, it has never before been monitored in a group of subjects spanning this length of time or followed-up after a year of teaching experience. While the mathematics self-concept of the BEd group has not yet improved significantly, it is possible that it will continue to improve in their fourth year of the program and their first year of teaching as it did for the DipEd.

Findings also indicated that improvement in attitude towards teaching mathematics with technology slowed dramatically during the first year of teaching. This could be explained partly by the lack of technology and technological assistance in most primary schools, but interview data may reveal other reasons for this phenomenon. While BEd students' attitudes toward teaching with technology improved significantly at each round of the questionnaire, further analysis revealed that much of this improvement was attributable to an improved attitude toward the use of calculators. Computers remained a source of anxiety for BEd preservice teachers, despite having used them extensively to tutor school-aged children during mathematics education tutorials. This has significant implications for the integration of computers into teacher education programs.

If by improving preservice teachers' attitudes and self-concepts to mathematics we can initiate some impact upon classroom instructional practices and ultimately on children's attitudes and achievement levels in mathematics, then the findings of this longitudinal study are certainly reassuring. The results show that it is possible for teacher education programs to shift the attitudes and mathematics self-concepts of preservice teachers to more positive ones and that these shifts can continue to improve well into the first year of teaching. Whether a plateau will eventually be reached by this continual shift in attitudes or some other trend will manifest itself, can only be determined by subsequent rounds of the questionnaire and data collected from interviews. In addition, the fact that teacher attitudes toward mathematics and mathematics self-concepts continued to improve after one year of teaching experience - but at a slower rate than evidenced during their initial teacher education - highlights the importance of ongoing professional development during teaching rather than "one-off" teacher preparation.

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Do Algebraic Skills Improve with Higher Learning?

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Many students enrolled in first year university mathematics displayed consistent misunderstanding with some essential algebraic skills. Analysis of an algebraic test given to first year university students in 1996 highlighted five major categories of concern. Further algebraic tests were given to 1997 first year university mathematics students, second year university mathematics students and senior secondary students. The five categories were confirmed as major areas of consistent algebraic difficulties. Many of these difficulties did not seem to improve with higher mathematical learning.

Introduction

Understanding is important for success with algebraic skills. Many students know variables only as placeholders for numbers and are totally unprepared to regard algebraic entities in different ways (Barbeau, 1995).

How can we expect students to interpret solutions to problems when basic algebraic difficulties, based on a lack of understanding of early concepts that are assumed to be mastered prior to university, can be their largest stumbling block?

In 1996, two groups of first year university students (N=526 and N=88) sat an algebraic test three weeks after enrolment in a general mathematics course. Detailed analysis of these scripts led to the detection of five topic-independent categories of incorrect responses (Boustead, 1996). Four of these categories were described by Rotman as exclusive pre-algebra areas of arithmetic (Rotman, 1991). Rotman labelled the categories *"understanding the order of operations agreement"*, *"understanding the properties of numbers, especially fractions"*, *"understanding the structure of obtaining solutions"* and *"understanding the meaning of symbols"*. The last category was relabelled by this author as *"false generalisations"*. A final category of incorrect responses added after the analysis of the 1996 algebraic test was labelled *"understanding the range of possible solutions"*. Three of these categories were also mentioned in an analysis of a multi-choice algebraic test given to 40 science students in 1992 (Kaur & Sharon, 1994).

This paper is an extension of the 1996 analysis of the algebraic test sat by first year mathematics students. Do the same five categories still exist and are the categories consistent with a similar cohort in 1997? Are many of the weaknesses apparent in the five categories already present in senior secondary mathematics? If so, are there any automatic improvements in algebraic skills with higher learning?

The Study

In 1997 first year students who gained between 50% and 75% in the Bursary Mathematics with Calculus examination and who enrolled in a university mathematics paper (N=540), sat a one and a half hour, 28-item algebraic test in their second week of lectures. Calculators were not permitted and the students were provided with a table of relevant formulae. Students were encouraged to display all working. A third of these students completed the test within one hour.

A week later, two senior secondary mathematics classes from two different schools (N=16, N=21) and two second year university mathematics classes (N=142, N=198) sat similar six-item algebraic tests based on the categories determined from the 1996 and 1997 first year university algebraic tests. Calculators were permitted in all these classes and the tests took about 15 minutes to complete. Again, students were encouraged to display their working.

Each question in the six-item test was adjusted to the level of the students' mathematical knowledge. For example, a second year university student would be

expected to solve $m^2-8m+4=0$ as part of their work on second order differential equations, but a senior secondary student would be more familiar with the same question in terms of x rather than m.

A comparison was drawn between the three mathematics levels: senior secondary, first year university and second year university levels. The analysis was based on a random sample of 100 scripts from each of the first and second year classes and all 37 senior secondary scripts.

The Categories Confirmed

An analysis of the 1997 algebraic test for first year university mathematics students showed that in each of the five categories mentioned in the "introduction" between 19% and 29% of the first year university students gave similar types of incorrect responses. This confirmed that the five categories were still relevant in 1997.

Although there were a variety of other algebraic errors, especially in questions involving integration, differentiation and complex numbers, many of these incorrect responses could be placed in one of the five categories. Nevertheless, one other possible category was temporarily considered. Some students appeared to have difficulty understanding what the question was asking. For example, 9% of the students did not understand the meaning of "Find $\left|\frac{1}{x+iy}\right|$". Some students believed that the question was asking them to convert the denominator into a real value by multiplying by $\left(\frac{x-iy}{x-iy}\right)$. However, this difficulty was not widespread or consistent enough (>10% of students) to form a sixth category and any difficulty seemed to be limited to unfamiliar notation.

Understanding The Order Of Operation Agreement

Students need to be aware of the basic order of operations in arithmetic encountered in late primary or early secondary level, and the extensions or limitations of these concepts into algebra. For example, in algebra different variables added or subtracted within a bracket cannot be combined. This means that calculation within a bracket may not necessarily be the first order of operation. The most common incorrect response in this category in the 1996 test occurred when a negative value was multiplied into a bracket containing more than one variable. Students tended to deal with brackets by ignoring them. For example, - (x + y) would become -x + y.

Aim: For each cohort a question was selected to specifically monitor whether every variable inside a bracket was multiplied by a negative value outside the bracket.

Question:

Senior secondary $3x-(\frac{1}{4}y+5z)+1=0$

First year university $1-(1+\frac{x}{2}-\frac{x^2}{3})$

Second year university $6x-2z\frac{\partial z}{\partial x}+3y-(4z-4x\frac{\partial z}{\partial x})=0$

Results:

class	correct	no answer	failed to multiply all items	other (esp rearranging)
Senior secondary	0.48	0.12	0.22	0.18
University Year 1	0.48	0.05	0.28	0.19
University Year 2	0.51	0.24	0.17	0.08

Table 1: Average proportion of students for each type of response to "understanding the order of operations agreement".

Table 1 shows that approximately half the students in each of the three cohorts gave the correct answer. On average, 22% of senior secondary students, 28% of first year students and 17% of second year students made the same mistake. They ignored the

brackets. These results were slightly higher than for the 1996 analysis of first year university students.

The slightly lower proportion of students making this same error in second year university (Column 4, Table 1) was expected since students weak in first year university mathematics were unlikely to pass into second year. However, almost a quarter of the second year university students did not give an answer or show any working despite having been exposed to partial derivatives at the end of the previous year.

Senior secondary students may have exhibited fewer errors than first year university students because of algebraic skills revision done as part of the secondary course a few weeks before the 1997 algebraic test was sat.

Understanding The Properties Of Numbers, Especially Fractions.

Fractions are encountered initially in primary school and the processes learned at this level can impact later on algebraic skills. First year university algebra predominantly involves working with rational functions. In the 1996 algebraic test given to first year university students, the major areas of concern were in simplification and addition/subtraction of rational functions.

Aim: For each cohort two questions were selected. The first question was designed to monitor whether students could simplify a rational function where one element on the numerator matched the denominator. A second question monitored how successful the students were at adding two rational functions.

Simplification Of Rational Functions:

Question:

Senior secondary: $\frac{2x^2-2x}{2x}$

First year university: $\frac{ab-a^2b^2}{ab}$

Second year university: $\frac{2n+2n^2}{2n}$

Results:

class	correct	no answer	Incorrect response A	Incorrect response B	Incorrect Response C
Senior secondary	0.76	0.08	0.00	0.12	0.04
University Year 1	0.76	0.00	0.01	0.08	0.11
University Year 2	0.88	0.01	0.04	0.04	0.01

Table 2: Average proportion of students for each type of response to "simplification of rational functions". (A, B and C refer to the type of error described below.)

Overall, 16% of senior secondary students, 20% of first year university students and 9% of second year students exhibited difficulty when simplifying a given rational functions (Table 2). The proportion of students making errors in 1997 were similar to the 1996 analysis.

Incorrect Response A *in Table 2*: $\frac{2n}{2n}+\frac{2n^2}{2n}=1+2n$. Students assumed that $2n^2$ was the equivalent to $(2n)^2$. There were slightly more second year university students who gave this type of incorrect response than from the other two cohorts (senior secondary and first year university).

Incorrect Response B in Table 2: $\frac{\cancel{2n}+2n^2}{\cancel{2n}}=n$. The element on to the left of the + has been cancelled to zero by the denominator. The proportion of students committing this error was highest amongst the senior secondary students and second highest for first year university students.

Incorrect Response C in Table 2: $\frac{2n+2n^2}{2n}=2n^2$. Similar to Response B except that the $2n$ on the denominator and numerator are cancelled. First year university students dominated with this type of error.

Incorrect cancellation of rational functions was a concern for senior secondary students and first year students. Since the proportion of students committing this error was lower for second year university students, this type of incorrect response may improve with higher mathematics learning.

Addition Of Rational Functions:

Question:

Senior secondary: $\frac{1}{x-3}+\frac{1}{2x}$

First year university: $\frac{1}{x+1}+\frac{1}{x}$

Second year university: $Y\ (s^2+2s-8)-1=\frac{1}{s-3}$

Results:

class	correct	no answer	Error in finding denominator	Error in finding numerator	other (esp rearranging)
Senior secondary	0.56	0.16	0.27	0.01	0.00
University Year 1	0.69	0.01	0.27	0.03	0.00
University Year 2	0.36	0.17	0.37	0.04	0.06

Table 3: Average proportion of students for each type of response to "addition of rational functions".

All three cohorts displayed difficulty obtaining the correct denominator in a rational function (Table 3). Nearly a third of both secondary students and first year students, and over a third of second year university students, experienced this difficulty. Even if the denominator was correct, a small proportion of students in each cohort then made errors calculating the correct numerator. Again, 17% of second year students and 16% of senior secondary students chose not to attempt the question.

Understanding The Structure Of Obtaining A Solution.

Basic algebraic manipulation involves keeping an equation balanced throughout a calculation and understanding the contextual meaning of the symbols being used (Teppo & Esty, 1995). In the 1996 analysis it was noted that difficulties in this category became obvious whenever students manipulated an equation, used the quadratic formula or completed the perfect square. Values could be correctly in place initially, but calculations then led to errors in manipulation.

Aim: To see if students, once they substitute in the correct values into a formula or equation, can continue their calculation without making errors that alter the structure of the equation.

Question:

Senior secondary: $x^2-8x+4=0$

First year university: $x^2-6x-15=(x+a)^2+b$

$3-8x+x^2=0$

Second year university: $m^2-8m+4=0$

***Results*:**

class	correct	no answer	quadratic formula incorrect	complete perfect square, *b* incorrect	other
Senior secondary	0.71	0.03	0.19	0.05	0.02
University Year 1	0.60	0.06	0.17	0.13	0.04
University Year 2	0.54	0.24	0.14	0.02	0.06

Table 4: Average proportion of students for each type of response to "structure of obtaining solutions".

When students chose the technique to solve the quadratic equation, most (students) preferred the quadratic formula rather than completion of the perfect square technique. Only a few of the students who chose to use the latter exhibited errors in their calculation. Students who preferred the quadratic formula but were required to complete the perfect square (especially for the question given to first year university students) often made errors in calculating b in $(x+a)^2+b=0$. Of those students who preferred the quadratic formula between 14% and 19% substituted the values correctly into the formula but their common errors involved either shrinking of the divisor line (8% to 12%) or miscalculation within the square root (Table 4). Shrinking of the divisor line can be illustrated in the following example: $x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$ became $x=-b\pm\frac{\sqrt{b^2-4ac}}{2a}$.This error can also overlap the category "understanding the properties of numbers, especially fractions".

Generally, accuracy in calculations did not improve with higher learning and first year university results were similar to the 1996 analysis. Unfortunately about a quarter of the second year students did not attempt to solve the quadratic equation.

False Generalisation

Students can apply a previously learned technique to a context that may appear to be similar but is actually different. The technique the students use is inappropriate for the new context (Kaur & Sharon, 1994). For example: $\frac{(1+t)^2}{t^2}=\frac{1+t}{t}$. Here, the student used subtraction of indices to this situation when the 'base' variables were different.

In the 1996 analysis with first year university students it was found that if the questions were given as "Solve (3x-4)(x+1)=0" and "Solve (3x-4)(x+1)=2", 93% solved the first equation correctly but only 57% solved the second equation correctly. Some of the students (21%) tried to use the same method as the first . That is, (3x-4)=2 or (x+1)=2.

Aim: To monitor whether students use inappropriate rules or techniques in apparently similar but different contexts.

Question:

Senior secondary	Solve $(x-2)(x+3) = -4$
First year university	Solve $(2x-1)(x-1) = 5$
Second year university	Solve $x(x-2)(x+3) = -4x$

***Results*:**

class	correct	no answer	Incorrect x-2=-4 or x+3=-4	Ignored right hand side	other (esp. error in expanding or factorising)
Senior secondary	0.62	0.03	0.14	0.07	0.14
University Year 1	0.34	0.06	0.29	0.03	0.28
University Year 2	0.66 - 0.29*	0.20	0.05	0.01	0.08 +0.29*

Table 5: Average proportion of students for each type of response to "false generalisation".

* In Table 5, some of the second year university students (29%) cancelled the x's on both sides of the equation and thereby ignored a possible solution of $x = 0$. This error is discussed in the next category.

The first year university students exhibited the highest proportion of incorrect responses and most of these (29%) used the same technique as they would when the right hand side of the quadratic equation was zero (Column 4, Table 5). The results were similar to the 1996 analysis when 21% of the students made the same error. Even though many of the students in second year university mathematics may have remembered this type of question from the 1996 algebraic test the previous year, 20% of these students did not attempt the question.

The dramatic increase in proportion of first year university students who made the same error (29%) could be attributed to the change in context in which such a question is given. At school, students are reminded to rearrange equations so that zero is on the right hand side. Such calculations are usually taught as an isolated topic. At university this is a skill that does not get any reinforcement and the equation is usually within a different context. In the case of the 1997 algebraic tests, the question was out of the context in which the algebraic skill would have been learned.

Understanding The Range of Possible Solutions

Students need to look for solutions other than the most obvious. When students become familiar with complex numbers and functions, they need to consider complex solutions as well as real solutions. In finding the values of x when $\sin x = \frac{1}{2}$ students need to consider solutions other than $0 \le x \le \frac{\pi}{2}$. When locating all critical points, students need to be aware of solutions they could be eliminating with inappropriate cancellation of variables. On the other hand, when graphing a function such as $f(x) = \sqrt{x^2 - 4}$, students often incorrectly include the negative values of $f(x)$. Therefore this category is more than just looking for all possible solutions, but more about understanding which solutions are relevant.

For specific contexts students can be 'trained' to find negative as well as positive solutions for their algebraic calculations. Most students are likely to get this type of calculation correct. For example, $(x-2)^2 = 9 \Rightarrow (x-2) = \pm 3 \Rightarrow x = -1$ *or* $x = 5$. The question was given to a class of average senior secondary students (N=16). Ten of these students gave the correct answer while four students ignored the negative solution and gave their answer as "5".

However, in not so familiar contexts most students do not consider other possible solutions. The following question was given to the second year university students as well as to a class of accelerated senior secondary students (N=21).

If $x > y$, is it true that $\frac{1}{x} < \frac{1}{y}$?

Results:

class	correct	no answer	negative not considered	both negative for x and y	other
Senior secondary	0.04	0.01	0.86	0.08	0.01
University Year 2	0.06	0.01	0.81	0.10	0.02

Table 6: Average proportion of students for each type of response to "understanding the range of possible solutions".

Most students did not consider that x or y could be negative as well as positive. Of the students that did consider this possibility 8% of senior secondary students and 10% of second year university students assumed that both variables could only be negative. Very few students (4% and 6% respectively) considered the possibility of x being positive and y negative (Table 6).

The first year university students were given a question that involved the solution of a trigonometric function. The question was similar to that given in the 1996 test and the

results were comparable. Solutions outside the $0 \leq x \leq \frac{\pi}{2}$ range were not considered by 20% of the students.

So, although students can be 'trained' to consider other possible solutions within one context, this does not necessarily mean that the students understand what they are finding or will think of applying the same ideas to other contexts.

Conclusion

Analysis of the 1997 algebraic test given to first year university students concurred with the analysis of a similar algebraic test given in 1996. Categories of algebraic weaknesses and proportion of students displaying those weaknesses were similar for both years. The categories "understanding the order of operations", "understanding the structure of solutions" and "understanding the property of numbers" are revised briefly in algebra at the beginning of the final year of secondary school, and are assumed by the university to be mastered before entrance to university. The other categories of "false generalisation" and "understanding the possible range of solutions" are skills acquired in senior secondary school and early university. Coping with these skills require a good grasp of the other categories as well as linkage of ideas and choice of appropriate mathematical tools.

Algebraic difficulties in most of the categories were exhibited by senior secondary school students. These difficulties became more widespread and consistent with first year university mathematics students, and slightly less widespread (in some categories) with second year university students. One possible explanation for this trend is that some students could have learned their algebraic skills at school in a isolated, surface learning fashion with little understanding or extension of concepts. Such students would have difficulty coping with situations in which they have choices in mathematics or are required to link ideas. Combine this with a four month break from mathematics before university begins and some students begin university mathematics with gaps in skills.

By the time the student reached second year university there was evidence of slight improvement in categories such as order of operations, simplification of rational functions, structure of equations, false generalisation. However, similar weaknesses still exist, probably because of emphasis on concepts and assumption of prior algebraic skills.

Students do not appear to markedly improve their algebraic skills through the learning of more advanced mathematical concepts. The evidence is that weaknesses in algebraic skills at school are unlikely to be corrected automatically at university without active intervention by the student. The basics learned in pre-algebra arithmetic and early secondary school could therefore play a crucial role in mathematical success of many students at a later date.

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The resolution of uncertainty in mathematics classrooms

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This paper addresses negotiation as a social process related to the practices of mathematics (and science) classrooms and associates the need for negotiation with the occurrence of uncertainty. It is our argument that one pathway to knowing is via the resolution of uncertainty, that the process of resolution is fundamentally negotiative, that negotiation is mediated by language, that language presumes intersubjectivity, and that the matter of intersubjectivity is meaning. Data is presented illustrating the role of intersubjectivity as both mediating agent in the resolution of uncertainty, and as product of the negotiative process. Empirical evidence is reported regarding the occurrence of student uncertainty with regard to academic content encountered in classroom settings and the means by which these uncertainties are resolved.

Introduction

The research reported in this paper was undertaken as part of the Classroom Learning Project. This project has as its ultimate goal the modelling of the learning process as it occurs in classroom settings. One immediate task undertaken within the project has been to establish an empirical foundation for the constructs from which a model of learning might be constructed. Several of these constructs are best understood as metaphors: negotiation, abstraction, and mediation, for instance. Each such metaphor carries a burden of associative meaning that may or may not facilitate our understanding of the phenomenon or cognitive object signified by the construct. It is a characteristic of the discourse of the educational researcher that such terms are frequently gerunds (verb/nouns), open to interpretation as both process and product, but having the potential to confuse the distinction between the two. One question that must be asked of any theory employing such terms is whether the construct (abstraction, for instance) is intended to signify an activity (a process) or an outcome (the product of a process).

The phrase "negotiation of meaning" has been used to describe the means by which learning occurs in social settings. Earlier papers (Clarke, 1993, 1994, 1996) have examined the legitimacy with which negotiation is employed to characterise a cognitive process associated with learning. This paper addresses negotiation as a social process related to the practices of mathematics (and science) classrooms and associates the need for negotiation with the occurrence of uncertainty. It must be stressed that negotiation may be triggered by other states besides uncertainty: disagreement, for instance. However, this paper is focussed on documented instances of uncertainty, and the means by which such uncertainties were resolved.

In the analysis reported here, the social goal of resolving uncertainty is taken as the starting point, and the means by which any resolution is achieved is documented. In this account of classroom learning, intersubjectivity enters as a mediating agency, essential to the negotiative process, whereby uncertainty is resolved, and new knowings are constructed. Whereas Voigt's recent analysis starts from the learner's subjectivity and models the establishment of intersubjectivity (Voigt, 1995), this account starts from evidence of students' intersubjectivity and documents alternative forms of uncertainty and the process whereby resolution is achieved; a process in which intersubjectivity has a central role.

Research Methodology

The origins and rationale for the research methods employed in the Classroom Learning Project are outlined elsewhere (Clarke, in press). The current data base for the project consists of videotape records of fifty-five secondary maths and science lessons obtained using two cameras. One camera was directed at the teacher, while the other camera was focussed on a group of about four students. The teacher's utterances were recorded through a radio microphone, and a single microphone was used to record the conversations of all four students. The two video images were mixed on site to produce a composite picture in which the students occupy most of the screen with the teacher image superimposed in a corner of the screen. This

combined image was recorded onto video-8 tape using a compact videorecorder attached to a laptop computer. The researcher, seated at the rear of the classrom, was able to listen simultaneously to both student conversations and teacher utterances and to record field notes onto a word processing document on the computer. The field notes were "time-tagged" to corresponding events in the video record using *CVideo* software (Roschelle, 1992). The field notes enabled the researcher to document impressions of significant classroom episodes and learning events as well as to provide reference markers for the subsequent interviewing of student subjects. The research procedure was developed in an attempt to study learning in legitimate classroom settings, while minimizing the need for researcher inference regarding student thought processes. To infer student thought processes and the significance of classroom events on the basis of videotape data only seems an unjustified extrapolation. The video-stimulated recall interview has become an accepted research technique (see, for example, Anthony, 1994), and technological advances such as *CVideo* further enhance its effectiveness. On-site production of the linked field notes and split-screen video record allow the researcher to conduct such student interviews immediately after the lesson being studied.

The videotape record was used in the interview to stimulate student recall of classroom events. The use of the *CVideo* software enabled the researcher to locate within the field notes reference to actions of the student which seemed to be of significance either to the researcher or to the student. Having found a particular item in the word document, the software could be used to find very quickly the corresponding moment on the video record. This was then played back and discussed. In such interviews, students reconstruct their motivations, thoughts and actions, prompted by the video record of the classroom events, in full knowledge of the ultimate outcomes of those actions. In this situation, the reconstructed account provides both an explanatory interpretation of the video record and a reflective commentary on the relationship between action and outcome. The audio record of the interview provided a third source of data.

For the purposes of the analyses reported in this paper, six "integrated data sets" were generated from three mathematics and three science lessons. Each integrated data set is a single text document incorporating transcriptions of videotape and interview dialogue together with the researcher's field notes, time-tagged to the video record, and supplemented by copies of student written material. Instances in which there was evidence of student uncertainty were identified and these episodes were then categorised according to the nature of the uncertainty and the means (if any) of resolution.

Uncertainty and Resolution

It is our argument that one pathway to knowing is via the resolution of uncertainty, that the process of resolution is fundamentally negotiative, that negotiation is mediated by language, that language presumes intersubjectivity, and that the matter of intersubjectivity is meaning. Before presenting our empirical data, it is essential that we establish a theoretical framework with regard to each of the constituent terms in the preceding sentence.

Negotiation

Negotiation has been characterised in some detail elsewhere as a cyclic process of refraction (construal), reflection, and representation, the goal of which is consensus (Clarke, 1996). Cobb and Bauersfeld define the negotiation of meaning succinctly as "the interactive accomplishment of intersubjectivity" (Cobb & Bauersfeld, 1995, p. 295). Lave and Wenger, in associating learning with participation in practice, assert that "Participation is always based on situated negotiation and renegotiation of meaning in the world" (Lave & Wenger, 1991, p. 52). Since classrooms represent legitimate sites of situated mathematical practice, this perspective supports the need for the empirical documentation of negotiative processes in the classroom.

Meaning

The presumptions of meaning are community, purpose and situation. It is futile to discuss the meaning of a word or term in isolation from the discourse community of which the speaker claims membership, from the purpose of the speaker, or from the specific situation in which the word was spoken. Indeed, it is not the word that has meaning, but the utterance. Our understanding of meaning has been usefully informed by consideration of the writings of Bakhtin and his colleagues and interpreters.

> In all these cases, we are dealing not with the isolated words as a unit in language, nor with the *signification* of this word, but with the completed utterance and its concrete *meaning*, the content of this utterance (Bakhtin, 1979, quoted in Todarov, 1984, p. 53).

Since the data with which we work consist of utterances of one form or another, and since the focus of this paper is the resolution of uncertainty, Baktin's statement that "Meaning always answers some questions" (Bakhtin, 1979, quoted in Todarov, 1984, p. 54) is particularly useful. It identifies meaning with the need to know and, by implication, with the resolution of uncertainty.

The Role of Intersubjectivity

Language is constitutively intersubjective (Todarov, 1984, p. 30). As noted in the introduction to this paper, a level of student-student and student-teacher intersubjectivity is prerequisite to the negotiative processes by which the resolution of uncertainty is attempted. The role of intersubjectivity in negotiation can be argued successfully from the dependency of negotiation on language (or at least on some form of communicative process), but this does not excuse us from the obligation to present empirical evidence of intersubjectivity in action. An understanding of the means by which the resolution of uncertainty might be achieved requires an understanding of intersubjectivity as a phenomenon of social interaction. To establish this point, consider the following transcript (All utterances are by students. K and L were subsequent interviewees, S19 and S20 were not):

Episode 1

Line	Transcript
1	S19: It says how many sheets of graph paper would you need to show one million one millimetre squares.
2	L: To show one million, you know you don't divide it by one hundred, because there's more than a hundred one millimetre squares. I mean you're going to find the area of this.
3	K: What?
4	L: You've got to find the area of this, there's more than one hundred one millimetres.
5	K: That's right. I was doing length by—oh screw that.
6	L: One hundred one millimetre squares.Take length—
7	K: Um, there's how many down here?
8	L: And along that side there is—
9	K: How many are there down there?
10	L: There's a hundred one millimetres there.
11	L: No, there wouldn't be.
12	K: There wouldn't be, that's not right.
13	L: It'd be 250.
14	K: Yeah.
15	L: Yeah, there'd be 250, length of graph.
16	K: OK, so it would be length times width [inaudible]
17	L: And uh, 250 millimetres. Width—
18	K: What's width?
19	L: That's—
20	K: That's 10, 20, 30, 40, 50—
21	L: 18, 180.
22	K: Times 180. OK here we go. 250 [inaudible] 45 thousand.
23	K: Equals 45 thousand.OK, that's 45 thousand millimetres. What's a million divided by 45 thousand and times it by that?
24	L: OK. Therefore there are 45 thousand mm squares.
25	S20: 45 thousand mm?
26	L: Yeah.
27	S20: 45 thousand.
28	K: 22.2.
29	L: Sheets. Of graph paper.

Evidence for the enactment of intersubjectivity should assist us in locating intersubjectivity within the learning process as either agency or as outcome or as both. In the above transcript, there are several indicators of intersubjectivity. First, much of the actual text is incoherent as written text: sentences are ungrammatical or incomplete; pronouns are used without textual clues as to their referrents; single word utterances are frequent. Communication in this form can only be sustained if the participants share understandings of the referrents of the pronouns or key words, and of the processes, actions, or relationships suggested (but not stated) by the sentence fragments and the participants' gestures. Further evidence of intersubjectivity can be seen in the manner in which one speaker will complete the sentence of the previous speaker, as occurs in lines 28 and 29 (we are using "line" as a shorthand for "utterance"). In this sense, it is useful to distinguish the overt text in the form of the literal transcription from the implicit text being co-constructed by the participants, the existence of which can only be inferred on the basis that the interaction appears to have been both purposeful and successful.

Identifying Uncertainty

We are employing "Uncertainty" to refer to a behavioural state. Examples are provided below regarding several forms of student uncertainty with regard to academic content encountered in classroom settings for which we have empirical evidence.

- Uncertainty regarding a *fact* or correct term;
  ["Why is it V, what does V have to do with speed?"]
- Uncertainty regarding the correct form of an action or *procedure*;
  [D: Can you tell me how to work that one out?
  M: You get eight four oh which is the number of students, divide five, times two.]
- Uncertainty as to the *meaning* of a term or an activity;
  ["How can the graph go below zero? You can't have a minus ten *speed*"]
- Uncertainty regarding a form of *representation*;
  ["It's the velocity, not the way it bounces"]
- Uncertainty as to the *correctness* of the student's opinion, understanding, method, or outcome.
  ["Tell me if you think this looks sort of logical"]

This categorization of Uncertainty is unlikely to be exhaustive with regard to all possible classsroom settings or situations. Rather, the categories provide an adequate classification of our existing data set.

Resolution

Resolution is yet another term open to interpretation as either process or product. The categorization system, that we have employed in this paper, is based on the dominant authority appealed to in the *process* of Resolution. For the purposes of this discussion, Resolution is taken to involve one of four forms, but these are not mutually exclusive, and a particular episode may involve student appeal to several authorities in seeking resolution of the same uncertainty.

Resolution via an appeal to *empirical evidence*;
Resolution via an appeal to a *person* (a knowledgable other, such as the teacher or another student);
Resolution via an appeal to a *text*;
Resolution via an appeal to *prior experience*.

An adequate characterisation of Resolution as outcome would require a more extensive data base and analysis than has been possible to date, and as such is beyond the scope of this paper. It is possible, however, within the scope of this paper, to anticipate and even to document some likely associations between Uncertainty and Resolution. Since, theoretically, each of the categories of Uncertainty may be resolved by any of the above means of Resolution, it is possible to consider the interaction of Uncertainty and Resolution. That is, does the knowing which is prompted by student uncertainty as to a procedure and which is resolved by appeal to a peer differ in a significant manner from the knowing which is prompted by student uncertainty regarding a representation and resolved by reference to a text (for example)? We conjecture that these differences might well be profound.

Results And Discussion

Examination of the six integrated documents revealed twenty-eight classroom "episodes" in which one or more students expressed uncertainty. The episodes comprised the dialogue and activities that students engaged in as they approached, worked on and reached completion of a particular classroom activity, such as a problem solving task. Thus, each episode was a coherent unit of activity unified by a single purpose. Within each episode there were identifiable "negotiative events" consisting of a number of utterances—questions, statements, answers—where a particular form of uncertainty was expressed and sometimes resolved. For example, a string of utterances relating to uncertainty about a mathematical procedure may be resolved by interaction with peers and teacher as a student participates in classroom activity. Forms of associated uncertainty and resolution are illustrated in more detail in the example below.

Uncertainty as to procedures and outcomes resolved by appeal to peers.
In this episode, year 8 students had been asked to find the height of a stack of one million sheets of paper. Uncertainty about procedures and outcomes was resolved through peer interaction. Appeal to peers was in frequent evidence in our data and the following episode is a particularly rich example of the features of this form of classroom negotiation (K and L are student interviewees; S20 and S22 student non-interviewees; and, T is the teacher).

Episode 2

	Line	Transcript
E	1	L: [writing] 500 sheets. Height equals.
V	2	K: OK, question 2 [Find the height of a stack of one million sheets of paper].
E	3	L: Does everyone understand what we did with number one?
N	4	K: No, but. Anyway. 500 sheets.
T	5	L: And how many sheets do we need?
	6	K: 500 sheets of what? 500 sheets.
1	7	L: Their height equals 5.8.
	8	K: I know. But we've got to do it all together so.
	9	L: One point oh times ten to the power of 6 divided by 500.
	10	K: Oh yeah sure what are you talking about? What are you talking about?
	11	L: Finding out how many five hundreds there are in a million.
	12	K: How many five hundreds there are in a million. That would make one thousand. How many thousands are there in a million? That would make a thousand, two thousand. What? [to another S]. I have a lot to say. This is called skill. This is what you do. You divide 500 into a 100 which is 2. Then you do a hundred [correcting herself] A thousand.
	13	L: Times five point eight. Shush.
	14	K: Which is 2. Then you do a thousand into a million which is 1000, so 1000 times 2 is 2000.
	15	L: Eleven thousand six hundred.
E	16	T: With your working out folks I want you to tell me what you are mult—Matthew— what you are multiplying by, and you simply put a little arrow telling me what and why.
V		
E		
N	17	K: What are we doing? Is it a million sheets of paper though?
T	18	L: Yeah. One point—
	19	K: We're doing a million sheets of paper.
2	20	L: Yeah, you need—
	21	K: Yes we do. We do, shut up.
	22	L: Therefore, I did that wrong.
	23	K: 2000 times 5.8 is eleven, six, zero, zero.[ie 11,600]
	24	L: And then?
	25	K: Which means—which could make it 11 point 6 metres, right?

Episode 2 (continued)

	Line	Transcript
E	26	L: Or on the - yeah. 'Cause you take off one to get the centimetres, and another
V		one, yeah.[pause] All right. And you've got to point out what the units are,
E		right?
N	27	K: You've got to point out the what is?
T	28	L: You have to show what to multiply too.
	29	[S22 says something to K, K laughs]
3	30	S20: That's not how you know, you look like you know what you're doing and
		you just do it.
	31	K: Exactly, you go into a state of total concentration, it lasts about 2 seconds,
		that's when you get the answer, and then you don't know what you're doing, so
		it doesn't matter. 500 sheets equals point, 5 point 8 centimetres. I don't even
		understand what I wrote.[pause]
E	32	K: Why do you divide a million by 500 to get that answer?
V	33	L: Because you know, you know what the height is.
E	34	K: Tell me what I'm doing here, tell me what I've done.
N	35	L: All right. You know that 500 sheets equals 5.8 centimetres. Oh, I can't see it.
T	36	K: Get some other glasses.
	37	L: Now, we need to know - we need to know the height of a million sheets of
4		paper. Therefore you must divide a million by 500 and times that number by 5.8.
	38	K: [writing] Equals 2 thousand. Sheets of paper. OK, 2000 times 5.8
		centimetres equals 11,600 centimetres, equals 11.6 metres of paper.[bell rings]

The above interaction exemplifies several features of negotiative exchanges between peers:

- The occurrence of "negotiative events" within an "episode"
  Event 1 combines the refinement of intersubjectivity within the group with L's first solution attempt (1 to 15).
  Event 2 revisits the procedure employed in Event 1 (16 to 25).
  Event 3 involves the negotiation of appropriate units of measurement (26 to 31)
  Event 4 reviews the procedure again and links it to the task (32 to 38)
- Student construal of the requirements of the didactic contract (Brousseau, 1986) (Line 8)
- Self-correction (a form of internalised negotiation) (Line 12)
- Completion by one student of a statement commenced by another student (12 and 13)
- Review of purpose and restatement of task (17 and 19)
- Regulatory or organizational interaction (Lines 3, 13 and 21)
- Evidence of intersubjectivity as mediating agency (Line 26)
- Student self-evaluation (metacognition) (Lines 12, 17, 22, 31, 34)

The transcript above may be considered as one episode in a lesson consisting of many episodes. An episode is defined by a consistent purpose such as the solving of a particular problem. Each such episode may involve several negotiative events. A negotiative event is defined by an identifiable intermediate purpose; a purpose whose realization is an intermediate goal within the encompassing episode. Each negotiative event may be composed of several utterances, each with its own immediate purpose. An understanding of a given episode requires an understanding of each constituent level of negotiation: the negotiative event and the utterance. Several other examples could be provided, but it is hoped that Episode 2 serves as an appropriate illustration both of the uncertainty-resolution relationship as it is enacted in classroom settings, and of the analytic procedure being employed.

Resolution not achieved

There were a number of instances where students were not able to resolve their uncertainty within the time allowed by the lesson or in the interview following that lesson. Not surprisingly, more superficial forms of uncertainty (such as procedural uncertainty) were usually cleared up within the lesson. Uncertainty about more complex and difficult material was less readily resolved.

Conclusions

It is a preliminary goal of the Classroom Learning Project to provide an empirical base for the constructs in current use in our theorizing regarding classroom learning. As the body of empirical data grows, we begin to see the nature of the constructs which our theories of learning must employ, and we begin to see the relationships between th e constructs. The data reported in this paper are intended to contribute to our understanding of intersubjectivity as both an outcome of classroom negotiation and as a mediating agency in the resolution of student uncertainty in mathematics classrooms. This paper reports evidence for the relationship between negotiation and intersubjectivity as associative, with the role of intersubjectivity identified as both a prerequisite for the classroom negotiation of meaning and the means by which the negotiative process is sustained and social closure of classroom activity signalled through group acknowledgement of the resolution of uncertainty.

It has been demonstrated that the resolution of uncertainty in mathematics classrooms can have a negotiative character. Student classroom activity within a single lesson can be interpreted as occurring in a hierarchy of levels of purpose: the lesson, the episode, the event, and the utterance, where each level can be associated with a specific purpose. Other data, not reported here, documents the purposeful resolution of student uncertainty over several lessons, and even over several weeks. Resolution over such periods of time goes beyond the confines of classroom activity and is not considered in this paper. The resolution of student uncertainty within the classroom can provide the pretext for purposeful activity for at least two of the proposed levels: the episode, and the event, and this has been demonstrated empirically in this report.

It is our suggestion that that the resolution of uncertainty in relation to academic content represents one form of learning occurring in classroom settings. While the resolution of uncertainty is only one of many perspectives that may inform our analyses of classroom learning, it not only provides insight into the negotiative process in operation in the classroom, but suggests at least two areas for further investigation. First, the nature of the cognitive objects which are the focus of student uncertainty, and, second, the forms of knowing that can be associated with particular uncertainty-resolution combinations. A further area for investigation is the tracking of the resolution of uncertainty concerning more complex and difficult academic content over longer periods than those addressed in this paper. It is intended that all of these matters will be addressed in subsequent papers.

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Is an Equilateral Triangle Isosceles? Student Perspectives

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Identified relationships between figures, based on their properties, is an important characteristic of Level 3 thinking in the van Hiele theory. However, there has been no research directed at how such relationships, leading to class inclusion, evolve. This study, involving indepth interviews with 24 secondary students, addresses this issue by considering their attempts at grouping seven different triangle types. The results reveal important features about perceived relationships between figures, and the existence of a developmental path.

The van Hiele theory (van Hiele, 1986), which consists of five levels of cognitive development provides a framework from which to view students' thinking processes in Geometry. The third level of thinking is characterised by the identification of relationships between properties of known figures which acknowledge the dependence of properties upon each other, and the existence of a network of relations between these figures (Pegg, 1995). The description of Level 3 highlights the notion of class inclusion as an essential characteristic at this stage of cognitive growth. Van Hiele (1986, p.95) described this aspect as requiring a student to "build up a network of relations in which the figures are interconnected on the basis of their properties".

Studies have highlighted that ideas of class inclusion are not easily grasped, and their acquisition remain an important hurdle to be overcome before formal deduction (Mayberry, 1981; Fuys, Geddes and Tischler, 1985; and, Pegg and Davey, 1991). In the light of this, it is surprising that no study has specifically targeted the notion of class inclusion for investigation. This paper addresses this by exploring, via task-orientated interview sessions, students' attempts at identifying relationships between different triangles.

Two research questions guided the study. They are:
1. What was the nature of the links students formed when grouping different triangles?
2. Was there evidence of some developmental pattern in the different responses?

Design

This paper reports the results of one aspect of a larger study developed to explore aspects of class inclusion. Twenty-four students, six from each of Year 8 to Year 11, were selected from two secondary schools in Armidale. The students were of above average ability and there were equal numbers of males and females.

The purpose of this study was to have the students identify and justify relationships among seven different triangle types, namely, acute scalene, obtuse scalene, right scalene, acute isosceles, obtuse isosceles, right isosceles, and equilateral. The format of the interview is contained in Figure 1 and includes student tasks and the questioning focus common to each interview. This format was chosen as it enabled the students to work with familiar recalled information, supplemented information, individual tree designs, and discussion involving prompts and probes from the interviewer. The continual revisiting of the same relationships, as drawn on different maps by the students, provided a vehicle for extracting further information, as the maps were used as a catalyst for discussion concerning the reasons for the existence of links (or relationships). The investigation required the analysis of responses to questions (ii), (iv), and (v) which deal with the relationships between the seven triangle types. The analysis of the responses required the development of a diagrammatical summary which combines the information gathered in student maps and interview transcript form.

Triangle Relationships

(i) Int: I would like you to write a list of all the triangle names you can think of. Begin with acute angled scalene. Draw each triangle.
(ii) Int: Design a tree diagram which links the different triangles. Draw a sketch to link each type.
(discussion follows concerning the reasons for links and/or lack of links)
(the following three points are addressed if required)
(iii) Int: There are some triangles that we can add to this list. (provide triangles not recalled)
Draw a sketch of each new triangle.
(iv) Int: Design a second tree diagram incorporating all the triangles on the list.
(discussion follows concerning the reasons for links and/or lack of links)
(v) Int: Return now to your first map. I would like you to add the new triangles to your original tree.
(discussion follows concerning the reasons for links and/or lack of links)

Figure 1: Summary Interview Structure

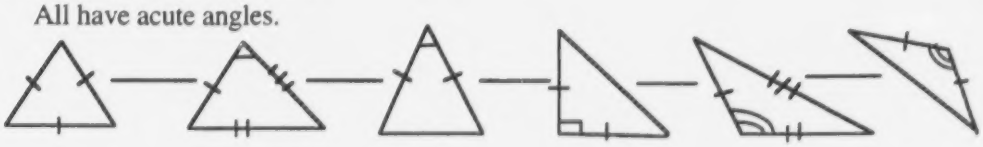
Results

Overall, the students found the ideas familiar, but the questions were seen as non-routine. There were several types of responses identified. The codings take into consideration the types of relationships described and the justification of these relationships. Below is a detailed analysis of the main categories of responses including a description of one typical student response for each type incorporating a diagrammatical representation of their response and relevant samples taken from the interview transcript.

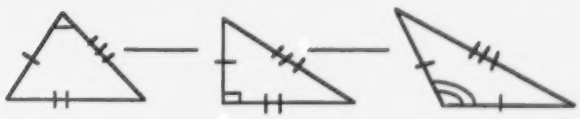
Type A

This type of response indicates the use of a single similar feature or property to relate the triangles together, such as containing acute angles, unequal sides, or equal sides. Only one feature or property is used in each grouping, and the groupings change according to the property or feature that is the focus. Hence, a class of shapes has not formed an identity of its own. There was only one response coded as Type A and this is summarised in Figure 2 below.

All have acute angles.



All have unequal sides.



All have at least two sides equal.

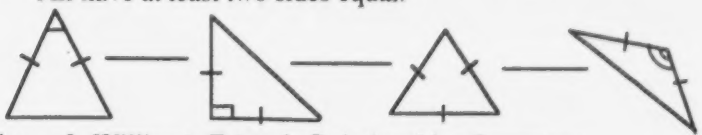


Figure 2: William's Triangle Relationships Summary

The following excerpt conveys the justifications for the relationships based on similar features or properties described by William. It illustrates that the relationships form spontaneously, and groupings change as often as the identifying property or feature changes. There were no links formed across the groups unless the student was prompted.

Int: What is that link there for? (right angled isosceles to obtuse angled scalene)
William: Because it has got acute angles.
Int: Anything else?
William: Um no. [pause] There I have put all the unequal sides together.

Int: No more links there?
William: No...
Int: What have you done up on this top row?
William: They all have three sides and they all have at least one angle that is an acute angle.
Int: What have you done on the next row?
William: Um they all have uneven sides.
Int: And here?
William: They have, all have, at least two sides the same.

Overall, a Type A response forms groups of triangles based on a single similar feature or property. In William's case the features were acute angles, unequal sides, and having at least two sides equal. The groupings change as frequently as the identifying characteristic, without the formation of dominant relationships. Triangles can belong to a number of groups. Hence, classes of triangles are not a workable identity.

Type B

These responses include the formation of three mutually exclusive classes of triangles, these being, scalene, isosceles, and equilateral. Each of these classes represents a unit which has a specific name to encapsulate the similarities of the group. The similarities include one, or a combination of the following; side properties, angle properties, and axes of symmetry. None of the eight responses within this type makes a link between the class of isosceles and the equilateral triangle.

Megan's response (see Figure 3) represents the best of a Type B response as it includes the three classes (standard for this Type) and relationships that exist due to angle-type links. Megan's justifications for classes of shapes are based on the following similar properties: (i) the equilateral triangle has three sides and three angles equal; (ii) the isosceles class of triangles has two sides equal, two angles equal, and one line of symmetry; and (iii) the scalene triangles have no sides equal and no angles equal.

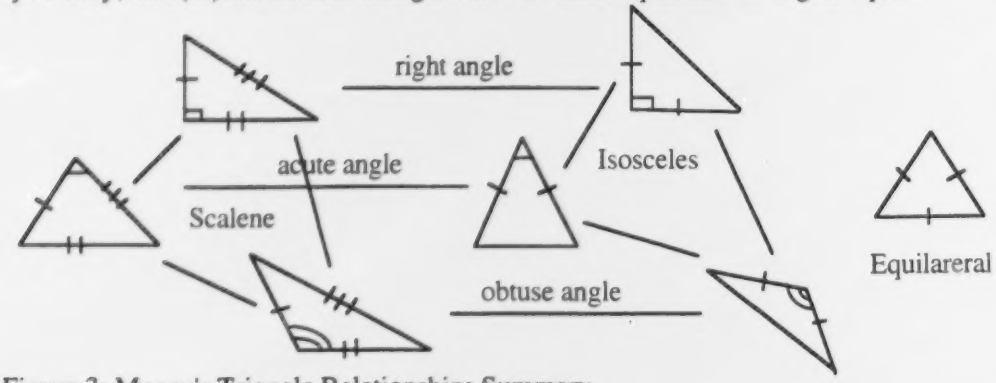


Figure 3: Megan's Triangle Relationships Summary

A link is not made from the isosceles class to the equilateral class based on acute angles as the equilateral is described as having specifically three angles equal and therefore it is not possible to link the classes for any reason. The links across classes are only made according to angle types with the exception of the equilateral. This is illustrated by the following excerpt.

Megan: ... The equilateral triangle is on its own ...
Int: Do you think that the isosceles and the equilateral belong together?
Megan: Oh no not really.
Int: Why don't they?
Megan: Because that one (one side of isosceles triangle) there can be different. The only thing that they have in common is that they have some equal sides and equal angles. An isosceles only has two equal sides and two equal angles whereas an equilateral has three equal sides and three equal angles. With these scalene triangles each side can be all different lengths and I think that sometimes they can be right angled and the same as the isosceles (can also be right angled).

Megan: Um this one is a right angled scalene um I don't want that link between the equilateral and the isosceles. Those isosceles are linked because they both have a line of symmetry.

In summary, Type B responses all contain three triangle-type classes, i.e., scalene, isosceles, and equilateral. The name of each class carries meaning and a triangle must contain all the characteristics of the class, and no others, to belong to that class. Three categories differentiate between those responses with completely isolated classes and those that describe angle-type links between classes. Category 1 comprises of triangle-type classes only. Category 2 comprises of three triangle-type classes and the addition of right-angle links. Category 3 comprises of three triangle-type classes and the addition of right-angle, acute-angle, and obtuse-angle links. None of the responses in Type B make a link to the equilateral triangle.

Type C

These responses are consistent with Type B responses described above and hence, a summary diagram is not provided. The small difference is based upon the addition of tentative links made between the equilateral triangle and the isosceles class of triangles. The similarities are noted between these triangle types but the differences identified do not allow the link to be made. Three responses were coded as Type C.

The discussion concerning the appropriateness of constructing a link between the equilateral and the isosceles classes indicates the awareness that the equilateral triangle also has two sides equal and/or two angles equal, however, a link is not made based on the equilateral triangle containing three sides and/or three angles equal. Mark's discussion illustrates the formation of three triangle-type classes and indecisive comments concerning a possible link between the equilateral and the isosceles triangle.

Mark: So I can link up the scalenes because they have no sides that are equal. Um, the equilateral um, it can be linked to there, it is not really linked it is sort of tacked on the end.
Int: Do you think that link there exists? (isosceles to equilateral)
Mark: Well I can't see how it can link I should cross that one out.
Int: So why are these others linked? (isosceles triangles)
Mark: They all have two sides that are equal.

When questioned again about the appropriateness of a link between the equilateral and the isosceles class of triangles, Mark described both the similarity and the differences between the classes, and ended with an inability to make a decision concerning the relationship.

Int: Now you still have the equilateral linked on there. What do you think about that?
Mark: Um, since these are both 45 degrees there and no they are all 60 degrees on that one, um these turn out to be equal, um I don't know maybe two angles are equal on both of them.

Overall, the responses in this group contain the three triangle-type classes and angle-type relationships across classes with the exception of the equilateral triangle. The distinguishing feature of these responses is that the similarities between the equilateral triangle and the isosceles triangles are noted but the differences identified do not allow a link to be made.

Type D

This group of responses made definite links between the equilateral and isosceles triangle classes based on similar properties. Eight responses were coded as Type D. Each of the responses in this group include three triangle-type classes with the addition of a link between the isosceles and equilateral classes. The relationship is based on both classes containing two equal sides and/or two equal angles, although, the equilateral triangle is not yet described as a subset of the isosceles class of triangles.

Beth's response as summarised in Figure 4 is typical of a Type D response including three angle-type links. She described the triangle-type classes in terms of similar side properties. The link between the equilateral and the isosceles class of triangles exists on the basis of both classes containing two sides the same length. Beth also acknowledged that the equilateral triangle is acute, and links the equilateral triangle with the acute isosceles triangle for this reason.

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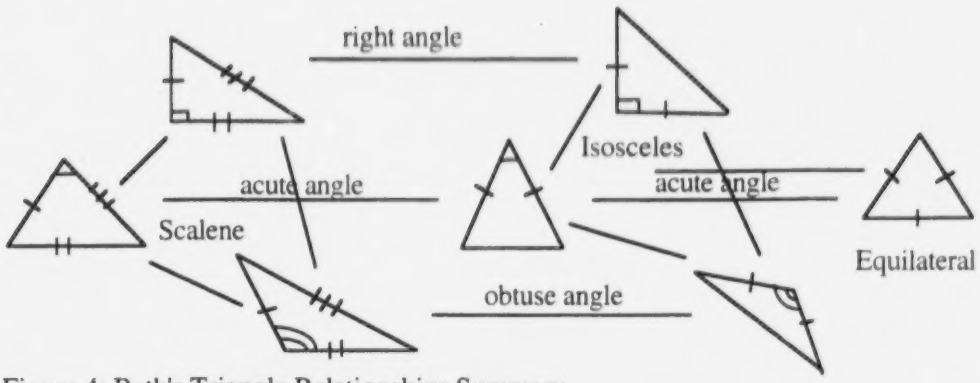

Figure 4: Beth's Triangle Relationships Summary

Int: And why do they (isosceles triangles) link to your equilateral?
Beth: Because they (angles on equilateral triangle) are all acute.
Int: Any other reason why the equilateral links to the isosceles?
Beth: Um because it has got two of the sides the same on both of them. I can link that with that because they are all acute angled.

In summary, Type D responses make a significant link between the equilateral and the isosceles class of triangles based on similar properties. At no stage are equilateral triangles described as a subset of the isosceles class of triangles but a relationship exists between the two classes within Type D. The four categories differ from each other through the gradual addition of angle-type links across the scalene, isosceles and equilateral classes. Category 1 contains no angle-type links. Category 2 acknowledges the existence of angle-type links in discussion but does not allow them to cross the established triangle-type classes. Category 3 contains right-angle links across triangle-type classes only. Finally, Category 4 adds relationships based on right-angle, acute-angle, and obtuse-angle similarities across the three triangle-type classes.

Type E

This group of responses is consistent with a Type D response, however, they include statements concerning the equilateral triangle's relationship to the isosceles class as not based on similar properties only, but due to the equilateral triangle being a form of an isosceles triangle. There are two responses coded within this category. Although both responses include this information there are differences in the responses which have placed them into two categories. Category 1 suggests that the equilateral triangle may be a subset of the isosceles triangle but it is not yet fully accepted. Category 2 accepts this notion but does so without justification.

Nathan's response is characteristic of a Type D Category 4 coding with one differing element. The distinguishing feature that separates this response from Type D responses is Nathan's suggestion that the equilateral triangle may be a subset of the isosceles triangle class, although this has not yet been accepted. This is represented diagrammatically by the dotted boundary in Figure 5.

When Nathan was asked to provide a reason for the link between the acute isosceles and the equilateral triangle he began by explaining that both triangles have acute angles. Nathan paused before continuing and went on to explain that both classes of triangles do have two equal sides. When prompted to place the equilateral within the isosceles class of triangles Nathan restated the similarity between them but was hesitant to accept the equilateral triangle as a subset of the isosceles class of triangles.

Int: Now why do you have those two (equilateral and isosceles) linked?
Nathan: Because they both have acute angles. Um [pause] That is because they have all got acute angles.
Int: Any other reason why the equilateral and the isosceles link?
Nathan: Because on those two the sides are equal like that. Yeah they are both isosceles I think.

Int: So do you think the equilateral is isosceles?
Nathan: Yes I think so. It has got two equal sides like an isosceles.

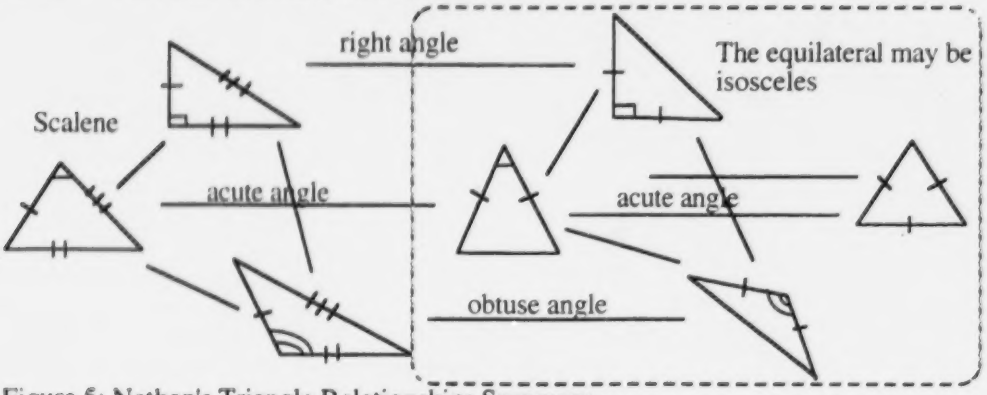

Figure 5: Nathan's Triangle Relationships Summary

Overall, both responses contain relationships between the triangles and justifications similar to the Type D responses. The difference observed is the ability to discuss the possibility that the equilateral is a subset of the isosceles class of triangles. Although Nathan was hesitant to accept the notion, his attempt at justifying this relationship sets the response apart from a Type D categorisation.

Type F

A typical Type F response describes the isosceles class of triangles as containing a subset, namely, the equilateral triangle. The equilateral triangle is identified as a form of the isosceles triangle, and the student is able to justify the equilateral triangle's existence within this class. It can also be argued why an equilateral is an isosceles triangle but that the isosceles triangle is not an equilateral. There was one response coded as Type F. The diagram developed from the information given by Sally is contained in Figure 6.

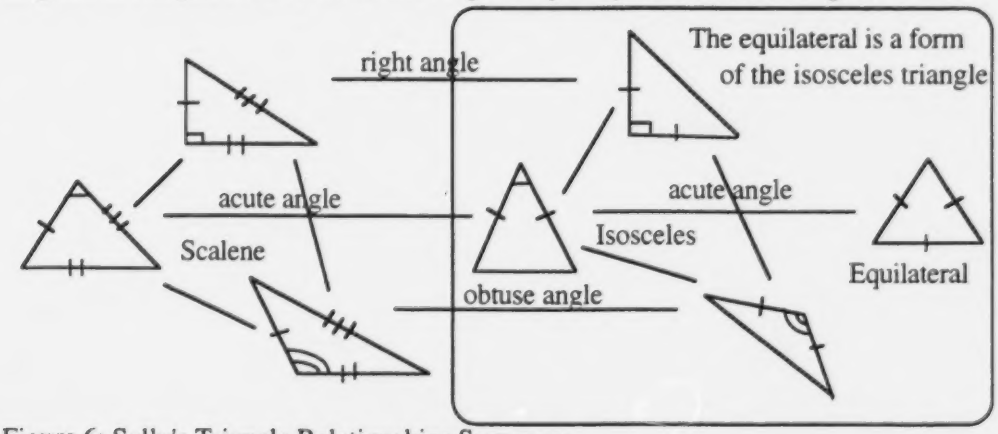

Figure 6: Sally's Triangle Relationships Summary

Sally's explanation of the tree diagram illustrates the clearly defined relationships that exist between the classes of triangles. Sally included the notion of class inclusion as an important feature of her tree design and was able to justify this on the basis of properties.

Sally: Um, all the triangles begin the tree. Then I differentiate between the side length with two or more equal sides and sides are not equal.
Int: So you have ended up with the equilateral and the isosceles on the same branch. Do you see those two triangles linking?
Sally: In that they have equal sides and equal angles. You could say that the equilateral triangle is a form of the isosceles triangle in that it does have two equal angles and two equal sides.

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Sally acknowledged the angle-type relationships in her discussion and explained that it is another method of grouping the triangles. Sally went on to explain that an important element of the design is to make sure that the equilateral class branches from the isosceles branch. This concept has become a dominating feature of the relationships between the different triangles.

Sally: I have done much the same thing but I think that it works better in that it includes the equilateral and the isosceles on the same branch.

In summary, the Type F response includes subsets within classes based on similar properties. The equilateral is clearly identified as a form of isosceles triangle. The student is able to justify this class inclusion and argue that the converse is not true.

Type G

This type of response also makes explicit that the equilateral is a subset of the isosceles class of triangles. The difference lies in the acquisition of further conditions upon these subsets. There is one response coded as Type G and the relationships are summarised in Figure 7.

Adam explained that the equilateral is a "special isosceles" based on properties, namely, two equal sides. Adam developed the relationships further as the link between the acute isosceles and the equilateral becomes significant. A greater link is acknowledged between these two triangles based on the equilateral triangle's existence within the isosceles class and containing acute angles.

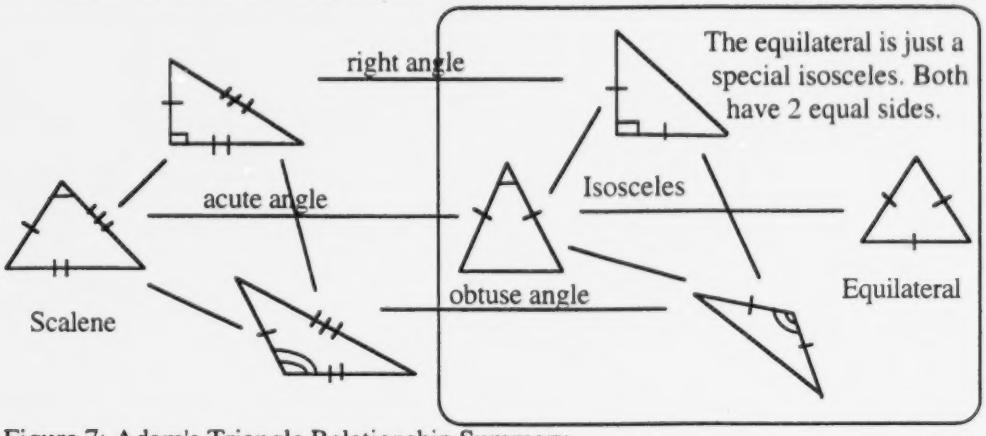

Figure 7: Adam's Triangle Relationship Summary

Adam: The equilateral can be linked to the isosceles.
Int: How come?
Adam: Because these triangles both have two equal sides and the equilateral is just a special isosceles. Some right angled triangles can be isosceles. [pause] ...
Int: What is the main reason these are linked together? (isosceles and equilateral)
Adam: They have the same things that these have.
Int: Do you see a link to all those?
Adam: No just the acute ones. Because 60 is less than 90.

Figure 7 developed as a summary of Adam's response illustrates the difference between the Type G and Type F response. Although Sally provided more detailed justifications for the relationships described, she did not acknowledge the significance of the relationship between the acute isosceles triangle and the equilateral triangle. Adam stated that the equilateral triangle is a subset of the isosceles class of triangles and made a major link to the acute isosceles when considering concurrently angle-type links, triangle-type classes, and the notion of class inclusion.

Summary

The students' responses were coded into seven types. These can be summarised as:

Type A A single similar feature is identified to relate triangles. Triangles appear in more than one group depending upon the identifying feature for each group.

Type B Scalene, isosceles and equilateral classes of triangles are formed and they are characterised by name and related by similar properties. No link is made between the isosceles and equilateral classes. The three categories separate responses on the addition of angle-type links across the isosceles and scalene classes.

Type C Three triangle-type classes are formed with angle-type links across classes. Similar properties are noted between the isosceles and equilateral triangles but differences described do not allow the link to be made.

Type D Relationships are formed across the equilateral and isosceles classes of triangles based on similar properties, but the equilateral triangle is not regarded as a subset of the isosceles class. The four categories within this type separate the responses on the addition of angle-type links across the three classes of triangles.

Type E Similar to a Type D response with indecisive statements concerning the possible inclusion of the equilateral triangle within the isosceles class of triangles, or statements concerning this notion of class inclusion without justification.

Type F The equilateral triangle is a subset of the isosceles class of triangles with justification based on properties.

Type G Similar to Type F but the subsets formed acquire further conditions. The relationship between the acute angled isosceles and the equilateral triangle becomes significant and can be justified.

Conclusion

The results presented in this paper provide a detailed description of student understandings of class inclusion as it relates to different triangle groups. Van Hiele (1986) described class inclusion as Level 3 thinking but implies that this notion is developed once all Level 2 properties become a totality. The student responses provide evidence which reveals that class inclusion is not a 'hit or miss' notion. Instead, there appears to be a developmental path which enables students to gradually acquire this skill. Further research is needed to explore this aspect with respect to quadrilaterals, and then to see if some general framework may help interpret the categories identified.

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What is the Object of the Encapsulation of a Process?

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Several theories have been proposed to describe the transition from process to object in mathematical thinking. Yet, what is the nature of this "object" produced by the "encapsulation" of a process? Here we outline the development of some of the theories and consider the nature of the mental objects (apparently) produced through encapsulation and their role in the wider development of mathematical thinking. Does the same developmental route occur in geometry as in arithmetic and algebra; what about axiomatic mathematics? What is the role played by visualisation?

Theories of encapsulation/reification

In recent years there has been great interest in the encapsulation (or reification) of a process into a mental object as a fundamental method of cognitive development in mathematical thinking. Piaget focused on the idea of a process becoming a mental object, noting how "actions and operations become thematized objects of thought or assimilation". Dienes (1960), following Piaget, used a grammatical metaphor to formulate how a predicate (or action) becomes the subject of a further predicate, which may in turn become the subject of another. Davis formulated the same basic idea a quarter of a century later:

> When a procedure is first being learned, one experiences it almost one step at time; the overall patterns and continuity and flow of the entire activity are not perceived. But as the procedure is practised, the procedure itself becomes an entity – it becomes a *thing*. It, itself, is an input or object of scrutiny. All of the full range of perception, analysis, pattern recognition and other information processing capabilities that can be used on any input data can be brought to bear on this particular procedure. Its similarities to some other procedure can be noted, and also its key points of difference. The procedure, formerly only a thing to be done – a verb – has now become an object of scrutiny and analysis; it is now, in this sense, a noun.
>
> (Davis, 1984, pp. 29–30)

He went on to distinguish a *visually moderated sequence* where each step is written down and prompts the next until the problem is solved and an *integrated sequence*, conceived as a whole which may itself be organised into sub-procedures. He also used the term "procedure" as a specific algorithm for implementing a "process" in an information-processing sense (Davis, 1983, p. 257).

At this time information-processing was focusing on the way in which a procedure which could be used as an input to another procedure could be conceived as a "conceptual entity" (Greeno, 1983).

The notion of the transformation of a process into an object took new impetus in the work of Dubinsky (1991) and Sfard (1991). Sfard hypothesised two approaches to concept development, one *operational* focusing on processes, the other *structural*, focusing on objects.

> A constant three-step pattern can be identified in the successive transitions from operational to structural conceptions: first there must be a process performed on the already familiar objects, then

> the idea of turning this process into a more compact, self-contained whole should emerge, and finally an ability to view this new entity as a permanent object in its own right must be acquired. These three components of concept development will be called interiorization, condensation, and reification, respectively.
>
> Condensation means a rather technical change of approach, which expresses itself in an ability to deal with a given process in terms of input/output without necessarily considering its component steps.
>
> Reification is the next step: in the mind of the learner, it converts the already condensed process into an object-like entity. ... The fact that a process has been interiorized and condensed into a compact, self-sustained entity, does not mean, by itself, that a person has acquired the ability to think about it in a structural way. Without reification, her or his approach will remain purely operational. (Sfard, 1992, pp. 64–65)

Dubinsky (1991) and his colleagues (Cottrill *et al*, in press) formulate the encapsulation of process into object as three stages of a four part theory with the acronym APOS. An *action* becomes repeatable as a recognised *process*, is then encapsulated as a mental *object* to later become part of a mental *schema*.

The notion of action and process are characterised in a manner reminiscent of the notions of visually moderated sequence and integrated sequence of Davis:

> An *action* is any physical or mental transformation of objects to obtain other objects. It occurs as a reaction to stimuli which the individual perceives as external. It may be a single step response, such as a physical reflex, or an act of recalling some fact from memory. It may also be a multi-step response, by then *it has the characteristic that at each step, the next step is triggered by what has come before*, rather than by the individual's conscious control of the transformation. ... When the individual reflects upon an action, he or she may begin to establish conscious control over it. We would then say that the action is interiorized, and it becomes a process. (Cottrill, *et al*, in press, (italics ours))

The action becomes a process when the individual can "describe or reflect upon all of the steps in the transformation without necessarily performing them." A process becomes an object when "the individual becomes aware of the totality of the process, realizes that transformations can act on it, and is able to construct such transformations."

The final part of the APOS structure occurs when "actions, process and objects ... are organised into structures, which we refer to as schemas." Here the suggestion is made that

> . . . an individual can reflect on a schema and act upon it. This results in the schema becoming a new object. Thus we now see that *there are at least two ways of constructing objects — from processes and from schemas.*
>
> (Cottrill, *et al*, in press, (italics ours))

By considering the developments of concepts in simple arithmetic of whole numbers, Gray & Tall (1994) reviewed how a lengthy procedure such as "count-all" (count one set, count another, put the sets together and count all) became compressed into a shorter procedure "count-on" (with various variations such as "count-both", "count-on-from-

larger), whilst also developing other techniques, such as remembering "known facts" and "deriving facts" from a combination of number facts and counting. This shows a number of different procedures being used to carry out essentially the same process in increasingly sophisticated ways. Symbols such as 4+2 occupy a pivotal role, as the *process of addition* (by a variety of procedures) and as the *concept of sum*. Soon the cognitive structure grows to encompass the fact that 4+2, 2+4, 3+3, 2 times 3, are all essentially the same mental object. They therefore defined:

> An *elementary procept* is the amalgam of three components: a *process* which produces a mathematical *object*, and a *symbol* which is used to represent either process or object.
> A *procept* consists of a collection of elementary procepts which have the same object. (Gray & Tall, 1994)

This notion of procept proved to be present throughout a large portion of mathematics. Tall & Thomas (1991) had already noted that, for many children, an expression such as $2 + 3x$ may be conceived as a process which cannot be carried out until the value of x is known – a reinterpretation of the notion of "lack of closure" discussed by earlier authors. Gray and Tall (1994) also noted the peculiar case of the limit concept where the (potentially infinite) process of computing a limit may not have a finite algorithm at all. Thus a procept may exist which has both a process (tending to a limit) and a concept (of limit), yet there is no procedure to compute the desired result.

Table 1 shows a summary of the discussion so far. It does not intend any direct correspondence between the stages of the theories, simply that each passes through a development of growing sophistication from some kind of process/procedure usually performed step-by-step and ending with an object/concept that can be manipulated as an entity in its own right. The intermediate stages specified in each line are not intended to correspond directly, for instance, Greeno's "input to another procedure" is essentially the same as conceiving a programming procedure as an entity, whereas Dubinsky's characterises the individual's ability to take control of a repeatable action, and Sfard's focus is on the ability to think of the process as a input/output without needing to consider the intermediate steps.

Piaget (50s)	action(s) ...	...	thematised object
Dienes (60s)	predicate ...	...	subject
Davis (80s)	visually moderated sequence ... each step prompts the next	integrated sequence ... seen as a whole, can be broken into sub-sequences	a thing, an entity, a noun
Greeno (80s)	procedure ...	input to another procedure ...	conceptual entity
Dubinsky (80s)	action ... each step triggers the next	interiorized process ... with conscious control	encapsulated object
Sfard (80s)	interiorized process ... process performed	condensed process ... process self-contained	reified object
Gray & Tall (90s)	procedure ... specific algorithm	process ... conceived as a whole, irrespective of algorithm	procept symbol evoking process or concept

Table 1 : The transition between process and object

The wider literature of the various authors suggests further similarities and differences between their ideas. For instance, there seems to be broad agreement that a function as a process is determined as a whole by input-output, regardless of the internal procedure of computation. Thus the functions $f(x) = 2x+2$ and $g(x) = 2(x+1)$ are one and the same as processes—even though the arithmetic procedures to compute them have a different sequence of operations. The intermediate stage(s) intimate how (one or more) specific procedures become seen as a single process without needing to carry out the intermediate steps.

What seems more problematic is to explain *precisely* what is meant by the "object" hypothesised to be constructed by reification/encapsulation.

What is the "object" of encapsulation?

Dörfler questioned the nature of the object formed by encapsulation:

> . . . my subjective introspection never permitted me to find or trace something like a mental object for, say, the number 5. What invariably comes to my mind are certain patterns of dots or other units, a pentagon, the symbol 5 or V, relations like 5+5=10, 5*5=25, sentences like five is prime, five is odd, 5/30, etc., etc. But nowhere in my thinking I ever could find something object-like that behaved like the number 5 as a mathematical object does. But nevertheless I deem myself able to talk about the number "five" without having distinctly available for my thinking a mental object which I could designate as the mental object "5". (Dörfler, 1993, pp. 146–147)

Support for this view comes from interviews on a video produced by Gray & Tall (1993) in which individuals were asked the following two questions:

> What does the word "triangle" mean to you?
> What does the word "five" mean to you?

The first was invariably met with a description or definition of a three sided figure as if the individual had a clear mental picture of what was being described, sometimes adding other properties, such as "the angles add up to 180°". The second invariably caused the difficulties described by Dörfler, with some individuals describing "five objects", or a property such as "it's one bigger than four", yet not being able to describe what the term "five" meant of itself. However, all were secure when asked to *operate* with the number "five", for instance, that "five plus five" evoked the response "ten".

We hypothesise that the distinction between the notion "triangle" (which most would consider as an "object") and "five", which Dörfler suggests is not, is the difference between what we would term a "perceived object" and a "conceived object". The first occurs based on perceptual information — seeing a triangle, physically cutting out a triangle, touching it, feeling the corners, counting the edges. The focus is therefore on specific physical manifestations of the notion of a triangle. The second occurs when there is reflection on perceptions and actions, so the focus is no longer on the specific physical manifestations but on the actions/processes performed upon them .

Piaget, as usual, has made pertinent comments in this direction, long before any of the rest of us. He distinguishes between *empirical abstraction* deriving knowledge from the properties inherent in real-world objects and *pseudo-empirical* abstraction deriving knowledge from the processes which the individual performs on the objects.

A procept, such as number, is a conceived object formed by pseudo-empirical abstraction. It focuses on the process of counting, building to the idea that different ways of counting the same set give the same result, leading to the concept of number.

Even though there may be no mental object corresponding precisely to the number "5" as there is with a perceived object, there is a huge cognitive structure built up allowing the individual to use the symbol 5 *as if* it refers to an object. The number "5" has a *concept image*, in the sense of Tall & Vinner (1981), consisting of "all the mental pictures and associated properties and processes" related to the concept in the mind of the individual.

There is in practice an intimate connection between conceived objects and operationally defined concepts:

> With regard to icons, Piaget's distinction between the "figurative" and the "operative" would seem to be of some importance. Number is not a perceptual but a conceptual construct; thus it is operative and not figurative. Yet, perceptual arrangements can be used to "represent" a number figuratively. Three scratches on a prehistoric figurine, for instance, can be interpreted as a record of three events. In that sense they may be said to be "iconic" but their iconicity is indirect.
>
> (von Glasersfeld, 1987, p. 233).

We suggest that the total cognitive structure of the concept image of number, with its power to *manipulate* the symbols and to think of their properties, gives number an object status. What matters more is not what it *is*, but what we can *do* with it.

The "non-existence" of an "object" corresponding to a number is not as strange as at first seems. Consider, for example, the notion of "animal", which includes cats, dogs and gorillas. When we mentally picture an "animal", which one do we "see"? The name "animal" is a signifier which can be used to signify any of a wide number of particular instances but we appear to fail to have a *single* mental object which is "*an* animal." Nevertheless, to paraphrase Dörfler, although we may fail to have a unique mental object for "an animal" we deem ourselves able to talk about it.

The scope of the process-object construction

Once the possibility is conceded that the process-product construction can be conceived as an "object", the flood-gates open. By "acting upon" such an object, the action-process-object construction can be used again and again.

> . . . the whole of mathematics may therefore be thought of in terms of the construction of structures, . . . mathematical entities move from one level to another; an operation on such 'entities' becomes in its turn an object of the theory, and this process is repeated until we reach structures that are alternately structuring or being structured by 'stronger' structures. (Piaget, 1972, p. 70)

The various proponents of process-object construction claim a different scope for the application of their theories. For Dubinsky, following Piaget, a process is any *cognitive* process. The notion of "permanent object" arises through "encapsulating the process of performing transformations in space which do not destroy the physical object" (Dubinsky *et al*, 1988, p. 45). A "perceived object" in our sense is, for Dubinsky, formed by process-object encapsulation. He also sees the theory of encapsulation applying equally well to the logical construction of formal concepts in advanced mathematics. He further acknowledged that there may be *many* cognitive processes involved in the construction of a mental object, used in an increasingly coherent manner, leading naturally to his later assertion that objects can also be formed by encapsulating schemas. Dubinsky therefore offers a single, unified theory of encapsulating cognitive processes as cognitive objects.

However, Piaget's distinction between *empirical abstraction* and *pseudo-empirical* abstraction suggests a further subdivision of the growth of knowledge into constructions focusing on *objects* being acted upon, and those focusing on *actions* (and their subsequent symbolisation and conception as encapsulated objects).

Sfard's theory of the complementarity of operational and structural sees "two sides of the same coin", one focusing on operations and the other on the structure of objects and relationship between them. This is applied throughout mathematics. For instance, she classifies solutions to the question: Find the number of boys when "the boys outnumber the girls by four", as operational in a form such as "add four to the number of girls" and structural as "x=y+4" (Sfard, 1995, p. 21). In an apparently similar style, Kieran (1992) considers the (arithmetical) evaluation of an expression such as $2x+3$ for a numerical value of x as "operational" but their manipulation as "structural". But there is a subtle distinction in meaning, for what is "structural" for Kieran becomes "operational" at a higher level in the sense of Piaget and Sfard, once the expressions are thought of as manipulable objects.

Further information on the distinction between the two types of conception include the idea that a structural conception is "supported by visual imagery" whilst an operational conception is "supported by verbal representations" (Sfard, 1991, p. 33). Thus the graph of a function is considered structural, whilst the algebraic expression for a function may be structural or operational. A set-theoretic definition of a function as a "set of ordered pairs" (with its graphical interpretation) is also considered as "structural" (ibid., pp. 5, 6). When proof by induction is considered, Sfard (1989, p. 151) classifies them as follows:

- *Operational*: for a property $P(n)$, prove $P(1)$ and that $P(k)\Rightarrow P(k+1)$ for all k to establish the truth of $P(n)$ for all natural numbers n,
- *Structural*: given a set $S\subseteq N$, prove $1\in S$ and $k\in S\Rightarrow k+1\in S$, to establish $S=N$.

To highlight the structural way in which the visual can be used to suggest theorems, Sfard (1994) provides empirical evidence in which mathematicians use visual and spatial metaphors to provide them with "intimate familiarity" with structure that gives "direct insight into the properties of mathematical objects."

Skemp (1979), separates out the act of *building* a concept from its more formal *testing*. For instance, Archimedes used his "Method" to imagine lines making up a surface to "build" formulae for area, and then "tested" them using "proof by exhaustion" for formal publication. In the same way many mathematicians use intuitive structural conceptions as private constructs before releasing formally presented theories for public scrutiny. This once again suggests that the "structural" properties of objects could be usefully further subdivided, on the one hand through the teasing out of properties of objects, and on the other, specifying some of these properties as *criteria* to use for the basis of formal deduction of properties.

Gray & Tall's notion of *procept* only occupies part of this scenario because their notion of "process" is a *mathematical* process represented by the mathematical symbolism. Neither the child's notion of "permanent object", nor the concept of "triangle" are procepts because neither has a symbol evoking either process or concept.

A theory of the cognitive development of mathematical concepts from childhood to professional mathematician is outlined in Tall (1995). It begins by noting (as did Sfard) that, when an individual performs actions on objects, the focus of attention may be either on the *objects*, or on the *actions* (and their symbolisation), or on a combination of *both*. A focus on the *processes* in arithmetic leads to their encapsulation as arithmetic procepts (Gray & Tall, 1994). Elsewhere procepts have different characteristics which may be described using different adjectives, for instance, *computational* procepts in arithmetic having built-in processes of computation, *template* procepts in algebra, with internal numerical evaluation for specific values of variables and external manipulation of algebraic expressions, or the *limit* procepts of the calculus.

A focus on *objects*, as in geometry, leads to a different sequence of events–teasing out the properties of the objects, making verbal descriptions, thinking about relationships, verbalising inferences, formulating verbal proofs, leading to a broad development after the fashion described by Van Hiele (1986).

Focusing on both operational processes and the properties of objects—either in turn or at the same time—gives a *versatile* approach (Tall & Thomas, 1991). This proves particularly valuable when computer software is available to carry out the processes internally, allowing the individual to focus either on the study of the processes, which they may carry out, or program, for themselves, or on the concepts produced by the computer (Tall & Thomas, 1989). Versatile approaches have proved successful using visual properties, such as viewing the "local straightness" of graphs to complement the process of symbolic differentiation to give a derivative function (Tall, 1985), and also using process/concept, such as studying the evaluation of expressions (carried out by the student) separately from the properties of (equivalent) expressions, evaluated by the computer (Thomas, 1988).

One final case of "process-object" construction, in the sense of Dubinsky, is the notion of a *defined* object. In the sense of Sfard, this has structural overtones because the definition specifies certain structural criteria. However, constructing objects from definitions is cognitively subtle. There may be well-known examples of the defined object which all seem to have a common property, and yet this property may not follow from the specified criteria. For instance, the notion of triangle carries with it the properties inherent in 2 dimensional space. Its angles add up to 180°. Yet if a triangle is defined formally in some other type of geometry, for instance the geometry on the surface of a sphere, then the sum of its angles may be different. Just try cutting the skin of an orange and see. Two cuts through the "north pole" at right angles to one another meeting a cut round the equator will give a spherical triangle with *three* right angles, adding up to 270°.

We ask, just as we have before, "what is the object obtained by the processes of definition and deduction?" For instance, if we write down the axioms for a mathematical group, in what sense is there an *object* which we call "a group"? We arrive at the same conundrum as before. We speak of "an animal" without necessarily having in our mind's eye a visual mental image for a "general animal", we know how to manipulate "5" without having a specific mental image for it as an object. Likewise we can build up the properties of a "defined object" by deduction from the given criteria (axioms). For instance, we might deduce that "there is only one, unique, identity element in a group". Theorems deduced in this way are then properties shared by all structures which satisfy the criteria. We just use the same linguistic conventions in speaking of "a group" as we do in speaking of "an animal". For instance, we can say "a group has a unique identity element."

We have discussed three kinds of object construction: *perceived objects*, which are general concepts arising from teasing out the properties of specific objects, *procepts*, which involve symbolising processes so that the symbols can be manipulated as objects, and *defined objects*, arising from specifying criteria from which properties may be deduced by formal proof. The "object" of the encapsulation of a process is a way of thinking which uses a rich concept image to allow it to be a manipulable entity, in part by using mental processes and relationships to *do* mathematics and in part to use a name or symbol to mentally manipulate to *think* about mathematics.

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The Effect of Instruction on Students' Generation of Diagrams

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Although the use of diagrams is advocated in mathematics, support for this instructional practice appears to be intuitive rather than evidentiary. A case study was used to evaluate the effectiveness of instruction in diagram generation with Year 5 students. The results suggest that although instruction can have a positive effect on students' diagram generation, the success of the program is dependent on the teachers' understanding of the role of diagram generation in problem solving and how diagram generation can be facilitated.

Using Diagrams in Problem Solving

Although the strategy *draw a diagram* is advocated in mathematical problem solving (e.g., Australian Education Council, 1991) for its assumed cognitive advantages (e.g., Nunokawa, 1994; van Essen & Hamaker, 1990), empirical justification for the use of the strategy *draw a diagram* is equivocal (Shigematsu & Sowder, 1994; Simon, 1986). Shigematsu and Sowder (1994) have asserted the need for more research and theory on this topic to validate and inform instructional practice. Visual literacy is a long neglected field (Balchin & Coleman, 1965; Box & Cochenour, 1994). Thus, the effective use of a diagram is a significant and urgent problem for mathematics education research.

The use of the diagram in problem solving assumes two roles: (a) as a representation of the *problem structure*, and (b) as the basis for the development of a *solution structure* (Carroll, Thomas, Miller, & Friedman, 1980). The problem structure is comprised of the relationships among the problem information and the presentation of that information, whereas the solution structure is comprised of the solution process and the solution (Carroll et al, 1980). As the solution structure is dependent on the solver's perception of the problem structure, the generation of an appropriate diagram is integral to a successful solution (Yancey, Thompson, & Yancey, 1989): "Generating a drawing does not guarantee that one finds the correct solution, but merely increases the chance that a problem will be conceptualized correctly" (van Essen & Hamaker, 1990, p. 309). Henceforth, the focus of this paper is on the generation of diagrams in problem solving.

As the correspondence between the diagram and the problem structure determines the value of a diagram for problem solving (Veloo & Lopez-Real, 1994), the diagrams generated by students should represent the structural features of the problem. Whereas novices base their representations on the literal features of the problem, experts incorporate relevant background knowledge into their representations (Chi, Feltovich, & Glaser, 1981). Notwithstanding the appropriateness of the diagrams which are generated, an additional concern is that students are reluctant to *draw a diagram* even when specifically directed (Veloo & Lopez-Real, 1994) and even consider the use of a diagram to be inappropriate (Shigematsu & Sowder, 1994).

Instruction in Diagram Generation

However, when instruction has been implemented the results have been disappointing (e.g., Simon, 1986), not unexpectedly, because of defects in: (a) the content of the instructional program, (b) the selection of tasks for evaluation, and (c) the evaluation criteria.

The content of an instructional program should focus on using diagram drawing to enhance conceptualisation because the advantage of generating a diagram relates to the conceptualisation of the problem (van Essen & Hamaker, 1990). Simon's (1986) instructional program comprised six diagram drawing subskills. Of her subskills, only (1) representing all relevant information, and (2) creating an integrated diagram are critical to the conceptualisation of the problem. In contrast, the subskills of (3) labelling completely,

(4) checking the accuracy of the diagram, and (5) drawing multiple representations are not critical. Indeed the final subskill of (6) verbalising what is represented and what needs to be represented may inhibit problem solving (Schooler, Ohlsson, & Brooks, 1993). Furthermore, an instructional program should also include the different types of diagrams that have unique problem structures: networks (ie., path or line diagrams), hierarchies, matrices, and a range of diagrams that exhibit part-whole characteristics (Novick & Francis, 1993).

The selection of tasks for evaluation was problematic in Simon's (1986) study as they do not appear to have been sufficiently challenging as indicated by high pretest scores and the acknowledged ceiling effect. Novel problems rather than routine problems should be used for evaluation because in novel problems the students need to derive the problem structure rather than simply implement the solution structure.

The evaluation criteria used by Simon (1986) were: (1) the type of diagram, (2) the accuracy of the diagram, (3) the completeness of the diagram, and (4) the labelling of the diagram. The type of diagram generated and the accuracy of the diagram are appropriate criteria because they are integral to the recognition and representation of the problem structure (Novick, 1996). However labelling is inappropriate because although the diagram may have been used as a cognitive tool, the student may have had no communicative intent. Completeness is also inappropriate because as the student conceptualises the problem his or her internalized representations may be more developed than his or her external representation (Hegarty & Narayanan, 1994). Frequency of diagram generation is an additional appropriate criteria because students are reluctant to use diagrams (e.g., Veloo & Lopez-Real, 1994). Furthermore, the extent of a student's autonomy in diagram generation is also an important criteria because there may be a change in a student's *Zone of Proximal Development* which can be discerned through scaffolding (Vygotsky, 1978). Scaffolding can be used to ascertain how much teacher support is required to assist the student to generate a diagram. Although the type of diagram, frequency of diagram generation, and autonomy of diagram generation can be readily established, ascertaining the accuracy of a diagram is problematic. As a problem structure can be represented diagrammatically in various ways, there is no single "correct" diagram. However as prototypes are useful for ascertaining expertise when the degree of similarity between the exemplars (i.e., diagrams) may be low (Sternberg & Horvarth, 1995), a *theoretical prototype* can be developed by applying the properties of the type of diagram (Novick, 1996) to the context of the problem. The accuracy of the diagram can then be established by comparing the congruence between the student's diagram and the prototype.

Method

A case study design (Yin, 1994) was used to investigate the effect of instruction on the *generation of diagrams,* which is one aspect of ongoing evaluative research on diagram use in problem solving (Diezmann, 1995; 1996). It was hypothesised that there would be an improvement in students' diagram generation as a consequence of an instructional program consisting of twelve half-hour lessons. In order to optimise the conditions for success, each of the issues discussed earlier in relation to Simon's (1986) study was accommodated: (a) *the content of the instructional program,* (b) *the selection of tasks for evaluation,* and (c) *the evaluation criteria.*

The informants in the study were 12 Year 5 students with a mean age of 10 years 3 months from a moderately sized parochial school in a lower socio-economic suburb in Brisbane. Tasks comprising isomorphic sets of five novel problems were presented to each informant during 30 minute interviews conducted before and after instruction. As the informants were not specifically instructed to use a diagram, those informants who did not spontaneously use a diagram were given further opportunities to generate a diagram through scaffolding in the form of *general prompts* (e.g., Is there any other way you could do the problem?) or *specific prompts* (e.g., Could you draw a diagram or picture?). Specific prompts were used to investigate whether the informants: (a) didn't think to generate a diagram, or (b) couldn't generate a diagram. The interviewer was known to the

subjects through prior classroom involvement and the interviews were video-taped and subsequently transcribed. *The Koala and The Frog* tasks (see Figure 1) are reported because there were distinctive visual differences between the diagrams generated for the pre- and post-instruction interviews respectively.

The Koala (Pre-instruction)	**The Frog (Post-instruction)**
A sleepy koala wants to climb to the top of a gum tree that is 10 metres high. Each day the koala climbs up 5 metres, but each night, while asleep, slides back 4 metres. At this rate how many days will it take the koala to reach the top?	A frog was trying to jump out of a well. Each time the frog jumped, it went up four rows of bricks, but because the bricks were slippery it slipped back one row. How many jumps will the frog need to make if the well is 22 rows high?

Figure 1. The Koala and Frog Tasks

Results and Discussion

The implementation of this study led to two assertions.

Assertion 1: After instruction, there will be a positive quantitative change in the diagrams that are generated independently. After instruction, the number of informants who spontaneously generated a diagram increased from four to eleven. However providing informants with verbal prompts also facilitated the generation of a diagram. Although no informants responded to a general prompt on the pre-instruction task by drawing a diagram, six of the eight informants who did not draw a diagram responded to the specific prompt. Three of these informants spontaneously generated a diagram, and the other three informants appeared uncertain but generated a diagram upon request. Their uncertainty appeared to be related to either a difficulty representing the problem structure or the perceived utility of the diagram. For example, Jon stated "I thought of drawing a 10 metre high, a 10 metre tree. I don't have enough room to do a 10 metre tree" and Candice shrugged her shoulders when asked: "Would it help (to draw a diagram)?" Two informants responded negatively to the specific prompt. On the post-instruction task, Adrian was the only informant who did not generate a diagram spontaneously. However he responded to a general prompt with the statement "By a line (diagram or network)", which suggests that although he did not generate a diagram, he at least knew the correct type of diagram to draw. This response was a stark contrast to his head shaking on the pre-instruction task when given a similar prompt.

As shown on Table 1, after instruction there was a positive quantitative change in: (a) the number of diagrams drawn, and (b) the degree of scaffolding required to produce a diagram or identify the type of diagram to be drawn. No negative quantitative differences were evident. However despite the quantitative improvement in diagram generation not all of these diagrams were accurate representations of the problem structure.

Table 1
Spontaneous Generation of a Diagram (n=12)

degree of scaffolding required to generate a diagram	*The Koala* task (pre-instruction)	*The Frog* task (post-instruction)
no scaffolding	4	11
a general prompt	0	1
a specific prompt	6	0

Assertion 2: After instruction, there will be a positive qualitative change in the diagrams that are generated independently. The appropriate type of diagram for these problems was a network (Novick, 1996). Although all diagrams generated by students could be classified as networks, some diagrams were structurally inaccurate when compared to the theoretical prototype. The prototype consisted of *nodes* depicting either the metre marks or the bricks, depending on the problem, and a depiction of *movement* between the nodes.

On the pre-instruction task eight of the twelve informants represented the *starting and finishing nodes,* whereas all of the informants represented these nodes on the post-instruction task. However on the pre-instruction task *the intervening nodes* (nodes between the starting and finishing nodes) were only represented by two informants. Deficits in the representation of the intervening nodes resulted in the informants having difficulty: (a) tracking movements, and (b) implementing the solution structure. However on the post-instruction task, every informant represented all nodes. The *organisation of the nodes* for the solution structure was also of importance. On the pre-instruction task the solution structure was compromised when an informant: (a) did not identify the intervening nodes, (b) used an approximate rather than a precise position, or (c) drew the nodes as the solution strategy was being implemented. Although none of these responses was evident on the post-instruction task, two informants had difficulty with the solution structure because: (a) the spaces between the nodes were too small for accurate use, and (b) an error was made in calculating the quantity of the intervening nodes. Thus although difficulties were encountered with the organisation of the nodes on both the pre- and post-instruction tasks, there was an increase in the number of informants who were able to organise the nodes effectively after instruction as shown on Table 2.

Instruction appears to have had a positive effect on the representation of *nodes* because more students were able to: (a) represent the starting and finishing nodes, (b) represent the intervening nodes, and (c) organise the nodes for the implementation of the solution structure (see Table 2). Helen's diagrams in Figure 2 exemplify how informants typically depicted the starting and finishing nodes in *The Koala* and *The Frog* tasks by showing the base and top of the tree or well respectively. Although she has used some intervening nodes (the branches) in *The Koala* task, her organisation of these nodes did not facilitate the implementation of the solution structure. In contrast, she represented all intervening nodes on *The Frog* task and the organisation of these nodes facilitated the implementation of the solution structure. The second aspect of the theoretical prototype was *movement* which is now discussed.

Table 2

A Comparison of the Use of Nodes on the Pre- and Post-Instruction Tasks (n=12)

Node Description	Pre-instruction Task	Post-instruction Task
starting and finishing nodes accurately represented	8	12
intervening nodes accurately represented	2	12
organisation of nodes for the solution structure	2	10

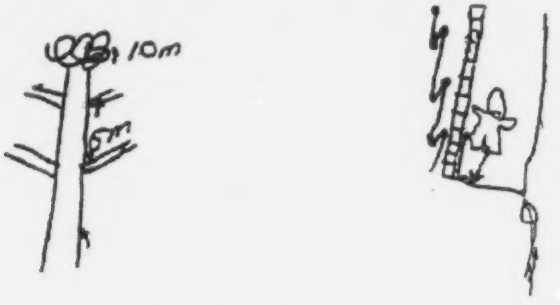

Figure 2. Helen's diagrams of *The Koala* and *The Frog* Tasks Respectively

On the pre-instruction task, *movement* was only represented by four informants, however all informants represented movement on the post-instruction task as shown on Table 3. Helen's diagrams (see Figure 2) exemplify the omission of drawn movement on the pre-instruction task, and the inclusion of drawn movement on the post-instruction task. However some of the informants who drew movement had difficulty with either the

direction of the movement or the amount of movement (see Table 3). After instruction no informants had difficulty with the direction of movement, however six students were still unable to represent the amount of movement correctly. Four of the six students made the same error as Ben whose diagram is shown in Figure 3. On *The Koala* task Ben identified the base of the well as one brick high. Elise, whose error was the same as Ben's, made a similar error on *The Frog* task. The apparent increase in the number of informants experiencing difficulty with measurement on the post-instruction task is not interpreted as a negative effect of instruction, because, with the exception of Elise, none of the informants who had difficulty representing the amount of movement on the post-instruction task attempted to represent movement on the pre-instruction task. The instructional program is interpreted to have had a positive effect on the generation of diagrams because there were increases in the number of informants, as shown on Table 3, who: (a) represented movement, (b) correctly represented directionality, and (c) correctly represented the amount of movement.

Table 3
A Comparison of Informants' Representation of Movement in the Pre- and Post-Instruction Interviews (n=12)

	The Koala task	The Frog task
the use of a representation to depict movement	4	12
direction of the movement represented correctly	3	12
amount of movement represented correctly	1	6

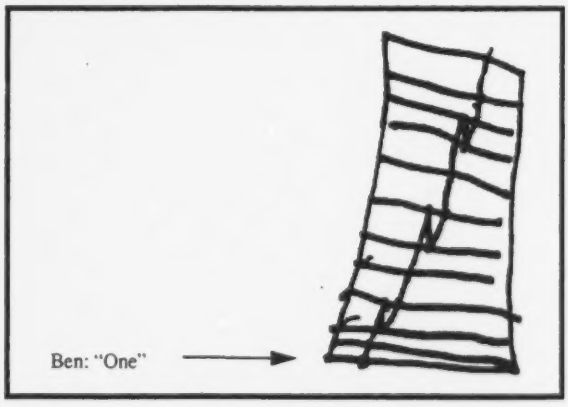

Figure 3. Ben's Error in the Representation of Movement on *The Frog* Task

There was also an improvement in the number of informants who represented both the movement correctly and the nodes correctly. Whereas no informants had a totally accurate representation prior to instruction, five informants' representations were totally accurate after instruction. Hence Assertion 2 is supported because there was an increase in the number of informants who represented: (a) the correct type of diagram, (b) either the nodes or movement accurately on the diagram, and (c) the nodes and movement accurately on the diagram.

The employment of a theoretical prototype was particularly useful for ascertaining the qualitative changes in an informants' diagrams. For example, Kate's diagrams on the pre- and post-instruction tasks, as shown on Figure 4, both represent all aspects of movement and the starting node, however on the pre-instruction task there are deficits in her representation of the intervening nodes, the ending node, and the organisation of the nodes, all of which were overcome on the post-instruction task. The contrast between Kate's elegant post-instruction diagram, as shown on Figure 4 and Helen's pre-instruction diagram (see Figure 2) exemplifies the expert-novice difference, in that the

expert's diagram (Kate) is a structural representation, whereas the novice's diagram (Helen) is a literal representation.

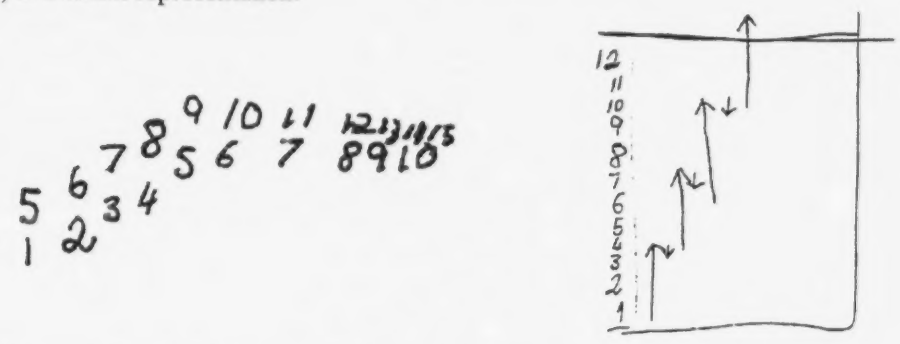

Figure 4. Kate's Diagrams for *The Koala* and *The Frog* Tasks Respectively.

Implications

The assumption of this study is that diagram generation is important because diagrams facilitate the conceptualisation of a problem (van Essen & Hamaker, 1990). The results of this study have five implications. Firstly, this study provides evidence that an instructional program can improve students' generation of diagrams. Secondly, specific quantitative and qualitative changes in diagrams can be identified using a theoretical framework to establish appropriate evaluation criteria. Thirdly, the success of the instructional program can be attributed to the theoretical framework that supports the role of the generation of diagrams in problem solving. Fourthly, quantitative changes should not be under-rated because they provide evidence that a student is becoming familiar with the domain of diagrams as a unique representational system. Representational systems are problem spaces in their own right *prior* to becoming useful "cognitive" tools (Landsman & Karmiloff-Smith, 1992). Finally, an improvement in students' generation of diagrams will require an informed and proactive stance by teachers and teacher educators, who may need to consciously re-educate themselves to include visual literacy in mathematics instruction.

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Year 8, 9 and 10 students' understanding and access of percent knowledge

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This paper reports on Years 8, 9 and 10 students' knowledge of percent problem types, type of solution strategy, and use of diagrams. Non- and semi-proficient students displayed the expected inflexible formula approach to solution but proficient students used a flexible mixture of estimation, number sense and trial and error instead of expected schema-based classification methods.

The application of percent in the real world cannot be denied. Percent discounts, profits, losses, savings, and increases, are an integral part of our society, as attested to by billboards, newspapers, advertisements, and retail marketing. There can be no question of the social necessity of having an understanding of percent and therefore its importance in the mathematics curriculum. However, percent is often misused or misunderstood when applied in the real world, as seen through errors made in media advertising (Watson, 1994).

According to Parker and Leinhardt (1995), one reason why percent is a difficult topic to learn and teach is that the notion of percent has changed and evolved from its roots in the market place into an elusive concise concept with multiple meanings. According to Parker and Leinhardt, percent can be all the following: (a) a number in that a percent can be written in an equivalent fraction or decimal form; (b) a comparison in the part-whole fraction sense (e.g., if a candidate receives 35% of the votes, this percent is the subset of people who voted for this candidate compared to the total number of votes cast); (c) a ratio comparison where the comparison is between two distinct sets (e.g., there are 400% more boys than girls); (d) a statistic for manageable interpretation (e.g., a state's employment rate of 8.5% is compared to the national average of 10%); and (e) a function when amounts are calculated according to a stated percent (e.g., interest rates, discounts, etc). The link between these many dimensions of percent, according to Parker and Leinhardt, is that of proportionality:

> The common thread woven through all these descriptions is that percent is an alternative language used to describe proportional relationships - a language that is unique, concise and provides a privileged notation system. (p. 444)

Knowing percent: Understanding mathematics requires, amongst other things, proficiency in arithmetical calculation but arithmetical proficiency alone is no guarantee of having mathematical understanding (Leinhardt, 1988). The recent literature reveals that understanding mathematics consists of having knowledge of a concept and process in a variety of forms and that the different forms of knowledge need to be connected to form a schema that is accessible in a variety of application tasks (Putnam, Lampert, & Peterson, 1990). Of pertinence to this paper are the knowledge forms of Resnick (1982), Skemp (1978) and Leinhardt (1986). Resnick categorised knowledge as being either syntactic (the correct performance of mathematical procedures) or semantic (the understanding of the meaning of those procedures), Skemp categorised knowledge as being either instrumental (knowledge of computational procedures) or relational (knowledge of why those procedures work) whilst Leinhardt categorised knowledge as being intuitive ('everyday' real world application knowledge which is normally acquired before formal instruction), concrete (associated with representation by appropriate concrete materials during instruction), computational (knowledge of the algorithmic procedures) or

principled/conceptual (knowledge of the principles that constrain or justify the algorithmic procedures and which takes place after instruction is complete). Thus, to the cognitivist, knowing mathematics various internalised representations of related mathematical ideas and connections between the representations (Putnam, Lampert & Peterson, 1990).

Understanding percent, then requires appropriate mental models to accommodate the various notions of percent as well as the procedures solving percent problems. As Parker and Leinhardt (1995, p. 47) stated: ". . . knowing percent both in school and out means understanding its multiple and often embedded meanings and its relational character".

However, to solve problems, students also need to access the knowledge they have constructed. Prawat (1989) argued that access to knowledge is determined by the learner's organisation and awareness of three factors: knowledge base (concepts, principles, rules, facts and procedures); strategic and metastrategic thinking (general problem solving heuristics and metacognitive processes, such as planning, monitoring, checking, revising); and disposition (habits of mind). In particular, performance on mathematical tasks is influenced by metacognition. Garofalo and Lester (1985) argued that mathematical knowledge is influenced by three metacognitive categories of person knowledge, "one's assessment of one's own capabilities and limitations with respect to mathematics in general, and also with particular topics or tasks" (pp. 167) including such affective variables as motivation, anxiety and perseverance; task knowledge, one's beliefs about the nature of the mathematical tasks; and strategy knowledge, awareness of strategies for guiding problem solving. Thus adequate percent knowledge consists of understanding the meaning of percent in its many dimensions together with knowledge of the principles which legitimise percent calculations, as well as metacognitive knowledge to enhance access to such percent knowledge.

Instructional approaches for teaching percent: The literature provides a variety of instructional methods for developing the concept of percent and for solving percent application problems. A common approach used to develop the concept of percent is to link percents to fractions and decimals (e.g., Brueckner & Grossnickle, 1987). However, the concept percent can also be promoted through linking percent to ratio (e.g., Brown & Kinney, 1973), through studying percent expressions as statements of proportion (e.g., Schmalz, 1977), through exploring the special language of percent, and through the exploration of patterns of simple percent calculations (e.g., Glatzer, 1984). Instructional approaches for solving percent application problems are also varied. In terms of mathematics structure, percent application problems are of three types, described by Ashlock, Johnson, Wilson and Jones (1983, p. 297) in the following manner:

Type I Finding a part or percent of a number (e.g., 25% of 20 is p);

Type II Finding a part or percent one number is of another (e.g., p% of 15 is 5); and

Type III Finding a number when a certain part or percent of that number is known (e.g., 20% of p is 6).

In their analysis of the literature on percent, Parker and Leinhardt (1995) stated that, by 1960, there were 5 distinct computational procedures for solving percent equations taught in schools. The five procedures can be summarised as follows:

(1) Traditional/cases - students classify the problem and apply a different procedure for each problem type (multiply the number by the percent as a decimal for Type I, divide

the numbers and translate the decimal answer to a percent for Type II, and divide the number by the percent as a decimal for Type III);

(2) Percent formula - "knowns" are substituted in the formula, P = BR (P is percent as a number, B is base number and R is percent as a rate) and the "unknown" is found by algebraic manipulation;

(3) Equation - "knowns" are categorised as factors or product and substituted in the formula, factor x factor = product (algebraic manipulation is used to find the unknown);

(4) Proportion - percent is considered as a common fraction with a denominator of 100 and is equated to a fraction made up from the two other possible numbers (i.e. a/100 = c/d), the problem is classified in terms of the unknown and this is found by algebraic manipulation or cross-multiply method; and

(5) Unitary - 1% of the "known" is calculated and then simple arithmetic computations are performed to calculate the required percent (e.g., 11% of 200 is thought of as the product of 1% of 200 and 11).

The literature offers various teaching approaches to give meaning to the variety of computational procedures. For example, representing percent problems with 10x10 grids (a large square divided into 10 rows of 10 small squares) or on number lines (from 0 to 100) have been suggested as a means for helping students visualise the computational procedures of percent calculations (e.g., Bennett & Nelson, 1994). Mnemonic strategies, which emphasise the key words "of" (meaning multiply) and "is" (meaning divide) have also been suggested to help students interpret percent problems and to order percent calculations (e.g., McGivney & Nitschke, 1988). However, these approaches rarely appear to address the multi-faceted nature of the topic of percent, cover all knowledges (decimal/fraction, ratio, and proportion), and encompass all percent meanings (number, fraction, ratio, proportion, statistics and function). Furthermore, Parker and Leinhardt (1995) reported that results of comparative teaching studies did not conclusively suggest that one approach was superior to another.

Assessment of percent knowledge: Parker and Leinhardt (1995) claimed that percent is a confusing topic in the mathematics curriculum for both students and teachers, and that basically, "percent is hard" (p. 423). Their claim is supported by the findings with respect to percent of the Fourth National Assessment of Educational Performance (NAEP) of Mathematics (Kouba et al., 1988) which provided evidence that students at the 7th and 11th grade levels appear to lack understanding of percent and have difficulty with percent applications (particularly Types II and III problems).

A recent study by Lembke and Reys (1994) looked at Years 5, 7, 9 and 11 students' conceptual and computational percent knowledge, before and after formal percent instruction, and showed a more promising picture of the percent knowledge students may possess. Lembke and Reys interviewed high- and middle-ability students in each of the four year levels and found that: (a) students in Years 5 and 7 (who had not received formal instruction in percent used a variety of intuitive strategies to solve percent problems; (b) older students (Years 9 and 11) utilised a percent formula for calculating percentages, often making careless errors; and (c) common benchmarks (100% is a whole, 50% is half, and 25% is half of a half of something) were used by students of all year levels as an aid to undertaking the calculations and to check the reasonableness of their calculations.

Implications for this paper: In a similar manner to the study of Lembke and Reys (1994), the study on which this paper reports was designed to analyse and categorise percent knowledge and solution strategies accessed by students of different proficiency with respect to percent problem solving, and to draw implications for appropriate instruction in the development of percent concepts and solution strategies. Specifically, the study focused on three year levels (Years 8, 9 and 10) and three categories of proficiency: proficient, able to solve all three types of percent problems; semi-proficient, able to solve type I problems but not able to solve types II and III problems; and non-proficient, not able to solve any type of problem. The following questions were a focus for the study:

(1) What knowledge do proficient, semi-proficient and non-proficient percent problem solvers possess and access?

(2) How do proficient, semi-proficient, and non-proficient percent problem solvers interpret and represent percent problems?

This paper reports on three aspects of this study, namely, proficient, semi-proficient and non-proficient percent problem solvers' knowledge of percent problem types, type of solution strategy, and use of diagrams.

Method

The methodology adopted in the study is qualitative. The research method is that of semi-structured Piagetian clinical interview and protocol analysis (Ginsburg, 1981).

Subjects: The subjects were a purposeful sample of eighteen students from a Year 8, a Year 9 and a Year 10 class from a Brisbane secondary boys school. The ninety students from these three classes were given the three types of percent problems to solve and, from their responses, were categorised as proficient, semi-proficient, or non-proficient. From this, two students per year level were selected at random from each of the performance categories.

Instruments: The instrument was a clinical interview. The tasks focused on students' understanding of percent problems (designed to be within the experience of the students) and the strategies the students used in solving these problems. The first task explored students' global schema of percent by asking about their knowledge of the three structural types of percent problems; the second identified strategies used by the students in solving percent problems by asking them how they solved the three types of problems; the third explored students' use of diagrams in solving percent problems by asking them to solve a problem with a diagram if they did not spontaneously use one.

Procedure: The students were removed from their class and interviewed in a separate room. The interviews lasted 30 minutes and were videotaped. The students had attempted the problems before the interview and the interview focused on recalling the methods they had used in solution. The probing of the students was based on contingent questions. If knowledge was detected that had not been used in problems, the students were questioned as to why it was not used.

Analysis: The interviews were subjected to protocol analysis (Ericcson & Simon, 1984). The grounded theory approach of Strauss and Corbin (1990) was used to determine patterns and commonalities.

Results

The eighteen students' responses to the three tasks are given in detail. The students are denoted as follows (the first number refers to their Year level):

Proficient students	8P1, 8P2, 9P1, 9P2, 10P1, 10P2
Semi-proficient students	8SP1, 8SP2, 9SP1, 9SP2, 10SP1, 10SP2
Non-proficient students	8NP1, 8NP2, 9NP1, 9NP2, 10NP1, 10NP2

Students' responses on number of percent problem types: Four of the six proficient student (8P2, 9P2, 10P1 and 10P2) and two semi-proficient students (9SP2 and 10SP2) identified the three types of problems from Ashlock et al. (1983). One non-proficient student (8NP2) identified two of the problem types, describing them as 'find the percent of a number given - what number is 25% of 60' and 'find what percent of one number another number is another - 15 is what percent of 60'; as did one semi-proficient student (8SP2), describing the problem types as 'finding the percent of a number, determining what percent of this number is this number, profit problems and loss problems'. The remaining 10 students (8P1, 9P1, 8SP1, 9SP1, 10SP1, 8NP1, 9NP1, 9NP2, 10NP1 and 10NP2) did not identify any of the Ashlock et al. problem types, and thought there were four or more types. The non-proficient students were particularly varied in their descriptions of problem types. For example, 'questions on maths tests, percent in the real world, percentages used to sell things, and percentages used for exporting' (8NP1) and 'those ones which you divide and multiply' (10NP2).

Students' solution strategies: The most widely used strategy for finding solutions to percent problems was the percent formula procedure. This was used by three proficient students (9P1, 9P2, 10P2), four semi-proficient students (9SP1, 9SP2, 10SP1, 10SP2) and all the non-proficient students. This strategy was used with a trial-and-error strategy (when the formula was forgotten) by two proficient students (9P1, 9P2) and three semi-proficient students (9SP1, 10SP1, 10SP2). All the non-proficient students did not attempt the problems for which they could not determine a formula.

Two proficient students (8P1, 9P1) used the unitary procedure, two proficient students (8P2, 10P1) used the traditional/cases procedure, while two semi-proficient students (8SP1, 8SP2) used the proportion procedure. Four non-proficient students (9NP1, 9NP2, 10NP1, 10NP2) used a key words strategy (the word "of" indicates multiply and "is" indicates divide).

Proficient students showed strong skills in mental computation and operation relationships (8P1, 9P1, 9P2, 10P1), conversions between percent, common fractions and decimal fractions (8P1, 10P1, 10P2), and benchmarking, approximation and estimation (8P1, 8P2, 9P1, 9P2, 10P1); they also showed some indication of ability to analyse the problems structurally (8P2, 9P1, 9P2, 10P1). Examples of bench marking, approximation and estimation are provided by the following: 8P1 said that the answer to '51 is 85% of what number?' had to be a little larger than 51 because of the relationship of 85% to the whole; 9P1 related '186 is what percent of 240?' to 120 out of 240 being 50%; while, for '28% of 150?', 10P1 said 28% is approximately 1/3, which means that the answer is close to 50. The proficient students used estimation with trial and error and knowledge of structure, for example, when given two numbers and asked to find a percent, 9P2 divided the smaller number by the larger and multiplied by 100, looked at the answer, and reversed what he had done when he thought the answer was unreasonable.

All semi-proficient students used some aspects of benchmarking, approximation and estimation to assist their problem solving but were not as skilled as the proficient students in mental computation. Some semi-proficient students were also skilful with conversions (8SP2, 9SP1).

Non-proficient students did not generally reveal flexible thinking; rather, they tended to follow routinised patterns of activity (e.g., convert the percent to a decimal) even if that strategy was not helping them to solve problems. Some non-proficient students did this before they even read the problem.

Students' responses with respect to use of diagrams: No students spontaneously drew diagrams in solving the percent problems. When asked to use this strategy to help solve the problems, all the proficient students, one semi-proficient student (9SP2) and five non-proficient students (8NP1, 8NP2, 9NP1, 9NP2, 10NP2) were able to draw diagrams that reflected the problem. The remainder could not draw a diagram and were not interested in doing so. Of the students who drew diagrams, four (8P2, 9P1, 8NP2, 10NP2) drew number lines, three (8P1, 10P2, 9NP2) drew pie charts, three (10P1, 9SP2, 9NP1) drew 10x10 grids, one (8NP1) drew rough rectangles, and one (9P2) drew diagrams of rivers and used the analogy of people crossing these rivers with respect to the problems.

Discussion and conclusions

Performance across categories: Proficient students generally knew that there were three types of problems. They could also represent percents in these problems with a variety of effective diagrams, although they were reluctant to draw them and tended not to use them to solve the problems. With regard to their solution strategies, they found it frustrating to discuss their procedures for solution and their preferred response was "I just do it!". They did not rely on the percent formula approach as much as less proficient students; they tended to use a variety of approaches. When an approach was not available (e.g., they forgot the formula), they tried another approach. In their solutions, they constantly estimated and benchmarked, manipulated numbers until the answers makes sense, converted readily between percents, common fractions and decimal fractions, and had a good understanding of the relative size of numbers in terms of relationships in the problem (in this, they tended to have the multiple meanings of percent as described by Parker & Leinhardt, 1995). They appeared to have good mental calculation skills and to understand the effect of operations (e.g., they reversed operations). Importantly, they also appeared to be able to analyse problems in terms of their meanings and be able to predict the operation to be used and the size of the answer relative to the numbers they had been given (e.g., when given problem '51 is 85% of what?', they could see that 51 was approximately 3/4 of the answer).

Except for one student (9SP2), semi-proficient students had no idea of the number of percent problem types and could not represent percent situations with diagrams. With respect to problem solutions, they were reliant on the percent formula approach although they were happy to use trial and error if they forgot the formula. Like the proficient students, they used benchmarking, approximation and computational estimation, but usually as a checking mechanism at the end of the solution rather than as an aid to analyse the problem and predict the size of the solution at the beginning of the problem. They were able to realise when an answer did not make sense, but were unable to construct alternative strategies to correct their mistakes or overcome difficulties.

Non-proficient students thought there were many types of percent problems, usually seeing surface features as constituting difference (e.g., percent to sell things and percent to import are different problems). Surprisingly, they were able to draw (when

asked) appropriate diagrams to represent the problem but were unable to use them for solution. With respect to solution strategies, they tended to try to solve problems by using the formula and key words approaches rather than examine the question as a whole; they looked for "of" for multiplication and "is" for division (following a syntactic approach, Resnick, 1982). They had little idea if an answer was sensible or if they had used the formula correctly. Unlike the proficient students, their repertoire of strategies was limited so that, if they forgot the formula (their main strategy), they were unable to access another strategy and consequently were unable to solve the problem.

Expectations of performance: Proficient students were able to access knowledge that enabled them to solve all three types of percent problems. Therefore it is likely that the proficient students had principled-conceptual and relational knowledge (Leinhardt, 1988; Skemp, 1978) and strategic and metastrategic thinking (Prawat, 1989), including self belief and strategy knowledge (Garofalo & Lester, 1985). Thus, it seems reasonable to expect proficient students to use solution strategies that efficiently translate their schema-based understanding of percent to solutions, that is, either the traditional cases or the proportion procedure. On the other hand, less proficient students were unable to access knowledge useful to percent problem solving. Therefore, it seems likely that less-proficient students had procedural and instrumental knowledge (Leinhardt, 1988; Skemp, 1978) of percent and lacked strategic and metastrategic thinking (Garofalo & Lester, 1985; Prawat, 1989). Thus, it seems reasonable to expect that less-proficient students would use rote procedures inflexibly in attempting to solve percent problems.

The students' responses in this study were as expected for non-proficient and semi-proficient students. These students were inflexible and formula oriented. The semi-proficient students showed more use of estimation and trial and error than the non-proficient students. The non-proficient students tended to focus on key words and to discontinue solution attempts if they could not determine an appropriate formula.

However, students' responses were not as expected for proficient students in that they did not strongly reflect a schematic understanding. Proficient students did show some indication of identifying problems by solution structure and they knew the problem types. They also used strategies and metastrategies and were confident in their solutions (as expected from the findings of Garofalo & lester, 1985). However, instead of a schema-based interpretation of problems leading to a classification approach to solution (e.g., cases, proportion), they tended to use a flexible mixture of benchmarking, approximation and estimation, and number and operation sense, along with a variety of strategies and some use of the trial and error strategy (i.e., what could be called a 'first principles' approach to solution).

Implications for teaching

The first implication is based upon the reasons for the success of the proficient students (and the lack of success of the other students), a success which lay in the proficient students' repertoire of strategies, flexibility with respect to strategy access, and number and operation skills. Therefore, the skills of benchmarking, approximation and estimation, conversions between percent, decimals fractions and common fractions, and number and operation sense (including mental computation) should be the focus of instruction for all students, along with the trial and error strategy.

The second implication is based upon the unexpected strategy use of the proficient students. Of the 90 students who were tested, only a very few could be categorised and selected as proficient. These few students did not translate their knowledge into efficient solution procedures (although they did translate their knowledge into effective

procedures). Therefore, there appears to be a need for students to be able to recognise problem type and to translate this directly to solution procedure. Instruction in proportion using a number-line diagram appears to offer the best opportunity for this because it does not rest totally on recognition of categories (as the cases procedure does).

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Pencil-and-paper Mathematics Tests Under the Microscope

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The effectiveness of a highly regarded short-answer pencil-and-paper test (*Mathematics Competency Test*, published in 1996 by the Australian Council for Educational Research) was investigated. In the study, 182 students—in Years 5 through 8, in 8 classes in NSW and Western Australia—answered the test questions and were interviewed. About 28% of responses were *either* (a) correct but students showed less than full understanding; *or* (b) incorrect, but students showed at least partial understanding, of the key concept/s) and skill(s).

According to *Measuring Counts*, a policy brief issued by the Mathematical Sciences Education Board (1993) in the United States, pencil-and-paper tests became popular because they provided an efficient means of assessing large numbers of people at low cost. There were, however, major criticisms of the tests: they could not tap students' ability to estimate the answer to arithmetic calculations, to construct geometric figures, to use calculators or rulers, or to produce complex, deductive arguments. Furthermore, the psychometricians' narrow focus on technical criteria—primarily reliability—worked against educationally sound assessment because it generated tests comprising many short items, and did not permit examinations composed of a small number of complex problems. This resulted in students being asked to perform large numbers of smaller tasks, "each eliciting information on one facet of their understanding, rather than to engage in complex problem solving or modelling" (p. 7).

In the 1990s there has been an awareness of the deficiencies of externally-set, pencil-and-paper mathematics tests (hereafter denoted ESPPMTs). Kilpatrick (1993), for example, called for less emphasis to be given to the psychometric aspects of assessment. According to Kilpatrick (1993), educators need to understand how people, "not apart from but embedded in their cultures, come to use mathematics in different social settings and how we can create a mathematics instruction that helps them use it better, more rewardingly, and more responsibly" (p. 44). To do that, he added, "will require us to transcend the crippling visions of mind as a hierarchy, school as a machine, and assessment as engineering" (p. 44).

Rhetoric and Reality

The Rhetoric

Over the past 10 years, in fact, much has been said, verbally and in writing, by advocates of more authentic methods of assessing mathematics learning and teaching (Black, 1994; Clarke, 1988, 1996; Clements & Ellerton, 1996; de Lange, 1996; Holt, 1993; Kilpatrick, 1993; Leder, 1992; Mathematics Sciences Education Board, 1993; National Council of Teachers of Mathematics, 1995; Niss, 1993; Romberg, 1993; Webb & Coxford, 1993). In the United States of America, the National Council of Teachers of Mathematics (NCTM) not only emphasised the need to develop and introduce new assessment methods in its *Curriculum and Evaluation Standards for School Mathematics* (NCTM, 1989), but it also published, in 1995, a special volume on *Assessment Standards for School Mathematics* (NCTM, 1995).

The Reality

Despite the rhetoric calling for more authentic assessment procedures to replace, or at least complement, ESPPMTs, mathematics educators and education bureaucrats have found it difficult to escape from a psychometric mindset in which the concepts of test reliability, validity, item discrimination, etc., are set in stone. Strident public

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demand for education accountability has given rise to rigid interpretations of "best-practice" education environments in which ESPPMTs have an important role to play. Best practice, many bureaucrats believe, can be most effectively achieved by specifying outcomes and behaviours indicating that these outcomes have been achieved, and then asking test experts to construct high-quality, highly reliable and valid, pencil-and-paper tests which will not only monitor the extent to which the outcomes are being achieved in different schools, but also (through the process of benchmarking) enable schools to be ranked according to "value-added" criteria (Cuttance, 1993; McGaw, 1995).

Teachers, education bureaucrats, politicians seem to be willing to allow *formative* assessments to be carried out by "alternative modes" (Clarke, 1988, 1996; Schmidt & Brosnan, 1996), but for *summative* tests, the demands of public accountability ostensibly require valid and reliable pencil-and-paper instruments, created by test experts, to be used (Black, 1994).

It is one thing for educators to criticise current practice and to propose new approaches, but it is altogether another thing to achieve large-scale change. Late in the 1990s ESPPMTs still continue to be widely used. Indeed, throughout the 1990s, education bureaucracies throughout the Western world, including Australia, New Zealand, and the United Kingdom, have been increasingly driven by societal demands for greater school and system accountability (Australian Education Union, 1995; Black, 1994; Garet & Mills, 1995; Olssen, 1993).

The move towards accountability led Shavelson, Baxter, and Pine (1992) to comment that in the final analysis the United States of America might be placing far too much weight on accountability to achieve its reform agenda. They argued that states with the strongest and most technically sound accountability systems had not achieved their desired reforms, and stated:

> Perhaps what is needed is far less account taking and far greater consideration and resources given to teaching and learning, especially for students drawn from diverse social, economic, cultural and language backgrounds. An increased emphasis on and bully-pulpit use of highstakes testing may, paradoxically, have a deleterious effect on U.S. education. Tactical, political "fixes" are not what is needed. Rather, we believe that a long-term, realistic approach to assessment—one that transcends politicians' terms of office—is. (pp. 26–27)

Political and professional realities render it almost certain that ESPPMTs will continue to be widely used across the world for assessing the mathematics performance of school children.

Yet, it is difficult to mount serious public arguments against ESPPMTs when the largest mathematics competition in the world, the Australian Mathematics Competition, uses such tests in our own backyard. So does the International Association for the Evaluation of Educational Achievement (IEA) which, with funds provided by governments around the world, has used pencil-and-paper tests in its three major international mathematics achievement studies.

Stake (1995), the veteran North American education evaluator, has recently stated that moves towards greater accountability in schools, with their attendant reliance and pencil-and-paper tests, are dangerous, in that they are likely to generate "overstandardisation, oversimplification, over-reliance on statistics, student boredom, increased numbers of dropouts, a sacrifice of personal understanding, and probably a diminution of diversity in intellectual development" (p. 213).

According to Stake, psychometricians' attempts to define different kinds of test validity are misleading because "testing as an activity and individual tests as a tool are neither valid nor invalid until the results are interpreted in some way," for "only the interpretations of test scores in particular situations can be said to be valid or invalid" (p. 172). Stake went on to say that research indicates that with increased testing and curriculum standardisation, teachers attend more to the so-called basics—the most elementary knowledge and skills—and less to what is needed to get their

students to achieve a deep understanding of even a few topics.

Research Into the Effectiveness of Externally-set, Pencil-and-paper Mathematics Tests of the Short-answer Variety

Despite the continuing assumption by education bureaucrats (and parents, politicians, and many teachers), that externally-set, short-answer pencil-and-paper mathematics tests provide a cost-effective, objective way of monitoring mathematics performance, it needs to be recognised that recent mathematics education research is calling into question the power of externally-set written examinations to provide proper education accountability.

Summary of Research into the Effects of Standardised Testing

Kindsvatter, Wilen and Ishler (1988) drew attention to a substantial body of literature which describes the effects of large-scale, standardised testing for accountability. Data indicate that the use of such tests does not raise standards, and teachers consider that standardised tests do not assist learning. In fact, there is no conclusive evidence to show that assessment of student learning from externally-set pencil-and-paper tests provides better quality data than could be provided by teachers' ratings of students (Mathematical Sciences Education Board, 1993).

Furthermore, it is not widely known that research has generated data suggesting that, despite enthusiastic claims to the contrary by psychometricians and education bureaucrats, performance data generated by ESPPMTs can be seriously misleading. There is, in fact, increasing evidence pointing to the conclusion that students who answer pencil-and-paper mathematics items correctly sometimes have little or no understanding of the mathematical concepts and relationships which the items were designed to measure, and that this applies even for so-called "valid" and "reliable" tests (Frary, 1985; Gays & Thomas, 1993; Hembree, 1987; Thongtawat, 1992).

Nevertheless, politicians and education bureaucrats still believe that ESPPMTs can satisfactory measure mathematical learning. A large research study into this question, jointly supervised by the authors and carried out in Thailand by Thongtawat (1992), found that the proportion of students who gave correct answers to multiple-choice mathematics items but who did not understand the mathematical concepts and relationships involved in the items, was much higher than that for corresponding short-answer but not multiple-choice items. Thongtawat also found that students who scored poorly on a test could sometimes have a good conceptual grasp of the material which the items covered.

The authors have previously carried out research designed to test the effectiveness of multiple-choice and short-answer pencil-and-paper tests (Ellerton & Clements, 1995). In a study in which 116 Year 8 students in 5 schools in 2 Australian states were interviewed, we found that both multiple-choice and short-answer questions were seriously ineffective in assessing student understanding. We concluded:

> We are particularly concerned about the increasingly widespread use of externally-set, multiple-choice and short-answer pencil-and-paper instruments. Testing regimes, based on crude, one-off, pencil-and-paper instruments are employed, and justified by reference to the need to make teachers, schools and systems more "accountable." The data in this paper suggest that claims that such tests can provide "quality assessment data" about students' and schools' mathematical performances are not justified. (pp. 12–13)

In fact, over one-third of responses to well-constructed questions in multiple-choice and short-answer formats were such that either (a) correct answers were given by students who did not have a sound understanding of the correct mathematical knowledge, skills, concepts and relationships which the questions were intended to cover; or (b) incorrect answers were given by students who had partial or full understanding. Almost 50% of the incorrect responses were given by students who

did understand, at least partially, the mathematics that the questions were designed to assess (Ellerton & Clements, 1995).

In the same paper, we questioned whether multiple-choice questions could ever be valid for assessing mathematical learning, pointing out that we do not know of any practising adult mathematicians who actually work in situations where they are regularly asked to choose one correct answer from four or five possibilities. The implication is that schools which use multiple-choice tests as a major method for assessing mathematical learning are in danger of continually reinforcing in their students' minds a flawed image of the nature of mathematics.

The Present Study

The Instrument

In our most recent ("in progress") research, we are investigating the effectiveness of the *Mathematics Competency Test,* published in 1996 by the Australian Council for Educational Research. This 46-item test has an excellent pedigree. Its three authors (P. E. Vernon, K. M. Miller, and J. F. Izard) are highly regarded educationists/psychologists/psychometricians; it is distributed by a highly regarded education research organisation; and the manual for the test indicates it has construct, content-related, and criterion-related validity, and an internal consistency index of 0.94 (Izard & Miller, 1996). The test, which is designed for individuals 11 years-of-age or older, uses short-answer but not multiple-choice items, and therefore the guessing factor is minimised.

The Sample

Thus far, 182 students who took the test have been interviewed by graduate research assistants. The students were in 8 classes in three schools—a middle-class independent school in New South Wales (72 students in four Years 5 and 6 classes were interviewed); a government school in the Western Suburbs of Sydney (47 students in two Year 8 classes); and a highly regarded government school in a working class suburb of Perth (63 students in two Year 7 classes). On average, each interview took about 40 minutes.

The Aim of the Study

The aim of our study was to determine, for each particular item, and for each particular student interviewed, the extent to which the student understood the main concept which that item was intended to test, and then to classify initial responses as being "matches" (when a correct response is associated with an adequate understanding of the key concept(s) being tested, or an incorrect (or no) response is associated with an inadequate understanding) or "mismatches" (when a correct response is associated with inadequate understanding or an incorrect (or no) response with an adequate understanding).

Following the classification of responses, the proportions of responses which were deemed to be "matches" were calculated for (a) each interviewee, (b) each class of students, (c) each school, and (d) the total group of interviewees.

The Extended Newman Interviews

The authors have previously developed and used (Ellerton & Clements, 1995) an extended form of the Newman error analysis technique (Ellerton & Clarkson, 1992; Newman, (1977) to investigate *both the errors and correct answers* given by students to items on pencil-and-paper mathematics tests. In the present study the 182 students were interviewed by 4 graduate assistants who had been trained, by the authors, to conduct extended Newman interviews. Although the students had originally attempted all 46 questions on the *Mathematics Competency Test,* extended Newman interviews were conducted for only 33 of the questions—the first 33 questions for students in Years 5 and 6 and, depending on a student's initial score on the test either Questions 1 to 33 or Questions 14 through 46 for students in Years 7

and 8 (students who obtained high raw scores were interviewed for Questions 14 through 46).

The aim of each interview was to ascertain the level of understanding associated with each of the responses by each student to the 33 questions on the *Mathematics Competency Test*. After the standard Newman questions had been asked, and answers given, the interviewer then carried out probes into issues arising from answers given to the questions and from answers given during the interview.

For each question, a student's "Level of Understanding" (LU) was assessed, by the interviewer, according to a 3-point scale: "0," if a student did not recognise, or had no grasp of the necessary concepts; "1," if a student recognised which concepts might apply but had only a limited understanding of the concepts; and "2," if the concepts and relationships were recognised and were understood. Criteria for making these decisions had been developed before the main study began and, following each set of interviews, the interviewers met with the authors to discuss any difficulties they had experienced in making LU classifications. These meetings resulted in LU classification criteria being clarified and sharpened.

Three examples of LU classifications: Question 20 on the *Mathematics Competency Test* stated: "A block of wood measures 6 cm by 5 cm by 4 cm. What is its volume?" Many of the students gave "120" as their written answer, without stating the unit. According to the "Scoring Key" in the *User's Manual,* this response had to marked as incorrect. However, in the interviews some of the students who gave this answer stated confidently, without any prompting, that the volume was 120 cm^3. These students were given a LU classification of 2 for the question.

Question 22 on the *Mathematics Competency Test* stated: "$3y + 2 = 14$ $y =$ " Most of the students interviewed did not know the convention that "3y" means "3 times y." Some guessed that this was the case, and gave the correct answer (y = 4). Others guessed that $3y$ meant $3 + y$, and gave the answer $y = 9$." When told, in the interview, that $3y$ meant "3 times y," many of these students had no trouble working out that y should be 4. These students had their initial response marked "wrong," but were given an LU classification of 2 (because, it was decided, the main point of the question was whether a respondent could solve the "equation" "If 2 is added to 3 times a number, and the answer is 14, what is the number?")

Question 31 on the *Test* showed three boxes which were labelled A: "200g $4.00" B: 250g $4.50 C: 300g $6.00 D: 500g $9.50, and respondents were asked to state which box—A, B, C or D—represented the best value? Many who selected the correct alternative (B), had either guessed, or, in the interview, provided an inappropriate reason. Often they changed their answer in the interview. Such students were given a LU classification of 0 even though their initial answer was correct.

Results

It is not possible to provide detailed analyses of results for individual questions or students in this paper—these will be given, however, in subsequent papers, and in a final report which will be written after the sample of schools and students involved in the study has been enlarged.

Overall, about 28%—that is to say, over one-quarter—of the original responses were "mismatches," in the sense that correct answers were given by students who lacked an adequate understanding of the key concept(s) being tested, or incorrect answers were given by students with full or partial understanding. No student had no mismatches, and more than half the responses of some students were "mismatches."

The proportion of responses which were mismatches for intact classes of students varied from 20% (for a Year 8 class) to 41% (for a Year 7 class). For the three schools, the percentages of mismatches were 23% (for the independent school), 21% (for the government high school) and 41% (for the government primary school).

Discussion

Although the *Mathematics Competency Test* has a reliability index of 0.94, over one-quarter of the students' initial responses were classified as "mismatches." Our data indicate that raw scores on the test do not permit informed judgments to be made about students' knowledge and understandings with respect to the mathematics covered by the test. A correct answer did not always correspond to "understanding" and, for an educationally significant proportion of responses, students who did understand the main concept(s) involved in a question gave incorrect answers.

The findings of our research fly in the face of continued widespread usage of pencil-and-paper tests to assess mathematical understandings in all parts of the world. There are strong vested interests which are propagating the view that ESPPMTs provide the most objective and cost-efficient means of measuring mathematical learning. In Australia, full cohort statewide testing is already taking place, and national testing is on the Federal Government's agenda. Yet, our evidence would suggest that if pencil-and-paper mathematics tests are being used (as is indeed the case) then it is inevitable that invalid results will be obtained.

Validity and reliability issues: Research into why elementary and junior secondary school students make mistakes on word problems on pencil-and-paper tests has consistently indicated that well over half of the errors are associated with difficulties with what Newman (1977) called Reading, Comprehension, or Transformation (Ellerton & Clements, 1992). Analysis of written responses, without any other form of analysis, does not reveal the proportion of correct answers given by students who have little or no understanding of the mathematics involved, or the proportion of incorrect answers which were given by students with good understanding.

This raises serious questions about whether ESPPMTs can ever generate "valid" assessments of mathematical understandings (especially for tests designed for primary and junior secondary students). At issue, in fact, is the meaning of the term "valid," given that the interviews revealed a large number of personal and linguistic variables which affected students' responses. Such variables are usually not recognised fully by test constructors who rely heavily on traditional psychometric considerations (such as within-test correlations—Izard & Miller, 1996, p. 28).

Izard and Miller (1996) recognised some of the limitations of their test, so far as validity was concerned. They stated, in the *User's Manual:*

> Although the test items are consistent with those used in similar instruments, they do not include some types of mathematical task. This is a function of the mode of testing; all other mathematics tests like this suffer the same deficiency with respect to construct validity. The authors of this test recognise that the mode of assessment with a written test provides little evidence of a candidate's skills in solving complex problems in context, undertaking investigations, or carrying out particular practical mathematical tasks such as estimating distances in a local context, measuring using a variety of units, and manipulating space. Accordingly, a comprehensive assessment of mathematical competence will need to combine evidence of performance on traditional written test questions (such as those on the Mathematics Competency Test with judgments about the candidate's proficiency in these other tasks. (pp. 25–26)

This quotation is important, for it suggests that, *by themselves,* ESPPMTs cannot generate quality data for assessing mathematical performance.

So far as reliability is concerned, from our perspective our data indicate that the reported internal consistency index of 0.94 for the *Mathematics Competency Test* suggests that test scores will not only "reliably" generate much helpful information about a student's understandings, *but also much misinformation*—the only trouble is,

without interviewing individual students it is impossible to know what is helpful information and what is misinformation.

Mathematics education research and the use of pencil-and-paper test instruments: Given that recent research studies suggest that about 30% of all responses to questions on ESPPMTs are "mismatches" (Ellerton & Clements, 1995), it appears to be the case that mathematics education research which operationalises and measures a dependent or independent variable through a single application of a pencil-and-paper research instrument is on shaky ground. Since research studies conducted in many countries have relied heavily on such an approach, a review of what constitutes "good design" for quantitative studies in mathematics education research is urgently needed (see Clements and Ellerton (1996) for further discussion of this issue).

In Conclusion

We agree with commentators who fear that pencil-and-paper tests have driven, and continue to drive, other forms of school assessment (Kilpatrick, 1993; Resnick, 1987). Our own research, part of which is reported in this paper, indicates that even the most expertly constructed ESPPMT can never produce an accurate summary of what children really know, unless interviews are part of the data analysis process. Given time constraints, that is most unlikely. The fact that the percentage of mismatches varied considerably between schools in the study reported in this paper suggests that ESPPMTs should not be used for benchmarking school performance.

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Mathematics in the Vocational Education and Training Sector: A Case Study

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In an industry-driven VET sector major educational decisions are made, not by educators, but by representatives of industry. Mathematics curricula for scientific industries have been designed under a positivist, technicist paradigm without an adequate research base. These are unlikely to be responsive to industry needs for a variety of reasons which include the methodologies used by two separate projects which impact on the curriculum framework, the superimposition of CBT requirements, and the failure to address the issue of transferability.

Introduction

The Vocational Education and Training (VET) sector describes itself as being *industry-driven*. This means that, in keeping with government policy, major educational decisions are made, not by educators, but by representatives of industry. In this paper I will argue that mathematics curricula for science paraprofessionals have been established under a positivist, technicist paradigm Competency Based Training (CBT) without adequate attention to the messages of the mathematics education and other research communities. Although intended to be responsive to the needs of scientific industries it is unlikely that this will actually eventuate for a variety of reasons. These include the methodologies used by two separate projects which impact on the design of the curriculum framework, the superimposition of CBT requirements, and the failure to address the issue of transferability.

The National Training Reform Agenda (NTRA): For the last decade the VET sector has experienced radical changes as a result of was originally known as the NTRA. The sector underwent a metamorphosis from the Kangan philosophy of development of the whole person (Kangan, 1979) to what is now known as a *training market*, where the language of business and industry has replaced the language of education as students and/or industries become known as *clients*, teachers as *trainers*, curriculum *packages* are *delivered* not taught; diverse *products* come from a diversity of *suppliers* to meet diverse *demand* in a diversity of *locations*, in a *training market* which strives for "continuous improvement through greater innovation, improved client choice and enhanced efficiencies" (Harmsworth, 1996, p. 3). The communities of interest represented in the corporatist alliance between industry, unions, and governments have been influential in shaping current Australian educational policy, which, in part, advocates closer links between education and industry, ostensibly to improve Australia's economic productivity and industrial competitiveness (Kell, 1993; Sachs, 1991). Education has been harnessed as a tool of micro-economic reform and its values have been largely ignored (Stevenson, 1995). As well as being industry-driven, the VET sector has also become nationalised, with major implications for curricula.

CBT: Since 1995 VET curricula have been required to have national accreditation in CBT format in order to obtain government funding. This is despite the fact that, as Jackson (1993, 1995) argued, there have been two decades of research to indicate that CBT has not and probably will not improve learning in most areas where it has been applied. Jackson argued further that CBT be viewed not as an instrument of curricular reform but as a an administrative means of keeping programs accountable: It is a textually mediated environment where what counts is not what individuals *do*, but what they can be shown to have done. In contrast to earlier eras important decisions about curriculum and teaching have been made, not by the teacher, but through documents which speak with an authoritative voice, according to Jackson. She concluded that CBT as a technicist solution was unlikely to serve the long-term interests of workers. Mulcahy (1996) also argued that CBT, constituted from a range of formal tools (analytic techniques; competency standards; competencies; modules), attempts to shape vocational curriculum practice in particular ways. This will be discussed in more detail below.

Mathematics Curricula for Science Paraprofessionals

The National Mathematics Curriculum Project (NVMCP): With a seemingly endless range of mathematics units being offered across the TAFE sector throughout Australia, and diverse regulations in each state, the NVMCP was set up in 1992. It conducted an audit of mathematical skills included implicitly and explicitly in TAFE courses. To assist further curriculum development it recommended a set of topic packages be developed and mapped onto an associated topic network; topic packages could be assembled into modules or included in other mainstream modules as required. The NVMCP had already commenced when the legislation requiring CBT format was passed, to which it was then required to conform. Its stated intention was to develop a mathematics curriculum which was nationally consistent across all vocational areas.

The audit's methodology was to collect and categorise mathematics curriculum documents from each state's TAFE system, although there were difficulties with this process (Pantlin & Marr, 1992). The final product is an extensive atomised topic network composed of 94 topic packages, from fractions and decimals to numerical methods (for example) arranged in a more or less linear progression of pre- and co-requisites. Although practitioners were employed as writers, the parameters had already been set.

The National Laboratory Science Technician (Scitech) Curriculum Project: Aimed at science paraprofessionals, this project consulted extensively with industry personnel and conducted sophisticated task analyses, collecting a wide range of data through a variety of techniques, to develop competency statements, study pathways, and appropriate (CBT) curriculum materials (Johnstone, 1993). The project developed a model that required "learners to be exposed to a rich domain of specific knowledge and given the broad problem solving strategies to facilitate its transfer" (Johnstone, 1993, p. 69). Findings revealed that industry desires attributes such as: "efficiency, persistence, client focus, adaptability, flexibility, independence, team work, concern for ethics and quality, and responsibility for their own work" (p. 68). The final product was intended to promote integration of skills across the curriculum.

The Scitech Project made a selection of what were considered appropriate mathematics packages from the NVMCP to form two mathematics and two statistics modules. However, as Harris (1991) argued, lists of mathematics topics generated by industry frequently overlook important mathematical skills that are invisible to those workers actually using them (see also Hogan, 1996), because they are embedded in the work context. Harris (1991), referring to workers at lower levels, noted:

> It is the lists of assessable skills that people remember as 'mathematics.' . . . In a climate that accepts that mathematics-for-labourers is minimal, numerical skills and little more, every new curriculum initiative for such a defined group generates or revives lists of skills all remarkably similar to those of previous projects and previous ages. The status quo is maintained and the popular and industrial image of workforce mathematics is fossilized in a state that is disowned by mathematicians and rejected by most in mathematics education. (p. 142)

Buckingham (1995a, 1995b, 1996, 1996 October), Hogan (in preparation), and Zevenbergen (1995) have established that the topics mandated for workers at the basic levels of the Australian Qualifications Framework (AQF) do not reflect the actual mathematical needs of industrial workers. In fact such workers are likely to be using and understanding much more sophisticated mathematics and statistics, simply because they are contextually significant (see also Dowling, 1991), but may be unable to satisfy competency requirements on a basic skills test without the use of a calculator. Similarly, many of the mathematics topics prescribed for science technicians, albeit at higher AQF levels, do not accurately reflect the work they will be likely to undertake either on the job or in other classes. For example, there are nine assessment criteria for one learning outcome which requires the solution of vocational mathematics problems using linear and quadratic functions and their graphs, and eleven assessment criteria for another involving exponential and logarithmic functions. These criteria mostly use the verbs "solve" and

"sketch;" that is, skills which reproduce school mathematics techniques. At no stage are interpretive skills mentioned (except "interpret the meaning of a logarithm") and it could be assumed that such manipulative skills have been demanded by industry. Personal experience suggests that these skills are not used on the job, rather it is the interpretation of "messy" data, particularly the analysis of graphical data generated by sophisticated instruments including computers that will be useful. Ironically calculus units are strictly proscribed as they are not considered by industry to be relevant for workers at this level (I. Johnstone, personal communication, February 1996). Most of the suggested references are between 10 and 20 years old, while no reference is made to innovative work such as Barnes's (1991) Investigating Change series. This series has gender equity as one of its underlying principles, a concept barely addressed in these projects, or by the VET sector at large (see FitzSimons, 1997). In addition, the series uses a problem solving approach to understand and interpret patterns; collaboration, communication, and reflection are actively encouraged — exactly the kinds of attributes which are, as noted above, demanded by industry. In contrast to Johnstone's assertion, problem solving skills are not mentioned in the four mathematics/statistics units: The only problems are of the traditional textbook kind.

The selection of content relevant to students is confounded by the Scitech project having to simultaneously address the needs of six specialisations (animal technology, biological sciences, chemical laboratory technology, food science, geoscience, and medical laboratory technology), while the NVMCP packages have to be available to *every* industry group in the system. There is clearly a large distance from the detailed task analyses provided by incumbent workers or their supervisors to the final text which is mediated by the categorisation processes of the Scitech project as well as the NVMCP network, in addition to having to conform explicitly with detailed requirements of CBT (see, for example, Porter, 1993) which operates under a positivist paradigm. Such a circuitous route can only exacerbate the problem of accurately describing the mathematical ideas and techniques actually relevant to an occupation and transforming them into meaningful curriculum documents. This would appear to be one of the major problems (if not *the* major problem) of mathematics in the VET system, as it attempts to make curricula relevant to the system's *client group*: learners and (potential) employers. Billett (1996) has identified the problem as one of transfer of knowledge between different communities of practice: embedded in the social practice of the workplace, and disembedded in the sociocultural level of the TAFE college (see also Adler, 1996).

An example of text: The following is an application of the learning outcome: "Solve vocational mathematics problems using linear and quadratic functions and their graphs:"

> A researcher in physiology has decided that a good mathematical model for the number of impulses fired after a nerve has been stimulated is given by the quadratic function
>
> $$y = - x^2 + 20x - 60,$$
>
> where y is the number of responses per millisecond, and x is the number of milliseconds since the nerve was stimulated.
> (a) When will the maximum firing rate be reached?
> (b) What is the maximum firing rate?
>
> (ACTRAC, 1994, p. 29)

This example appears to be contrived, and makes no attempt to explain the physical model from which it was supposedly derived. Even students who are enrolled in the subject of *Anatomy & Physiology* can find no meaning in it; they strip away the words and proceed to solve the quadratic equation in a similar manner to the contiguous examples. It is clearly an example from a school mathematics text. One of the intentions of the Scitech project was for students to have transferable skills, yet the issue of transferability is the subject of much debate (Billett, 1996) and cannot be assumed to take place automatically. Examples such as the above almost preclude such a possibility.

Dowling (1996) argued that school mathematics textbooks have the ability to position students according to perceived abilities and to distribute different messages accordingly. It would appear that VET mathematics curriculum texts, although seeking to organise mathematics according to an industrial setting, actually organise it according to the principles that predominate within the recontextualising activity that is strongly allied to school mathematics. They also position adult students (young and old) as school children once more. Dowling (1995) argued that the kind of examples found in school mathematics texts mathematise everyday experiences with the implication that people's lives would be more efficient and effective if they operated as mathematicians. However, he noted that this assumption is not valid "because mathematised solutions always fail to grasp the immediacies of the concrete settings within which, as Jean Lave points out, problems and solutions develop dialectically" (p. 4). Willis (1992) also attacked the spurious claims to contextualisation through so-called industrial applications. There is clearly a need for mathematics curricula which address the needs of students not only to have a repertoire of mathematical skills but to know which mathematics to use, how, and when, and to what level of precision; to recognise when one's skills are not enough and to seek further help; to evaluate the reasonableness of one's calculations or judgements and follow up this evaluation in a constructive manner by iterating or abandoning the strategy adopted. In social settings as well there is a need for people to exhibit a personal confidence in their own abilities to solve problems they perceive as mathematical, in contrast to the generally held belief that they are more or less incompetent in this domain (Benn, in press; Coben, in press).

Relationship of mathematics to technology: Mathematics underpins the technological formation (FitzSimons, Jungwirth, Maaß, & Schloeglmann, 1996), but needs to be considered as more than simply a theory of techniques (Maaß, in preparation). Consistent with the situation in Europe as described by Maaß, current VET mathematics curricula rely too heavily on an instrumental approach: It would be appropriate for adult students to include a critical reflection on the uses to which mathematics is put. Skovsmose (1994) used the word *mathemacy* to encompass mathematical, technological, and reflective competencies. According to Wedege (1995), technological competence encompasses three components: professional, social and democratic competencies. These distinctions were based on two theses of Skovsmose. Firstly, the knowledge required to develop technology is different from the knowledge needed to analyse and assess it. Secondly, learning mathematics does not necessarily teach the learner how to use it; that is, technological knowledge cannot be reduced to mathematical knowledge. According to Wedege simply inserting mathematics into the curriculum for (potential) workers does not ensure technological competence.

Innovative Curriculum Programs

This paper has addressed some issues associated with prescribed mathematics curricula within a CBT context. (See also FitzSimons, 1996a, 1996b.) Achtenhagen (1994a, 1994b), Noss and Hoyles (1996), and Sefton, Waterhouse and Deakin (1995) provide models of innovative, integrative curriculum development and implementation in the business, banking, and automotive industries respectively. Achtenhagen's model has been developed in Germany, and is the result of many years' research and development — a luxury not available in Australia, where research in the VET sector is mainly concentrated on managerial issues. Although Buckingham's research has addressed generic numeracies in the workplace, there is no provision for research into the teaching of mathematics subjects within the college environment, particularly at the higher AQF levels. Professional development at this level is also virtually non-existent (see FitzSimons 1995a, 1995b).

Conclusion

This paper has investigated aspects of mathematics curricula for science paraprofessionals to illustrate some of the problems which exist in the VET sector, where a solid research base for the teaching and learning of mathematics is lacking. The sector

has also adopted a CBT model of curriculum and assessment despite the lack of research evidence that it will successfully achieve its intended outcomes, particularly at the higher AQF levels. The major problem for mathematics curricula appears to be associated with the identification of relevant mathematical content to reflect actual industrial needs and the subsequent didactical transposition into meaningful curriculum so that transfer will be optimally facilitated. In this way, it may be possible to begin to address the key competency *Using Mathematical Ideas and Techniques* in its fullest sense.

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The University of Melbourne's mathematics students: Who are they? Why are they here? Will they stay?[1]

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Students at the University of Melbourne participated in a larger study that aimed to explore the critical factors contributing to decisions to pursue mathematics at the tertiary level. Of particular interest and concern to the Department of Mathematics was knowing the composition of the student cohort, the students' reasons for choosing to study mathematics, and whether they would continue into higher levels. The findings and their implications are reported in this paper.

Background to the study[2]

Like other mathematics departments in Australian tertiary institutions, the Department of Mathematics at the University of Melbourne is concerned about the declining numbers and calibre of students pursuing mathematics courses. As part of a larger study exploring the critical factors which contribute to decisions about studying mathematics at the tertiary level, a survey questionnaire was administered to students enrolled in mathematics courses at three universities during 1996. Students at the University of Melbourne were involved and the data were gathered during Semester 1.

The literature suggests that a number of cognitive, affective, learning environment, social and cultural variables are implicated in students' decision making (see Forgasz, 1996). A range of relevant variables was included in the survey instrument administered to the students. Statistics consistently indicate that more males than females are enrolled in the most demanding mathematics courses offered at the grade 12 level in Victoria (Teese, Davies, Charlton, & Polesel, 1995) and across Australia (Dekkers, De Laeter, & Malone, 1991), and in tertiary courses in the "hard" sciences at both the undergraduate and postgraduate levels (Gaffney & Gill, 1996). These persistent patterns of gender difference are of continuing interest, concern and challenge. Gender was also included as a variable of particular interest in the research design. Australia's multi-cultural profile was also recognised in the study; students' national/ethnic backgrounds were examined.

The Study

The University of Melbourne's Department of Mathematics was particularly interested in knowing who comprised their cohort, why the students were taking mathematics courses and whether they would persist with their mathematical studies. The data gathered were explored to provide partial answers to these questions and the findings are reported here.

The survey questionnaire

The survey questionnaire comprised items aimed at exploring a range of factors implicated in decisions to pursue tertiary level mathematics. These included: perceptions of mathematics learning environments (school and university), beliefs about self as a learner of mathematics (affective and cognitive), beliefs about the influence of significant others (parents, teachers, careers advisers etc.), and reasons for studying mathematics (at grade 12 and at university). Biographical and background information were also gathered: for example, information about grade 12 schooling and indicators of socio-economic status and ethnicity. Data were entered into a database and analysed using $SPSS_{WIN}$.

The sample and sampling procedure

Due to funding and logistical considerations, it was decided to distribute the questionnaires to about 25% of the first year mathematics students (≈600), about 50% of the second year students (≈400) and to all third year students (≈100). The sample excluded students taking an Engineering degree (mathematics is compulsory), but did include students taking combined Engineering degrees (e.g., Science/Engineering). The

sampling procedure implemented was designed to cause least disruption to lecturers' and tutors' teaching programs. The questionnaires were mainly distributed through tutorials but also in lecture groups with small numbers. In some cases, students completed the questionnaires immediately; in others, they took them away and returned them the following week. In all, 374 (213M - 58%, 161F) completed questionnaires were returned; a response rate of ≈34%. The sample completing questionnaires at the University of Melbourne was representative of the total number of completed questionnaires gathered from the three universities participating in the study: 1072 students comprising 619 (58%) males, 448 (42%) females, 5 (sex not indicated).

Results

Who are they?

A profile of the University of Melbourne students is presented first. The vast majority (99%) were full-time students, under 21 years of age (96%) and had progressed directly from school to university (91%).

School backgrounds and socio-economic status: The school types attended for grade 12 studies and their locations were as follows:

School type

government	39%	co-educational	56%
Catholic	35%	single-sex	44%
Independent	21%		

School location

Victoria	94%	metropolitan	74%
interstate	2%	non-metropolitan	22%
overseas	4%	other	4%

Statistics on school type attended by Australian students have been relatively stable over the past few years. In 1993, about 72% of all Australian students attended government schools; in 1995 the proportion was 71% (Australian Bureau of Statistics Statsite, 1997). About 10% attended Independent (non-government, non-Catholic) schools in 1993 (derived from Ministerial Council on Education, Employment, Training and Youth Affairs, 1994). Among the sample of University of Melbourne mathematics students, the proportion of students from non-government schools was considerably higher than the Australia-wide data. The vast majority of government schools in Victoria are co-educational and many non-government (Catholic and Independent) schools are single-sex. The statistics on attendance at single-sex schools appear consistent with the school types attended by students. Since fees are associated with non-government schools, it can be inferred that many University of Melbourne mathematics students are from middle to high socio-economic backgrounds. About 29% of the students, however, were in receipt of financial assistance from the federal government through the *Austudy* scheme.

Ethnicity indicators: The results were as follows:

- 72% of the students were Australian born
- 49% had Australian born mothers; 48% had Australian born fathers
- 27% regularly spoke languages other than English [LOTEs] at home
- 10% who completed grade 12 studies in Australia had taken the alternative English program, *English as a second language* [ESL]

The data suggest that several students were themselves new migrants to Australia. Others were the offspring of one or more parents not born in Australia. The various waves of immigration to Australia appeared to be well-represented among the students studying mathematics at the University of Melbourne. Hence cultural/ethnic factors are likely to have played a part in shaping the students' decisions to study at the university and to study mathematics in particular.

Why are they here?

Students were asked to indicate why they had studied mathematics at the grade 12 level at school and at university in 1996 by marking one or more reasons from a list provided. Space was available to list other reasons. The percentages of students responding to each alternative are summarised below. Relevant statistics are also shown in brackets for the variables for which statistically significant gender differences were found.

Why mathematics was studied at school

keep options open	74%	(M: 70%; F: 80%)
wanted to	58%	
parents wanted	14%	
teacher encouraged	30%	(M: 26%, F: 36%)
career advice	25%	
like mathematics	54%	
good at mathematics	58%	(M: 64%, F: 51%)
to be with friends	3%	
pre-requisite for university	79%	
other choices worse	8%	
improve TER[3]	40%	
other	3%	

Why mathematics was studied at university

like mathematics	46%	
good at mathematics	36%	(M: 41%, F: 28%)
wanted to study mathematics	40%	
parents wanted me to	6%	
to keep options open	45%	(M: 40%, F: 52%)
lecturer/tutor encouraged me	4%	
career advice	14%	(M: 11%, F: 19%)
to be with friends	1%	
compulsory for another subject in 1996	31%	
pre-requisite for mathematics next year	27%	
pre-requisite for non-mathematics course next year	39%	
essential for career	28%	
the other choices I had were worse	9%	
other	8%	

The large proportions of students who had taken mathematics at school and at university because they *wanted to keep options open* were of particular interest. At both levels, the findings suggest that many students were uncertain about their future career paths, more so when choosing to study school mathematics than university mathematics. The significant gender differences on these items imply that the females were more tentative than the males. This interpretation is consistent with the finding that more females than males had taken mathematics at school and at university following *career advice*.

The proportions of students who indicated that they had studied mathematics at school because they *liked it*, and because they *wanted to* were disappointing. The explanations for these findings and for the small proportions of students who gave *careers advice* and *teacher encouragement* as reasons invite further investigation. A measure of success for the efforts to promote greater female participation in mathematics appears to be reflected in the finding that more females (36%) than males (26%) said that their *teachers had encouraged* them to study mathematics at school. On the other hand, it may be that females were more aware or were more willing than males to acknowledge the encouragement of their teachers.

School teachers appear to have been more encouraging of students to pursue mathematics (30%) than their tertiary counterparts. When first year tertiary students were omitted from the analysis, only 4% of the 148 students indicated that they had continued studying mathematics because their *tutors/lecturers had encouraged* them. While encouragement was not identified by many tertiary students as a factor influencing their

decisions, this does not mean it was not given. Clearly there is a difference between school and tertiary learning environments. The comparative sizes of the institutions and cohorts, and the differences in the learning settings (small classes compared to lectures) suggest that school teachers will know their students as individuals better than will lecturers. Regular tutorials are organised in the department with fewer than 20 students and offer better opportunities for one-to-one interaction. However, contact is restricted to 50 minutes per week. Previous research at the school level has identified personalisation as a critical learning environment factor implicated in fostering desirable student beliefs and attitudes towards mathematics (Forgasz, 1995). The extent of perceived personalisation within the department is an issue worthy of attention and further investigation.

A large proportion of students indicated that they had taken mathematics at school because they believed it would improve their TERs (40%). Of these students, only 45% also took mathematics because they believed they were good at it. It is of concern that such a large proportion of students who did not believe they were good at mathematics chose to study it at school in the belief that it would improve their TERs. In reality, simply attempting a grade 12 mathematics subject does not automatically improve the TER. A reasonable performance in *Specialist Mathematics*, the most demanding of the mathematics subjects offered, may result in an improved TER (Prof. Tim Brown, personal communication, 10 March 1997).

As reasons for studying mathematics at school and at university, the influence of significant others including parents and the peer group seemed smaller than might have been anticipated from the literature (e.g., Leder, 1992). It was interesting to note that a greater proportion of males than females had studied mathematics at school and at university because they believed they were *good at it*. This result was consistent with previous findings in Australia (Leder & Forgasz, 1992) and elsewhere (Leder, Forgasz & Solar, 1996).

Factors influencing decisions to study tertiary mathematics: The students were asked to indicate which one or more of a list of factors had influenced their decisions to study tertiary mathematics; the option *other* was also provided. Fewer than 5% of the students indicated that the following factors had influenced them: brother (1%), sister (1%), relatives other than parents or siblings (1%), friends (1%), and open days (2%). The results for the other factors listed, and significant gender differences, are summarised below.

teachers	11%	
careers advisor	7%	
father	12%	
mother	6%	(M: 3%, F: 9%)
own decision	68%	
open days	2%	
information booklets	6%	(M: 3%, F: 9%)
other factors	8%	

The students' *own decisions* were considered the most significant factor in deciding to study mathematics (68%). Yet, a rather large 32% of the students did not indicate that the decision had been their own. For many students, mathematics was a compulsory subject; perhaps, given the choice, some would not have chosen to study mathematics.

The data also suggest that teachers and parents had more influence on students' decisions than did careers teachers or advisors. While some males and females appear to have followed their fathers' advice, males seem less likely than females to be influenced by their mothers. Clearly, university open days and information booklets did not have much impact.

Will they stay?

Students responded on five-point scales (strongly agree to strongly disagree) to a number of items tapping their perceptions of tertiary mathematics and of the learning environment in 1996. The proportions of students agreeing or strongly agreeing with each item is summarised below.

easy	19%	lecturers approachable	63%
enjoyable	40%	tutors helpful	78%
challenging	84%	enough individual help	42%
interesting	49%	tutorials useful	74%
useless	10%	assessment fair	63%
well taught	50%	too much work	25%
understood work	54%	confident of passing	55%
lectures boring	43%		

Mathematics at the University of Melbourne was perceived to be challenging and useful by many students. About half the students found the content interesting and considered that the teaching was good but that lectures were boring. Most students were satisfied that their lecturers were approachable, that tutorials were useful and tutors helpful. The work load was not considered too high and assessment was thought to be reasonably fair. However, only about half the students felt that they understood the work, believed they received enough individual help, and were confident of passing.

A degree of challenge in a tertiary mathematics course is considered desirable. While some students (19%) considered the course easy, about 50% said they understood the work and were confident of passing, 63% that the assessment was fair, and a majority (75%) did not consider the work load too onerous. Hence, of the proportion who indicated that the course was challenging (84%), it is difficult to determine how many found their courses too difficult.

Of some concern is the low perception of the availability of sufficient individual help (42%) and that about half the students did not understand the work and were not confident of passing. These three factors may be inter-related. Opportunities for individual assistance are available through the tutorial system. In addition, lecturers and tutors are rostered for several hours per week throughout the year to help students in all subjects offered; more time is made available for subjects with large enrolments. The times and locations are well-publicised: at lectures, tutorials and on noticeboards. Experience has shown that this service is generally under-utilised except in the last few weeks prior to an examination period. The survey responses also imply that sufficient help appears to be obtained from tutorials. Why students do not avail themselves of the other avenues of available assistance remains unknown.

The survey indicated that mathematics had been enjoyed by 66% of the students at school but fewer (40%) indicated that they had enjoyed university mathematics. The differences between the two learning environment and a number of other factors, alone or in combination, may account for the decline in perceived enjoyment of the subject. As indicated by the survey results these may include: perceptions that the subjects (and/or content) are difficult and/or uninteresting; that lectures are considered boring and/or badly taught; that there is insufficient help available; and that students lack confidence.

Implications and Future Directions

Many of the findings discussed above may be implicated in students' decisions about initially selecting and later choosing to continue with tertiary level mathematics studies. The results have several implications for the University of Melbourne's Department of Mathematics, for schools and for the general educational community.

Implications for the University of Melbourne's Department of Mathematics

Before the present study was undertaken, a number of the issues that emerged from the findings of the present study had already been raised within the University of Melbourne's Department of Mathematics. Issues related to the quality of teaching, modes of delivery and assessment, and curriculum evolution, for example, had been discussed and remain high priority concerns. The study's findings highlight the need to re-visit these issues regularly, and confirmed anecdotal evidence and in-house survey results gathered over several years. While every effort has been made to provide ways for

students to seek personal help from lecturers and tutors, the findings imply that these may not always have achieved their purpose. To promote the available opportunities for assistance, frequent reiteration of their availability may be necessary. Continued monitoring of students' reactions to the organised program of assistance is also required and alternatives may have to be considered. Why so few students perceive that their lecturers and tutors have encouraged them to continue with mathematics at higher levels invites further investigation.

Over the years, the Department of Mathematics has encouraged and developed close ties with the school sector. For example, the *Melbourne University Program for Highly Able Students* [MUPHAS], in which students in grade 12 undertake first year university mathematics studies, has a growing number of students and participating schools. The results of the present study suggest other areas in which strengthening and broadening the relationships between schools and the university might benefit a greater cross-section of students. Included are improved careers advice, information booklets, and open days, and finding ways to prepare students better for the transition from school to university learning settings. These concerns are pertinent to all tertiary mathematics departments and schools. The *Australian Mathematical Society* is currently involved in a project to address a range of related issues.

Future directions: The next stage of the study

Students who completed the questionnaires were asked if they would be prepared to be interviewed at a later stage. An unexpectedly large number of students (≈90) volunteered. The interviews will be conducted during 1997. These in-depth interviews should provide some of the answers to the many unanswerable issues raised by the survey data and discussed earlier.

Final Words

At present, the higher education sector in Australia, and science-related faculties and departments in particular, are experiencing tough times. There are many forces competing for the attention, time and efforts of tertiary staff. Yet, the findings of this study serve as a sober reminder that there are several fundamental educational issues that should remain high on the agenda if the number and quality of students within mathematics departments are to increase. While economic considerations cannot be overlooked, cost-effectiveness should not overshadow other considerations. The quality of courses, of teachers, of the modes of delivery, of available personal assistance, of the learning climate, and of school-tertiary relationships may well be related to male and female students' subject choice decisions. It would appear essential for the shorter-term benefit of students and mathematics departments and for the longer-term scientific welfare of the nation that they continue to receive the attention they deserve.

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Notes

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2. Our thanks are extended to the staff and students at the Department of Mathematics at the University of Melbourne for their assistance and cooperation in the conduct of the study.

3. TER: Tertiary Entrance Rank, used to select for tertiary courses

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Relational word problems: A cross-cultural comparison

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Students' difficulty with mathematical word problems such as the 'students and professors' problem has been studied in mathematics education research. The difficulty may lie with the language or with choosing an appropriate schema. Translating these word problems into a language such as Chinese creates a new avenue of research, since the syntax of Chinese is different from the English version. This paper will discuss the results of part of a cross-cultural research masters thesis investigating problem processing of mathematical word problems in Chinese and English. The results suggest that while language appears to be a factor in the processing of relational word problems, the selecting of an appropriate schematic model is also an important factor.

Research has established that the cognitive effort required to process certain mathematical word problems is high in English (Clement, Lockhead & Monk 1981; Cooper & Sweller 1987; Lewis & Mayer 1987; and MacGregor 1991). Research in other languages on mathematical word problems has occurred in Europe (for example, Malle 1993), but it seems little has been done in Asian languages. Cognitive effort may not be so high in languages with different linguistic attributes which may allow for more efficient processing (than English) in certain circumstances.

Cross cultural language comparison between English and Chinese is interesting, particularly in mathematics, because, while the mathematics content is the same, there are notable differences in orthography and syntax. These characteristics of written language have been studied by researchers in cognitive psychology and psycholinguistics to determine the effects of cognitive processing in ordinary prose (Chen & Tseng 1992). In general, Flores d'Arcais (1992) has argued that the available evidence does not seem to indicate dramatic processing differences for words written in alphabetic or logographic scripts. However, Hoosain (1991) has argued that in a particular context and under certain circumstances cognitive advantage of Chinese over English is evident.

In the specific context of mathematics, the reading of sentences is a complex process. Students may read mathematical word problems and understand the sentences but have little idea how to solve the problem. Recent research on the problem solving process has involved two theories. Schema theory (Sweller 1993) associates the problem solving process with the semantic structure of the word problems. Dual coding theory looks at cognition as having two interconnected systems - one an imagery system and one a language system (Dawe & Anderson 1993). In the solving process, whichever theory is used, there appears to be a complex interaction between the words and the syntax, as well as the phonology, since, when solving many word problems, students often vocalise and subvocalise to help gain meaning. This interaction between the words, syntax and phonology may help to access the appropriate schema, building up an image to solve the problem. Many characteristics of a language thus assist in the access to problem processing in mathematics. However, in this paper, differences in syntax which may show differences in processing will be highlighted.

Many of the difficulties students encounter with mathematical word problems are those concerned with relational statements. Relational statements include transitive inferencing problems (sometimes called three term series problems or linear syllogisms) For example, *Ben is taller than George; Dave is shorter than George. Who is tallest?* Here there are two premises which are used to describe a linear ordering of the three terms (Ben, George and Dave). Each premise is converted into a two term array (eg B>G and D<G); then the two arrays must be integrated into a single three term array (eg B>G>D). This integration into the single array, according to Maybery, Bain and Halford (1986) imposes the greatest cognitive load. Sweller (1993) believes this to be intrinsic cognitive load since the load does not vary with restructuring. While we

cannot restructure the sentences much within English, the restructuring carried out when a translation to Chinese is undertaken may give a different cognitive load.

Relational statements are also included in compare problems such as: *There are five more women than men. There are 25 women, how many men are there?* The difficulty with these relational statements has been studied by a number of researchers in America, Australia and Germany (see for example Clement, Lochhead & Monk 1981; Lewis & Mayer 1987; Scholnick 1988; MacGregor 1991; MacGregor & Stacey 1993; Malle 1993; Thomas 1994). The most commonly accepted reason for reversal error was due to the syntactic translation of the problem, but MacGregor & Stacey (1993), Thomas (1994) and Malle (1993) claimed there are other factors involved as well. As problems become more difficult reversal error becomes more evident, suggesting there are two processes occurring, one related to language, the other related to conceptual understanding.

Research by MacGregor & Stacey (1993) on 281 secondary school students required students to formulate equations from problems which lend themselves to correct syntactic translation. However, their results showed a high rate of error. For the question *z is equal to the sum of 3 and y. Write this information in mathematical symbols*, only 57% of students were correct. For the question *s and t are numbers. s is eight more than t. Write and equation showing the relation between s and t*, only 27% of students were correct. They concluded that there was no support for the belief that syntactic translation "is a frequently used method for formulating equations and a fundamental cause of errors". They claimed that the students' reversal errors are due to their attempts to construct mental models of the problem. These mental models do not necessarily correspond with the conventional mathematical code (for example in the phrase *s is 8 more than t* students may associate the 8 with the number *s* since *s* is the bigger number). Malle (1993) also claimed the reasons for errors to be quite complex and that syntactic translation is only used for the simplest of text. His views reflect the idea of MacGregor and Stacey (1993) that the mental model the student constructs does not match the formal mathematical conventional model. So while studies in linguistics suggest that English students use word order as a cue to sentence processing and that students process as they read, in mathematical word problems, students, while trying to make sense of the problem, produce a cognitive model which may or may not match the original linguistic model. When students incorrectly write $6m = w$ for *there are six times as many men as women*, the equal sign may mean correspondence, not equivalence (Malle calls this an "Entsprechungsschema"). With this equivalence model, students, who are then asked to find the number of men, given the number of women, should be able to find the correct answer as long as they do not use the conventional mathematical algorithms, since the internal code of the student does not match the external mathematical one. The results from these studies suggest that the problem may not be purely a language or conceptual problem but a combination of both.

If, in solving the problem, the construction of a mental model is more important than the language, then this problem should be evident in both Chinese and English speaking students although the Chinese may have the advantage if they have better developed visual skills. There may also be differences due to the more practised mathematical skills of the Chinese students.

If however, in solving the problem, language is an important factor then the Chinese[1] version of problems involving reversal error may provide some clues as to the nature of the language problem since the Chinese syntax in the problem is noticeably different from the English version.

One major difference between English and Chinese is the structure of sentences and the status of the verb. Sentences which include verbs are the dominant sentence type in Chinese language (Liuxun & Liu 1993). However there are many common sentences in Chinese where the verb is omitted and the predicate consists of a noun, adjective, adverb or phrase. For example: an English sentence: *Ben is taller than George*, is translated into Chinese as*: Ben bei George gaode*[2] *(Ben compare George tall)*. Here the word *tall* is classified as a stative verb. This type of construction, which is common in Chinese when using comparisons, has implications for processing. The English reader is alerted to **Ben** then the relationship, ie **height**, with the suffix "er"

telling the reader a comparison is being made. In the Chinese version *Ben compare George tall*, the Chinese is alerted to **Ben** then the **comparison** of the two people then the relationship of **height**. The formation of a cognitive image in English is different from that in Chinese, presuming the reading continues from left to right in both languages. The English version creates an image of the tallness of Ben, while the Chinese version creates an image of the two people first. This difference may have important consequences for the development of an appropriate cognitive image in mathematical relational word problems.

The Chinese form of problems where reversal error occurs, tends to be more regular in the construction of these particular problem types. Chinese format in the following way:

Shuzi y shi shuzi x de 8 cheng

Number y is number x's 8 times (the number y is 8 times the number x)

which allows for correct and direct syntactic translation: $y = x8$. Moreover, the use of the word *de* is an indicator of possession and could be seen as a further clue to the positioning of the 8 with the x. However the syntax changes slightly in Chinese for addition /subtraction word problems:

nu huiyuan bi nan huiyuan duo wu

...female members compare male members more five (there are five more women than men)

In this question there is no verb but the adverb is used as a stative verb *duo* (more). (This construction is the same as the construction used in transitive inferencing questions such as *Ben compare George tall. (Ben is taller than George....)*. Here the adverb *tall* is always associated with the first noun.) While at first glance the construction allows for direct syntactic translation: $f = m + 5$, the construction of this type of compare statement may incorrectly associate the five with the female members giving an equation: $f + 5 = m$. The error may be in the correspondence model mentioned earlier as a possible model used in English. While there are a number of other differences between English and Chinese language, a study of the syntactical difference outlined above may give clues to the cognitive processes involved in solving mathematical word problems.

The purpose of the following experiment was to determine whether syntactic differences in relational statements between English and Chinese translate into differences in students' ability to solve these problems.

The Study

Subjects

Fifteen students from Taiwan who were enrolled in a preparatory program were selected for the study. The students from Taiwan had a mean age of 21.6 years (range from 17 - 32). They were preparing for entry into a business or commerce degree. Seven students had completed senior high school in Taiwan and their range of marks for mathematics was between 45% and 85%. Five students had completed high school at a different college with scores in mathematics ranging from 40% to 60%. Three other students were completing their studies in Australia.

All the 15 Australian students' first language was English and were in their first year of a commerce or business degree. The mean age was 19.4 years (range from 17 to 29). Seven students had completed Maths B at school with at least a sound achievement Another seven students completed the equivalent of Maths in Society with a range from low achievement to high achievement. One student had completed school in Malaysia but English was her first language.

Instruments

Two tests were used in this experiment and concentrated on the relational aspects of word problems. Test 1 consisted of ten context questions. Test 2 consisted of ten context free questions to match Test 1. The context free test was administered second so that it would not influence the results of the first test.

The questions were then translated into Mandarin (using traditional script) and input into a Macintosh computer by a Chinese translator. The authenticity and accuracy

of the printout was verified by a research fellow at the university of Taiwan who sets mathematics questions for school students. The final translations were then checked by Taiwanese students at the University of Southern Queensland to ensure clarity. To test clarity of translation, the initial translation of the test was also sent to an institute in China where the test was trialed, using a pen and paper test, on a group of 16 second year students in a business college. In this case, the students were given all the questions and were thus able to change their answers after finishing the test.

Students were asked to read the questions on the computer screen and respond on the computer. A trial test of 5 questions was initially given to the students to ensure they understood the instructions and procedure. Then the two tests were administered. For both tests, one question at a time was displayed on the screen with students pressing the return button when they wanted to go on to the next question. Students were given pencil and paper for any working. During the test, the writer observed the students and at the end of the testing, students were interviewed.

Results and Discussion

Full results of the experiment can be found in Galligan (1997). The results in Table 1 relate directly to relational word problems used - five from Test 1 and two from Test 2. In the experiment there were many errors in both cohorts (75% for the Australian and 68% for the Taiwanese). While the Chinese structure of these sentences is more regular this apparent syntactical aid did not appear to assist Taiwanese students' overall results. The following examples illustrate this.

In question 2.3 (*y is eight times z*) using MacGregor and Stacey's definition of syntactic translation (1993), six Taiwanese and four Australian students used this method. All of the Taiwanese syntactic translations were correct while only two of the Australian responses were correct. This also substantiates earlier experiments where no Australian student used syntactic translation (Galligan 1997). From the results of these experiments, it appears that Taiwanese students use syntactic translation more than the Australian students.

In question 2.8 (*x is five less than y*) the Australian version does not translate syntactically directly but the Chinese version does. While the numbers correct are the same (five for both in second experiment and four for both in an earlier experiment (Galligan 1997)), ten Taiwanese students and six Australian students had reversal error. Only four Taiwanese students used a syntactic translation ($x = y - 5$: all correct). However, instead of using a syntactic model the Chinese version may have prompted the compare model used in transitive inferencing.

The transitive inferencing question: *Ben compare George tall* has the same syntactic structure as *x compare y small 5*. In the former case the word *tall* is linked to *Ben*. Using this compare model would correctly link the *small 5* to the *x*. The seven students' error occurring when putting the two variables into an equivalence statement ($x - 5 = y$).

In question 1.4 (*6 times as many shirts as trousers)*, where correct syntactic translation was available for the Taiwanese but not for the Australians, only four of the Taiwanese correctly translated syntactically. The structure of this question is similar to question 2.3. It is interesting to note that 11 of the 13 mainland Chinese trial group responses chose this syntactic method to answer this question.

The result for question 1.4 is reflected in 1.7 (*eight more TVs than computers)* for the Australian students (3:3), but not for the Taiwnaese students (7:3). With question 1.7 only three Taiwanese students were correct. The Chinese structure of the sentence reflects question 2.8 where the preposition *bi* (compare) is used. However only 2 Taiwanese students had the incorrect solution $x + 8 = y$ and a further 2 students had $8x = y$. These two solutions could again suggest the linking with the model as mentioned in question 2.8. In this question the majority of errors occurred because of misreading of the question.

Table 1: Comparison of Australian and Taiwanese correct responses

	Australian (n=15)	**Taiwanese (n=15)**
Question	**Number correct**	**Number correct**
1.2 Ben is taller than George; Dave is shorter than George - Who is the tallest?	12	14
1.4 There are 6 times as many shirts as trousers in this shop. Use X for the number of shirts and Y for the number of trousers.	3	7
1.7 In a shop there are eight more T.V.'s than computers. Use x for the number of T.V.'s and y for the number of computers.	3	3
1.9 John is older than Fred. John is younger than Ken. Who is the oldest?	14	12
1.10 In a recipe for every cup of flour you use one third of a cup of sugar. Use x for the amount of flour and y for the amount of sugar.	5	0
2.3 Write this information in mathematical symbols: The number y is 8 times the number x	5	9
2.8 Write this information in mathematical symbols: The number x is five less than the number y	5	5

Question 1.10 (*for ever cup of flour you use 1/3 cup of sugar...)* produced different results. The syntactic pattern in Chinese in the previous questions (X compare Y adverb) was not seen in this question *(every use 1 cup flour then use three parts of one cups of sugar)*. Of the 15 Taiwanese students who were wrong in this question, 14 had reversal error. Of the 10 Australian students who were wrong, six had reversal error. There is something in the nature of this question when written in Chinese, which caused particular difficulties for the Chinese students. In Chinese, the syntactic and semantic structure of the question itself is different from the other relational word problems so this question does not easily fit into the schematic model used for the other problems. In the English version, the semantic structure is a little different (using the phrase *for every..)* but the syntactic structure is much the same as the other questions in which direct syntactic structure cannot be used. The other difference between the two translations is the different way the Chinese construct fractions. While Bell (1993) suggests conceptual advantage if fractions are expressed stating the denominator first, no advantage was found in the context of this experiment. The difficulty may lie solely with the language structure of the problem itself, the fraction component being irrelevant. However, it is a factor that cannot yet be ignored.

Structuring the language to allow for syntactic translation as in most of the Chinese questions, may appear to assist in the processing of these relational problems where syntactic translation is very clear as in question 2.3 and 1.4. However, it does not appear to aid processing where the translation is not as transparent. This reflects Malle's argument (1993) where he suggested syntactic translation is used only for the simplest of text.

The simple syntactic translation occurs in all the multiplication questions in Chinese but not in English. However this is not true for the addition questions. In the questions that involve multiplication, the Chinese had the verb *shi (is),* between the two things being compared (*shirts is trousers 6 times).* However, with the *more than* and *less than* statements the word between the two things was *bi (compare)* which is a preposition and the adverb such as *duo (more)* acts as the verb (*computers compare*

TV's more 8). For the 15 Taiwanese students in the two multiplication questions, there were 16 of the 30 questions correct, but for the two addition/subtraction questions only 8 of the 30 were correct. There was no such apparent difference with the Australian cohort (8 correct for both). The difference could be due to the different verb used - the *shi* prompting the correct model. The different cues used in sentence interpretation (ie the pre-verbal positioning to identify the subject in English and the post verbal cue to identify the object in Chinese) may allow Chinese readers to more correctly interpret the multiplication questions but a more extensive study would have to be undertaken to examine this more closely. Perhaps with an easy syntactic translation, Chinese may be able to pick up the model more easily than the English. There is also evidence that the Taiwanese students attempted to formulate equations more often than the Australian group (85% compared to 61%). Either the students have read the questions more carefully and hence knew they had to write an equation, or they are more expert in maths in that the word *fang cheng shi (equation)* prompted a response such as $x = 6y$ more often than their Australian counterparts. In their attempts to write more equations, the Taiwanese cohort may have used reversal error more often. But as other variables intervene, such as change in language, and order of presentation of questions, then similar errors occur in both languages. Moreover, when the question becomes more dissociated from a known model, error rates increase. Question 1.10 in the Chinese version is further from the model than the English version.

Conclusion

While the structure of relational word problems in written Chinese appears to advantage Chinese readers, in the experiments this advantage was not evident, providing further support for the suggestion of an appropriate cognitive model as the main cause of student error in relational word problems. The implication for educators may be an increased emphasis on developing in students an appropriate cognitive model to solve problems, rather than concentrating on the language difficulties of the problems themselves.

This study also provides a greater awareness of the difficulties and dangers in translating mathematical word problems and comparing results from different language groups. Just getting the language right does not necessarily mean the content conveyed by the language is the same. In cross - cultural mathematics tests, such as the recent Third International Mathematics and Science Study, this needs to be taken into consideration.

Notes

1. The Chinese language described here refers to Mandarin, the official language of mainland China and the the language used in the experiments was traditional Mandarin, the language used in Taiwan.
2. Full Chinese script version and tone marks for pinyin are not included here but are available from the author.

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The effectiveness of a professional development program based on action research

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This study documents a school-based professional development activity conducted in a secondary school in the Philippines where five secondary mathematics teachers and the head of school participated in a program of action research. Qualitative research techniques were used to provide rich information on the changes made in teachers' pedagogical knowledge, practices and beliefs. Within the parameters of certain constraints, it emerged that action research as a form of professional development led to important changes.

Introduction

Major changes to the curriculum in the Philippines in the past eight years have led to significant professional development programs, particularly in secondary education. These professional development activities were mostly in the form of seminars or workshops where teachers attended designated training centers. This usually required teachers to travel, which entailed time away from school, and expenses such as transportation costs. While benefits from in-service training of this kind exist, it appears that some writers are questioning such approaches (Castle & Aichele, 1994; Fullan, 1991). Alternate approaches to professional development that are less costly in terms of valuable resources and which have resulted in documented benefits may better suit the needs of the Philippines. In this context, an investigation was conducted into the effectiveness of school-based professional development program based on action research. In particular, the study aimed to determine how the teachers' pedagogical knowledge, practices and beliefs are affected by their participation in this form of professional development, and to identify the constraints in using this approach.

Method

Five secondary mathematics teachers from a public secondary school outside of Metro Manila took part in this study. Four of the teachers taught at one of the four year levels of mathematics offered at the school, and the fifth teacher (who was the department head) taught at the third year level of mathematics. The head of school also participated in the study as an observer and facilitator.

At the time of this study, only two of these participating teachers had attended a one-week seminar on secondary mathematics teaching using the new curriculum that had been fully implemented in 1992. For two of the teachers, it was their first time teaching mathematics in their assigned year level. In the past, they were either teaching mathematics at a different level or teaching in another discipline such as language. All the teachers were taking classes of 50-80 students and had either eight or nine 45 minute lessons a day, except for the department head who took four lessons per day. This meant that they were teaching at least six hours a day for five days.

Action research was chosen as both a vehicle for promoting teachers' professional development and as a method for determining possible change. Action research is a cyclic approach aimed at improving practice through planning, acting, observing, reflecting, and revising (Kemmis & McTaggart, 1988). Cohen and Manion (p. 194, 1994) said that for "inservice development of teachers— improving teaching skills, developing new method of learning, increasing power of analysis, of heightening self-awareness" action research is an appropriate method. Similarly, Kemmis and McTaggart (1988) argued that action

research is a method designed to achieve both change and learning from the consequences of the change.

The study was carried out in four phases during the period July 1995 to February 1996. The first phase involved the selection of a suitable school and participating teachers. The second phase required the assessment of teachers' perceived needs and conditions within the school. The third phase involved the implementation of the professional development program, and the last phase consisted of an evaluation of the project.

The selection of the school was based on two factors: the presence of a teacher who had attended the Philippines Australia Science and Mathematics Education Project (PASMEP) training; and the proximity of the school to the principal researcher's office or home. It was assumed that the presence of a PASMEP teacher would be helpful in facilitating the change which teachers would like to make in their school. PASMEP teachers have attended a wide range of professional development programs that have included a nine month training period in Australia and several training workshops at the University of the Philippines Institute for Science and Mathematics Education Development (UP-ISMED). As the time and distances of travel to schools in the Philippines can be extensive, the proximity of the school to the researcher's home or office was an essential consideration.

The assessment of the existing conditions, which was the second phase, was done through questionnaires, interviews and class observations. In the questionnaires that were administered by the researcher, teachers wrote information about themselves and their professional development needs. The department head and the PASMEP teacher who happened to be the head of school, completed a different set of questionnaires. Individual and group interviews were conducted by the researcher using the Filipino language to informally identify teacher's beliefs about their teaching practices. Classroom observations were carried out by the researcher for six weeks. In each of the first two weeks, each of the participating teachers was observed in at least three consecutive lessons. Visits to these lessons were announced; additionally, one of these lessons was recorded on video. These recorded lessons were viewed with the group and other teachers from the mathematics department where reactions were sought. The rest of the class observations were generally done by the researcher without informing the teachers when they would occur. A written description of all the observed lessons was made.

The third phase involved the group conducting an action research program aimed at developing teachers' professional practice. This phase involved attending a number of planning meetings. A brief outline of action research was discussed at the first meeting. The teachers then identified their own problems (Herrington, Sparrow & Swan, 1995) and agreed to initiate an action research program over three cycles that focused on increasing students' participation in mathematics through the use of practical work. To monitor and evaluate progress of the program, several documents were gathered. These documents were the teachers' diaries based on their lessons using practical work, their diaries when they participated in the meetings and their lesson plans. The researcher and, in some instances, the head of school made class observations. During three cycles of the action research, seminars were conducted in which the topics discussed were chosen by the group of teachers.

The last phase which was the final evaluation, was done after the third action research cycle. The group of teachers met to reflect on what was achieved through the project and on the impact of the project on their professional growth.

Results and Discussions

This section will consider results as they relate to determining the effectiveness of action research as form of professional development and the constraints in using this form of professional development in the Philippines. The effectiveness was looked at in terms

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the changes made by these participating teachers in relation to pedagogical knowledge, practices and beliefs.

A range of techniques was used to determine the effectiveness of the professional development program. Initially, each participant was given a questionnaire to identify perceived needs. Semi-structured and unstructured interviews were used, either individually or in groups to gather information on teachers' beliefs. In addition, the principal researcher, together with the head of school and in some cases the participating teachers made announced and unannounced classroom observations. Audio and video recording and also still photographs were taken in some of the classroom observations and group meetings to capture the situation and to provide a basis for discussions during meetings.

The qualitative data that were gathered in this study were analyzed using the NUD.IST (Non-numerical Unstructured Data Indexing, Searching and Theorising) computer program. Categorization of the data was guided by the research questions and an initial reading of the data. When the categories were determined the data were organized on an index "tree" and the nodes of this tree were explored to identify any patterns that emerged. The analysis provided a picture of change for each individual teacher and for the group as a whole.

The results of the analysis for the group as a whole and consequent discussion are described below in two parts: changes in teachers' pedagogical knowledge, teaching practices and beliefs; and the identified constraints to change.

Changes in the pedagogical knowledge, teaching practices and beliefs

In this study, pedagogical knowledge refers to the teacher's knowledge or awareness of strategies for teaching mathematics. Teaching practices refer to the action of the teacher as they carry out tasks such as teaching routine exercises, classroom management and the use of resources. The beliefs of teachers relate to beliefs about mathematics, the teaching and learning of mathematics and professional development.

In the first phase of the study, it appeared that teachers were familiar with exposition as the only strategy for teaching mathematics. They discussed this in one of the meetings.

Teresa:	What we are using, it's always the same.
Cindy:	That's it, just teacher exposition. You can't.
Researcher:	What else did you use, when I didn't observe you?
Cindy:	Only that, right?
Teresa:	Yes, because that's the only one.

(Planning meeting)

Classroom observations indicated that the teachers followed a similar expository approach. The teacher would talk in front of the class for most of the time, with occasional use of blackboard and chalk. Students participated by answering the teacher's closed-type questions or going to the board to show the solution to the assignment or practice work.

In the later part of the study, teachers mentioned the changes they had made. An interesting one was on the use of blackboard and chalk.

> Before when we didn't use activities, we just use chalk and board and sometimes manila paper.
>
> (Teresa, final group interview)

> At least I've learned other strategies aside from chalk and board, that's one thing. And also there's an inclusion of humor in mathematics. Not so much now that the students would think of me, "naku, math yan, patay"! [oh, my that's math, scary!].
> (Cindy, final individual interview)

> There's a big difference. It was good. Before, it was blackboard and chalk ... Before they [referring to students] were involved, but less and individually.
> (Art, final individual interview).

These teachers indicated that the use of blackboard and chalk had lessened because of the inclusion of hands-on activities for students. It was apparent that the teachers had become aware that there were alternative teaching strategies to exposition.

The beliefs of the participating teachers about mathematics, mathematics teaching and learning, and professional development were considered. In the initial interviews, some of the teachers mentioned that in mathematics teaching, the teacher is the "supplier" of knowledge and students are just the "receivers". However, this belief seemed to change as this study progressed. "We have this before that *I'm sir* so you can't do that to me. It was like that before. But now you can also accept from a student" (Teresa, Final group interview). This can be interpreted to mean that before, students should consider teachers as people in authority who cannot be questioned. Now the teachers seemed to welcome students' ideas. They talked further on this issue.

Bert:	A student could also give you knowledge, right?
Cindy:	Give and take.
Bert:	Yes, that can be, remember? Like that one ...
Cindy:	It's now give and take. But our role as a teacher, that's the basic ... Because, suppose to be it should really come from us. But this does not necessarily mean that we can't learn from the students. The teacher could also learn from the students as long as the students could learn from them.
Researcher:	So now, what do you think? How about you, Teresa?
Teresa:	Facilitator.
Researcher:	Facilitator. What do you mean by facilitator?
Teresa:	You have to guide the students.

(Final group interview on beliefs and practices)

It emerged from the above discussion that at this stage, they believed that students could learn from both their teachers and classmates. Teachers could also learn from their students. In this way, the teacher's role was viewed as a facilitator in the classroom. In learning mathematics, some of the teachers' beliefs in the early part of the study had to do with students' approaches towards learning. They believed that students should listen and be attentive to the teacher's discussion. They should also try to solve more problems or exercises and do more practice. Towards the end of this study, they appeared to have changed their beliefs on mathematics learning.

> Learning? Ah- they can easily understand because they saw it on the figures, in your examples, materials. They've seen it and they even got ideas from it. Before, math learning, when they entered [the classroom], what's on the blackboard, that's it. They just listen, they didn't have participation. Unlike now, they even volunteered [to participate]. (Alex, Final initial interview on beliefs and practices)

The teachers now appeared to believe that students could be encouraged to participate in the learning of mathematics by varying the teaching strategies such as the inclusion of activities. These teachers also changed their views, not only on how students learn, but also how they learn. With their involvement in this study, they were exposed to different forms of professional development activities. When asked on their comments about action research as a form of professional development, these were their statements:

> Cindy: Much better! Because the time allotment, more time to really absorb it. You could absorb what you do and really you could try it.
>
> Bert: But, ma'am [referring to the head of school], it's really better to have an in-service which is like this, the one we're doing. It seemed informal. As if we're just telling each other a story ... As if there's professional sharing, you're really reporting. As if the impact is different, it's "nakakatuwa!" [good feeling!]
>
> (Cycle 3 meeting)

These teachers seemed to view this form of professional development activity as better than the ones they were involved in the past because of the longer time allotted for them to work and reflect. Also, the informal way of organizing the group discussions seemed to suit them enabling them to talk freely about their ideas.

Constraints

In discussion with the teachers, four main areas on the constraints to change were identified: personal constraints, classroom constraints, system or institutional constraints and students' constraints.

It emerged that one of the most frequently mentioned personal constraints was the teachers' lack of experiences in using activities with their students. For most of the teachers, this was the first time they had used activities with their classes. Consequently, they found it difficult to prepare materials, to find a variety of activities, or to decide which ones were suitable for the type of students they had. Their attitude towards change could also be considered as one of the constraints. At times, resistance to innovations was observed. For example, an activity was suggested but was not tried. Their hesitation in trying to make changes in the teaching practices could also be attributed to their understanding of the subject matter.

> But sometimes I want to try, but then I became negative. It might be that my teaching will fail, right? It's a mistake! Or I would think if it's related to my topic. Sometimes I'd like to do something but I didn't do it because my mind is confused. Also sometimes I have something I'd like to teach, but it seemed I don't know. I would think, what if the students will question me about it? At times my students would ask me and I don't know, especially my students in Sampaguita, there are so many of them there that would ask you.
>
> (Teresa, Final individual interview on beliefs and practices)

Teachers may have also resisted change in not wishing to deviate in their "comfort zone". These teachers were used to a classroom environment where students sat quietly and appeared to be listening. Teachers were unaccustomed to noise being made from a group activity or students' discussion. In one of the teacher's diary, he wrote: "Students really knew how to evaluate logarithms. Only on the last part of the contest, the teacher was so irritated on the arguments of the students" (Bert, Class diary). Teachers also

appeared unaware of what could be changed in themselves when teaching mathematics. This was shared when the group viewed the video lessons that were taken during one of their teaching sessions.

> The one which is good first. Here, I changed my style with my students. I saw myself. The first thing, when there was this video session, you saw yourself. So, this is how I teach, that can be changed. [Laughter from the group.] That's true, yes! ... I realized it. Oh, there is still something to change in me! [Laughter from the group.] ... Sometimes it's funny and also frustrating when I saw myself. (Art, Cycle 1 meeting)

It appeared here that the teacher considered this a "good" way to critique himself and to identify the changes to be made in his teaching.

It also appeared that teachers' willingness to change their teaching practices was also hindered by these two classroom constraints: class size and the space to do it. The class size would vary depending on the year level. For a large class, a teacher would find it difficult to conduct group work or do an activity that enabled sufficient materials to be given to all the students.

> For example, you have a group activity, so you have to group the students first. Just in grouping them, all the time has been used ... The other constraint is the class size, it's too large. They're too many but the room is small. About seventy five students in a class on the average.
>
> (Teresa, Final individual interview on teaching and changes)

This large class size also created a problem of space for a teacher to monitor students' work during group activities and also to store the materials that they used in these activities.

Teachers indicated that time, materials, workloads and students' groupings are some of the system or institutional constraints to change. The 45 minutes allotted to a mathematics lesson was seen as inadequate. They said that some lessons could not be finished, more so if they used group activities. Some of them believed though that the use of activities was helpful in students' learning. However, although they have this belief, there is a tendency for them not to use it as often as they want to because they felt an "obligation" to finish the syllabus.

> We're following the DLC [Desired Learning Competencies]. If I couldn't finish the activity, the next day I make it as a review. So what happened, I wouldn't think of other activities because I'll be left behind. So as if that's a constraint. The constraint is really in time.
>
> (Teresa, Final initial interview on teaching and changes)

Further time pressure occurred with school activities such as sports meetings, periodic testing, a drugs symposium, or suspension of classes due to typhoons and flooding. The availability of materials was mentioned as another constraint. The school had no library facilities nor a mathematics laboratory when this study was conducted. It was not surprising then that teachers would mention this as one of the constraints to changing the way mathematics should be taught. It was even learned that mathematics textbooks for fourth year students were not available. To overcome this, the teacher formed groups of students where each group would "give" one mathematics book to be used by all. Also, the school did not provide teachers with the materials needed for the activities and sometimes buying it from their own budget seemed impractical. As one teacher remarked in one of the earlier meetings "That one, the one where you have to

connect the sticks. I said it's good to do it, but I can't, no materials. It's really frustrating!" (Cindy, Cycle 1 meeting). She made a similar statement in one of the final interviews: "And sometimes, too, there's one that I would like to use but the materials are expensive, so that's a real constraint" (Cindy, Final initial interview on teaching and changes).

Heavy workload was mentioned as a further constraint. This includes teaching loads and involvement in other school activities and roles such as scout chairman, sports trainer and so on. They indicated that their heavy workload, usually at least six hours of teaching a day, affected them physically.

> Ah, for me, mostly the constraints will still be on the preparation. For example, I'd like to do it, but you know that we have 9 loads, it's really difficult for me. We really don't have the time to do it. At home, supposed to be this is our time to rest, but we still prepare the activities. You even observed it, that really time is not enough.
> (Bert, Final initial interview on teaching and changes)

Students' background, attitudes and language emerged as further constraints to change. Students' content background created a problem for teachers to make the desired changes in their teaching of mathematics. They observed that some students lacked the necessary skills to understand the concepts that were discussed at their year level. Students' lack of exposure to different teaching styles also became a factor for teachers wanting to make changes. The usual classroom environment can be described as students appearing to be listening to teachers. So, changing the environment to allow for students working in groups and using manipulative materials for the activities, required an adjustment period for both the teachers and the students.

The observed attitude of students towards mathematics caused another problem for teachers wishing to make changes in their teaching. Teachers observed that students did not like mathematics and displayed negative attitudes towards the subject. "They don't have the initiative. What they know is that, it's math and they are scared of it" (Teresa, Initial group interview on beliefs and practices). Language was another difficulty that was mentioned. It was observed that there were many times when students could not proceed with a given activity because they could not understand the instructions. Teachers are often obliged to give further instructions, sometimes using bilingual language, that is in Filipino and English.

> Because the usual problem in Math, English is the medium of instruction, so sometimes you'll be forced to speak in Filipino [the national language] or else they would not understand.
> (Cindy, Initial individual interview on beliefs and practices)

Comprehension of instructions by the students was also considered by teachers as a constraint in activity work. Some of these teachers mentioned that this could be because the instructions were not clear or students were not used to following instructions on their own.

Conclusion

The results indicated that the involvement of these five teachers in the action research had an effect on their professional growth. There were noticeable changes in their pedagogical knowledge, teaching practices and beliefs. They all became aware that mathematics in secondary schools could be taught in different ways such as the use of practical work. They recognized alternatives to the use of blackboard and chalk and teacher-centered lessons. Teaching practices also accompanied this change. It was observed that they used a variety of resources and allowed students to do group activities. Teachers' beliefs also appeared to have changed. For example, they changed their view of

mathematics teaching from one based on a teacher-centered classroom to one where there was sharing of learning between a teacher and students. Additionally, teachers seemed to prefer an action research approach to professional development. Several constraints were mentioned by these teachers while they were implementing the changes. The teachers' main concerns arose from their students, classrooms and the educational system. Some of these concerns were also mentioned by Ulep (1993) as significant problems in the teaching of secondary mathematics in the Philippines. These identified constraints could be taken into consideration when planning for professional development programs or conducting further research on secondary mathematics teaching.

In the Philippines context, the budget for professional development activities is limited. With the approach adopted in this study, expenses were minimal. No accommodation costs and transportation costs were involved so access of everyone to this form of professional development activity was possible. In addition, interruptions to classes were minimal. It may be worth comparing the costs associated with different forms of professional development required to demonstrate equivalent degrees of professional growth as shown by these teachers.

Finally, the research has demonstrated that the action research process as described by Kemmis and McTaggart (1988) can be successfully carried out by secondary mathematics teachers in the Philippines and this form of professional development has positive effects on teachers' professional growth. This is quite different from many of the professional development programs that have been conducted in the Philippines. These programs were often held in venues where expenses for accommodation and transportation are needed, resulting in only a few teachers gaining access. Furthermore, the focus of the training may not be what is needed in the school where the teacher originates. It would be worthwhile then, for professional development designers to look more closely into other forms of professional development activities that directly address the needs of the teachers. Action research programs appear to offer a number of significant advantages.

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Students' Orientations to Learning Statistics – Profiles of Experience

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In this study I look at university students' orientations to learning statistics – how they feel about learning it, their conceptions of the subject matter and their approaches to learning it. I use complementary methods of analysis to understand the relationships among students' appraisals, conceptions, approaches and attainments on assessments. The findings reveal underlying dimensions of the variables and present dramatically different profiles of students' experiences. Students' learning is linked to a complex web of personal, social and contextual factors.

Introduction

The literature on statistics education shows that many students have a poor understanding of statistical concepts (for example, Garfield & Ahlgren, 1988). One aspect that is likely to contribute to students' difficulties in the subject is the way students constitute the subject matter as a whole — their understandings of the content presented and how they experience learning the subject in the context in which it is presented. Previous research on the quality of students' learning of mathematics or statistics at university indicates the need to explicitly address the relationship between the student and the learning context (Crawford, Gordon, Nicholas & Prosser, 1994; Gordon, 1995; Williams 1993). Hence, an important step in furthering understanding of student learning within the subject field, statistics, is to consider students' views of statistical knowledge and how these relate to other aspects of their learning and performance.

In this study I look at university students' orientations to learning statistics as a compulsory component of second year Psychology — how they feel about learning it, their conceptions of the subject matter and their approaches to learning it. Preliminary findings have been reported in Gordon, Nicholas and Crawford, (1996). In this paper I build on my earlier analysis. I use complementary methods of data analysis to understand the relationships among students' appraisals of learning statistics, their conceptions of the subject matter, approaches to learning and attainments on tests and examinations. Factor analysis is used to explore the relationship between the variables. Cluster analysis shifts the focus from variables to students, identifying groups of students with similar profiles of experiences in learning statistics. The findings show striking trends and inter-dependencies among the variables and reveal cogent differences in students' experiences. Vignettes from interviews with selected students are used to illustrate two different profiles. These indicate that while the subject matter presented, institution and teacher were the same for the students surveyed, the students' awareness about their learning links a complex network of individual, social and contextual variables.

Method

Second year Psychology students at the University of Sydney were asked to complete a survey on learning statistics. The survey was completed halfway through semester one, during the statistics lecture, with the kind cooperation of the lecturer concerned. The survey consisted of questions relating to demographic variables, such as age, gender and prior level of mathematics studied, three open ended questions and a Likert type questionnaire — the Approaches to Learning Statistics Questionnaire. The study was approved by the University of Sydney Human Research Ethics Committee. The three open ended questions were as follows:

Question 1 *Would you study statistics if it were not a requirement of your psychology course? Please give reasons for your answer.*

Question 2 *Think about the statistics you've done so far this year.*
- a) *How do you go about learning it?*
- b) *What are you trying to achieve?*

Question 3 *What in your opinion is this statistics course about? Please explain as fully as possible.*

These open ended questions were followed by the Approaches to Learning Statistics Questionnaire (ALSQ). The questionnaire was modified from the Approaches To Learning Mathematics Questionnaire (Crawford et al, 1995), which in turn was derived from the Study Process Questionnaire (Biggs, 1987). The ALSQ consists of two scales, the Deep Approaches Scale and the Surface Approaches Scale. An item from the Deep Scale (item 24) and one from the Surface scale (item 7) are shown below.

Item 24. I believe strongly that my aim in studying statistics is to understand it for my own satisfaction.

Item 7. I learn some things in statistics by rote, going over and over them until I know them by heart.

Interviews with selected students were carried out in order to flesh out and clarify the survey responses.

Participants

The survey was completed by 279 Psychology II students. Almost three quarters of them (203 students) were female. Most of the students were either 19 years old (48%) or 20 years old (18%), though a sizeable minority (16.5%) were at least 25 years in age.

The prior level of mathematics studied by the surveyed students was higher than is often assumed for psychology students. Almost one quarter (68 students) of those surveyed had studied mathematics at university, most of these having completed a first year mathematics subject at the University of Sydney. Of the rest, almost all (163) had written the final school examination in mathematics for New South Wales, called the Higher School Certificate (HSC), at one of the three highest levels of the five levels offered in mathematics. These three levels include the study of Calculus. Twenty six students had not taken mathematics for the HSC. These were almost all mature students, with twenty two of them being 25 years old or older.

All the surveyed students had completed a brief (about five weeks) introductory course on statistics in first year Psychology.

Results

In what follows, the term "Statistics" with a capital "S" will be used to denote the statistics component of second year Psychology being studied by the participants in this research project.

Willingness To Learn Statistics

The teachers of this topic have an unenviable task. Most students (73%) reported that they would not have studied Statistics, had they been given a choice. This finding complements research by Ainley et al (1994). They investigated the subject choice of Australian students in the final two years of school. They found that those who reported a high level of social interest, defined as interest in nurturing or helping others, had low interest and participation in mathematics and the physical science subjects.

The reasons reported most frequently by the students who expressed willingness to study Statistics, were that statistics was useful — for the study of Psychology or other subjects or higher degrees, for research or for general "real world" applications. On the other hand, students who were reluctant to study Statistics reported their reasons for this in terms of its lack of appeal — to them the subject was boring and hard and they did not enjoy it. Hence, rather than reflecting opposite sides of the same coin, the reasons given by students willing to study Statistics related to extrinsic factors, while the reasons given by the reluctant majority related to personal or intrinsic considerations.

This has important implications for the teaching of statistics as a service course. If educators stress the usefulness of the subject to their fields they may be preaching to the converted. Students' subjective evaluations in terms of interest and affective elements need to be addressed too.

Conceptions Of Statistics

Five categories emerged from the analysis of students' responses to the open ended questions, particularly Question 3. I called these categories NO MEANING, PROCESSES, MASTERY, TOOL and CRITICAL THINKING. Table 1 summarises the Conception

categories and illustrates each category by means of a representative excerpt from a survey. (See Gordon, Nicholas & Crawford, 1996, for more details.) With the exception of the first category (NO MEANING), the categories for students' conceptions form a hierarchy in terms of inclusiveness. That is, I am inferring that for categories 2 to 4, higher categories logically include the awareness expressed in lower categories. A response classified in a higher category indicates to me an awareness of the conceptions described in the lower categories, but shows added understanding. This is a logical hierarchy. Empirically, too, many students' responses indicated a conception of Statistics that encompassed two or more of these four categories. In these cases their conceptions were classified into the "highest" of the categories indicated. Finally, although the merit of the conception was not used to determine the hierarchy, the higher categories can be seen to express more educationally desirable conceptions of Statistics than the lower ones. Statistics is taught as a service course. The teachers of it hope that students will find it a useful tool and one that will have impact on the students' lives and continued learning.

TABLE 1
CONCEPTIONS OF STATISTICS : LEVELS OF AWARENESS

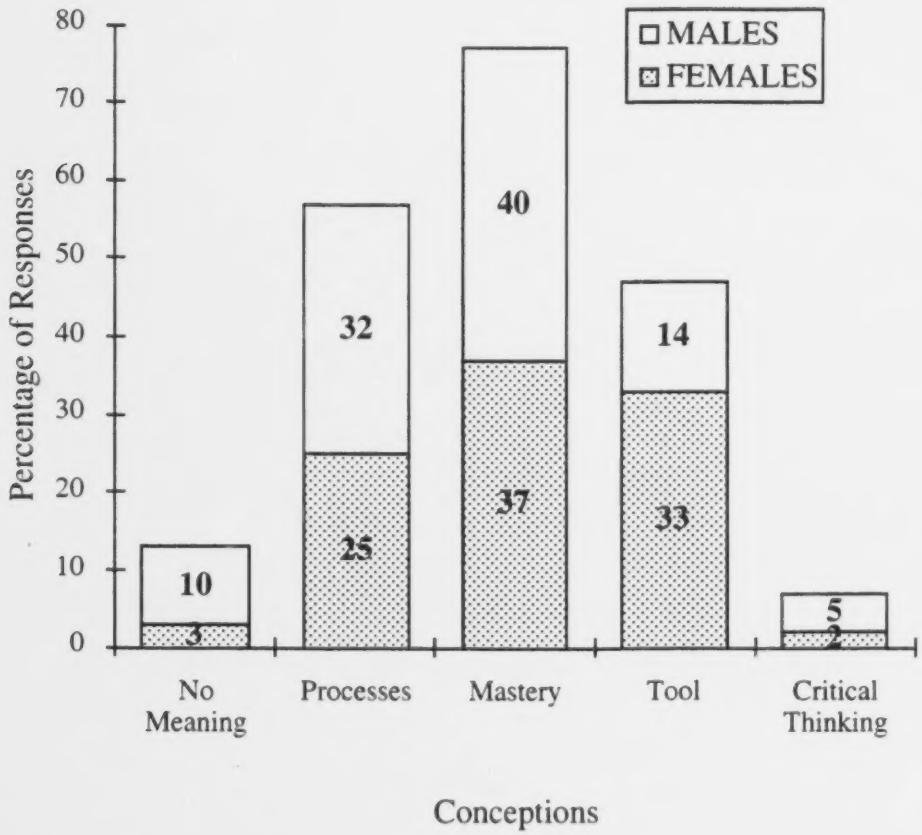
CATEGORY	VIEW OF KNOWLEDGE EXPRESSED BY CONCEPT	EXAMPLE OF RESPONSE
1. NO MEANING	imposed, irrelevant	*You tell me, I just learn.*
2. PROCESSES Mechanical techniques or algorithms	disconnected, for reproduction in assessments	*... You don't have to understand how it works, just be able to get the right answer.*
3. MASTERY reading and understanding material as presented in class	information to be accumulated and stored in order to meet the demands of assessments	*I am trying to achieve a basic understanding of the material & concepts & an ability to work out the problems. This will hopefully lead to a good result at the end of the year.*
4. TOOL knowledge for use	could be applied in real life	*It is trying to teach how statistics is used in everyday research in psychology. It's relevant to most future careers.*
5. CRITICAL THINKING a perspective on the ways in which data is used to make decisions.	a way of thinking and communicating; insight into the complexity & limitations of statistical theory	*Stats. is about methodology which is used as a comprehensive form of analysis to interpret and test theories psychologists create.*

Distribution of Responses The distribution of students' reported conceptions into the five categories is shown below in Figure 1. The percentages shown in the bar graph are presented separately for males (N=63) and females (N=186). As can be seen, the modal category for both males and females was MASTERY. Only six students (3 male, 3 female reported conceptions of Statistics which were classified into the highest category: CRITICAL THINKING.

The gender differences are particularly interesting here. Male students tended to see the Statistics in terms of academic exercises. Only 14% of them (9 students) compared to 33% of the females (62 students), perceived the subject as applicable to "real life" (TOOL).

FIGURE 1

DISTRIBUTION OF MALES' AND FEMALES' REPORTED CONCEPTIONS OF STATISTICS



Percentages rounded to whole numbers.

Performance In Assessments

The final mark for Statistics was the average of the following four components: the class mark and examination mark for semester one and the class mark and examination mark for semester two. The examinations both consisted of multiple choice questions while the class marks were based on open book quizzes. Males outperformed females in Statistics achieving a final mean score of 63.9% compared with the females' mean of 56.6% ($t=2.58$, $p<0.05$). This difference was due mainly to the superior performance of the males in the multiple choice examination in semester one. One finding explodes the myth that ability in the general area of Psychology and prowess in Statistics are mutually exclusive. There was a strong correlation of 0.62 ($p<0.001$) between students' final marks in Statistics and their average marks for the other areas of Psychology II.

Differences in Students' Experiences of Learning Statistics

Females scored significantly higher than males on the Surface scale of the ALSQ ($t=-3.2$, $p<0.01$). Scores on the Deep scale did not differ for males and females.

On average, scores on the Deep scale increased with each increase in category of Conceptions (from 1 to 5). The mean scores on the Surface scale decreased with each increase in Conception category, with the unexpected exception of the NO MEANING

group. I had anticipated that the students who conceived of Statistics as a meaningless and pointless subject would tend to adopt surface approaches to learning it. This proved not to be the case, although these students did have low scores on the Deep scale. This anomaly led me to inspect more closely the NO MEANING students' responses to the second open ended survey question, which asked them about their strategies for and intentions in learning Statistics. This revealed that these students evidently did not engage with the task. They were not attempting to memorise or rote learn the material presented, they simply made the minimal effort required. As one of them wrote in response to Question 2a, her strategy was to: "ignore it until I have to study for a test".

I then carried out a factor analysis to determine the separate constructs underlying a number of variables in the study. The following variables were included: Concept categories 2 to 5 (excluding the NO MEANING category, since it is not logically or empirically part of the hierarchy as supported by the previous finding), willingness to study Statistics (CHOICE: N0 1, YES 2), marks in tests and examinations (CLASS1, EXAM1, CLASS2, EXAM2), scores on the deep and surface scales of the Approaches to Learning Statistics Questionnaire (DEEP, SURFACE), Age in years (AGE) and gender (GENDER: MALE 1, FEMALE 2).

Four factors were extracted, using a principal components analysis with varimax rotation. These had eigenvalues 3.82, 1.63, 1.18 and 1.10 respectively. The four factors accounted for 70.3% of the variance (34.7%, 14.8%, 10.8% and 10.0%). The rotated factor matrix showing the loadings of the variables on the factors is shown in Table 2.

TABLE 2

ROTATED FACTOR MATRIX

	PERFORMANCE	ORIENTATION	GENDER	MATURITY
CONCEPT (4 categories)		78		
CHOICE (No 1, Yes 2)		71		
SURFACE		–67	31	
DEEP		67		
EXAM1	81			
EXAM2	82			
CLASS1	84			
CLASS2	83			
AGE				92
PRIOR MATHS		51		–51
GENDER (M 1, F 2)			93	

Decimal points and loadings less than 0.3 omitted. Decimals correct to two places.

As can be seen from Table 2, the first factor links the variables concerned with performance on assessment tasks. Interestingly, neither deep nor surface approaches loaded to any notable extent on this factor. The second factor shows the positive relationship among the variables for students' conceptions of Statistics (for levels 2 to 5), their willingness to study it, the level of mathematics studied previously and their scores on the Deep scale of the Approaches to Learning Statistics Questionnaire (ALSQ). Scores on the Surface scale of the ALSQ loaded negatively on this factor. The third factor indicates a link between females and surface approaches to learning Statistics, consistent with the finding reported earlier in this section. The final factor indicates that higher age is related to a low background in mathematics. This was the case, as described previously. I have interpreted the four factors as PERFORMANCE, ORIENTATION, GENDER and MATURITY.

In summary, an important outcome of this analysis was to establish links among: motivation to learn Statistics, higher levels of awareness about Statistics, better preparation in mathematics and deep approaches to learning, opposed to surface approaches. Performance on assessments emerged as a separate dimension to orientation. The factors derived are highly interpretable and consistent with findings arrived at by other methods.

Cluster analysis is a way of identifying subgroups of students (rather than variables) on the basis of similarity of scores on chosen variables. In order to complement the factor analysis by relating students' performances on assessments and their approaches to learning Statistics, clustering was done on the standardised variables (Z-scores) for: CLASS1, EXAM1, CLASS2, EXAM2, DEEP, SURFACE.

Profiles of clusters: Qualitative investigations (interviews and responses to open ended questions) suggested four different profiles of achievement and approaches. These were identified by cluster analysis (see Table 3). Differences between the means for the four groups were considerable and statistically significant ($p<0.001$) on the clustering variables..

TABLE 3

STANDARDISED MEANS FOR CLUSTERS ON ASSESSMENTS AND APPROACHES TO LEARNING STATISTICS

MEAN SCORES (Z scores)	CLUSTER 1 High Achievers Above Average on Deep scale (N=47)	CLUSTER 2 High Achievers Above Average on Surface (N=61)	CLUSTER 3 Low Achievers Above Average on Deep (N=42)	CLUSTER 4 Low Achievers Above Average on Surface (N=61)
*Final Statistics Mark	1.03 (77%)	0.62 (70%)	–0.36 (52%)	–1.12 (38%)
Z-Score: Deep Scale	0.98	–0.41	0.10	–0.36
Z-Score: Surface Scale	–1.06	0.38	–0.53	0.71

*Raw means in brackets. Mean mark in Statistics for surveyed students=58%, SD=18.5%.

The profiles of the first two clusters are particularly interesting. While both these groups of students achieved high marks in Statistics, attaining averages of 77% and 70% respectively, their characteristic approaches to learning were very different (see Figure 2).

Cluster 1 is characterised by students who reported adopting primarily deep approaches to learning Statistics, rather than surface approaches. Unlike the other three clusters, most (54%) of the students in this group expressed willingness to learn Statistics. The most common category for their conceptions was TOOLS (53.5%).

The students classified in cluster 2 evidently had a facility for learning Statistics, while lacking the motivation or awareness of the subject of their colleagues in Cluster 1. Most (77%) of them reported reluctance to study Statistics. Their modal category of Conception was PROCESSES (39% of them), followed by MASTERY (32%).

Clusters three and four performed, on average, at a much lower level than the first two groups, attaining averages of 52% and 38% respectively. The profile of the Cluster 4 students is particularly dismal with 93% reporting that they would not have studied Statistics given a choice. These students reported adopting surface approaches to their learning. Most of them (55%) perceived of the Statistics as being about MASTERY of statistical ideas and skills.

Most of the males were in clusters 2 and 1 (38% and 30% of them, respectively) while females fell mainly into clusters 4 and 2 (33%, 26%).

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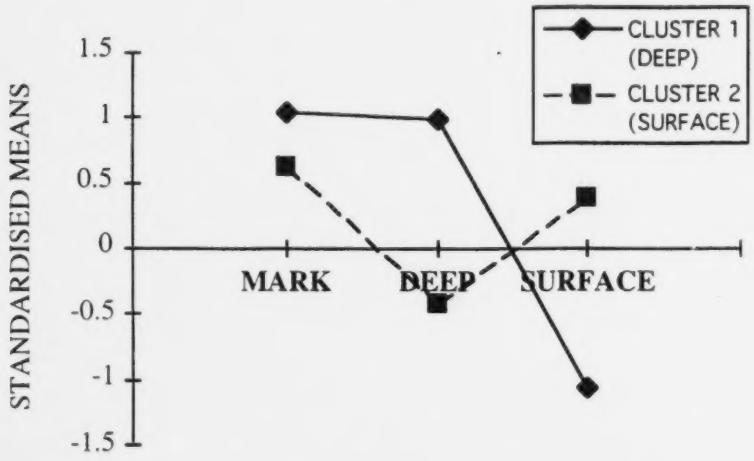
FIGURE 2
MARKS AND APPROACHES FOR THE TWO HIGH ACHIEVING CLUSTER GROUPS



MARKS AND APPROACHES

Two Vignettes From Student Interviews: The profiles described above are of clusters, not individuals. Hence any individual learner may fit some but not necessarily all the characteristics of the cluster group into which she or he has been placed. The following excerpts are from interviews with two high achieving students. Tilly achieved a final mark of 87.5% on Statistics, while Ben attained 95%. As the extracts show, these two students had, however, quite different approaches to learning.

TILLY (Cluster 1)

> When I'm trying to summarise my lecture notes I try to integrate all the information I have on each set topic. And understand that and therefore know how to apply it. What we're doing this semester is a lot more real statistics, how you really apply statistics to what you're doing. I think that's how psychologists would go about testing and researching these things. I want to learn it because I'm going to need the knowledge. And also — in a lot of ways it overcomes my frustration I had with maths. last year. So it's a triumph, almost, to have overcome that big barrier to statistics that I had built up last year.

BEN (Cluster 2)

> It doesn't so much interest me, but its easy going. You don't have to muck 'round doing reports and stuff. You can just learn it. And it's sort of half relevant to what I'm doing, and that's fine. It doesn't bother me a great deal if I don't understand, say, the exact theory behind different distributions and stuff like that. It doesn't really concern me a great deal. As long as I understand the basics and be able to just get through the questions.

Summary And Conclusion

This paper described the relationships among a number of variables in this study. These included demographic variables, the students' reasons for their willingness or reluctance to study Statistics, the categories for the students' conceptions of Statistics, their approaches to learning Statistics and their assessment results.

Most of the students were female and had studied mathematics at a level which included Calculus. The students were overwhelmingly adverse to studying Statistics, because they found it boring or difficult, though the subject was acknowledged as being

necessary for Psychology. A range of conceptions of Statistics were held by the students, but most thought of the subject as being about understanding the prescribed content and solving the problems presented in class, unrelated to the practice of psychology and isolated from the wider world.

The findings of this study present evidence of relationships among students' appraisals of learning Statistics, their conceptions of it and their approaches to learning it. It also shows that students' in the same class have strikingly different experiences of learning Statistics. In particular, qualitatively different orientations to and outcomes of learning Statistics were identified. Unfortunately the majority of the students were unwilling to learn statistics, had perceptions of the subject mainly in terms of accumulating knowledge for assessment purposes and surface approaches to learning it. While some of these students did well in tests and examinations it is unlikely that many of them will transfer their knowledge to other environments. For a small group of able students, however, the picture is far more satisfactory. In addition to achieving high marks, these students were motivated to learn statistics, showed an awareness of the applications of the subject beyond the classroom walls and adopted deep approaches to learning it.

This suggests that we need to take a systemic view of the learning environment. That is, we should consider not only the content and the presentation of statistics, but also how the students perceive it and the context in which it is taught. This is best summed up by Leont'ev: "Meaning mediates man's reflection of the world" (Leont'ev, 1981, p126). That is, individuals filter their experiences through their awareness of purpose or significance and it is this awareness that we need to understand in order to educate students of statistics.

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Constructing Initial Algebraic Understanding: The Proposed Research.

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An initial framework to be used as part of a study that is exploring how students construct initial algebraic understanding is proposed. As a starting point, Pirie and Kieren's dynamical model of growth of mathematical understanding is proposed. A qualitative ethnographic case study approach to be used in the research is described.

Introduction

The Concepts in Secondary Mathematics and Science [CSMS] project (1975-79) investigated students' understanding of letters (Kuchemann, 1981). Approximately 3500 students were interviewed and tested by means of a "pen and paper" test. As a result, Kuchemann reached the conclusion that many students' progress in algebra was limited by the fact that they never moved beyond viewing letters as numerical placeholders. A follow-up project, Strategies and Errors in Secondary Mathematics [SESM] (1980-83), reinforced these findings (Booth, 1984) when it investigated the reasons underlying errors made in school students' algebra.

These projects and subsequent studies highlight that research into teaching and learning of beginning school algebra is of major interest. Some of this research has focussed on the transition between arithmetic and algebra. Concepts such as cognitive gap (Herscovics & Linchevski, 1994) and didactic cuts (Filloy & Rojano, 1989) have been proposed to explain some of the difficulties and obstacles that students encounter in developing algebraic concepts. One particular component of such research is how the initial concepts of school algebra may be taught more meaningfully. A number of alternatives have been proposed. These differ from "chalk and talk" methods that seem merely to manipulate letters. Some of these newer proposed approaches focus on expressing generality in the context of everyday situations through the use of patterning activities and emphasise that algebra is a language (Mason, Graham, Pimm, & Gowar, 1985). Other approaches introduce algebraic concepts by using manipulatives such as match sticks, counters and cups in conjunction with pattern building (Lowe, Johnston, Kissane, & Willis, 1993; MacGregor, Pegg, Redden, & Stacey, 1994; Quinlan, Low, Sawyer, & White, 1987). Still other programs of introducing algebra indicate that functions should be present in algebra teaching from the outset and not left till late in the instructional sequence (Yerushalmy & Schwartz, 1993).

The Research

The literature highlights the need for more inquiry into students' thinking in the area of beginning algebra. How students construct their initial algebraic understandings is one question that continues to seek explanation. The research proposal aims at contributing to the cognitive research literature in this area of early algebraic concept formation. It anticipates exploring how students construct their initial algebraic understanding and aims at developing a theoretical framework to explain such construction. It attempts to investigate the mathematical thinking of individual students in the area of beginning algebra. In order to achieve this it is necessary that the design of the research be different from that of those which applied batteries of pen-and-paper tests to large numbers of students. The research design focuses on monitoring a small group of students for an extended period of time. This research, therefore, tracks a group of about ten students over a two year period as they proceed from the primary school (year seven) to the secondary school (year eight). It is in year eight that formal school algebra is introduced to them for the first time. The setting for this research is the naturalistic setting of the classroom.

A Case Study Approach

Since this study attempts to provide a fine-grained analysis of how students construct their initial algebraic understanding, an ethnographic case study appears appropriate. In this research, case study is chosen since it is the approach that is called for "when the phenomenon under study is not readily distinguishable from its context" (Yin, 1993). The research effort of this study is directed at describing and explaining students' understandings in the context of the classroom itself and will be qualitative in nature. How students construct their initial algebraic understandings is of major concern to this research and it is for this reason that case study is chosen as the preferred strategy. Additionally, the researcher is entering the naturalistic setting of the classroom and consequently has limited control over the events. "Case studies are the preferred strategy when 'how' or 'why' questions are being posed, when the investigator has little control over events, and when the focus is on contemporary phenomenon within some real-life context" (Yin, 1994, p. 1).

Examining, categorizing, tabulating, or otherwise recombining the evidence to address the initial propositions of a study is the essence of data analysis. Analysis of case study evidence, unlike statistical analysis, has few fixed formulas to guide the researcher. "Much depends on the investigator's own style of rigorous thinking, along with the sufficient presentation of evidence and careful consideration of alternative interpretations" (Yin, 1994, pp. 102-103). Since "the ultimate goal is to treat the evidence fairly, to produce compelling analytic conclusions, and to rule out alternative interpretations" (p. 103), it is important to have a general strategy. Two such types of general strategy, as identified by Yin, are relying on theoretical propositions and developing a case description.

The first of these two general strategies, relying on theoretical propositions, applies when theoretical propositions lead to the case study. The objectives and design of the case study are based on these propositions, which in turn invite research questions. These propositions may even lead to novel insights. They guide the data collection and the case study analysis. Data appropriate to the building and growth of algebraic understanding of students becomes the focus in this research on algebraic concept formation. "Theoretical propositions about causal relations - answers to 'how' and 'why' questions - can be very useful in guiding case study analysis" (Yin, 1994, p. 104). The second general strategy, developing a case description, involves developing a descriptive framework for organising the case. It generally serves as an alternative when theoretical propositions are absent.

In this research, the first of these two general strategies, relying on theoretical propositions, is the most appropriate one. The personal experience of the researcher in the classroom, teaching introductory algebra to year eight students contributed to an original theoretical proposition that the use of concrete representations may be a useful instructional method of introducing algebra to students in a meaningful way. This in turn has helped formulate the aims of this research. They are to explore how students construct their initial algebraic understanding as well as develop a theoretical framework to explain this construction. The results of this research hopefully will contribute to informing future teaching practice in the area.

Within this general analytic strategy, specific analytic techniques need to be considered. Two sets of such specific analytic techniques have been identified (Yin, 1994). They are dominant modes and lesser modes. The dominant modes are: pattern-matching, explanation-building, time-series analysis, and program logic models. The lesser modes of analysis are: analysing embedded units, making repeated observations, and doing a case survey (secondary analysis across cases). These lesser modes need to be used in conjunction with one of the dominant modes.

The pattern-matching mode of case study analysis compares an empirically based pattern with a predicted one. Explanation-building, the second analytic strategy, is a special type of pattern-matching. "The goal is to analyze the case study data by building an explanation about the case" (Yin, 1994, p. 110). This is mainly relevant to

explanatory case studies. The third analytic strategy is directly analogous to the time series analysis conducted in experiments and quasi-experiments. This strategy, time-series analysis, traces changes over time and may be possible "if the events over time have been traced in detail and with precision" (Yin, 1994, p. 113). The last analytic strategy is that of program logic models and is a combination of pattern-matching and time-series analysis. "The analysis deliberately stipulates a complex chain of events (pattern) over time (time series)" (Yin, 1994, p. 118).

It is expected that within the current research the main dominant analysis mode will be explanation-building. Being mainly relevant to explanatory case studies, this mode of analysis is very appropriate for this research. Glaser and Strauss (1967) have suggested a procedure for exploratory case studies that is similar to this analysis mode. However, the goal of explanation-building is to conclude the case while for Glaser and Strauss it is to build and develop ideas for further study. The final explanation may not be fully stipulated at the beginning of this explanation-building mode of analysis and in this regard differs from pattern-matching. "The case study evidence is examined, theoretical positions are revised, and the evidence is examined once again from a new perspective, in this iterative mode" (Yin, 1994, p. 111). As the data are collected and examined in this study, it may be appropriate that, in addition to explanation-building, other modes of analysis may be used according to the nature of this data collected. Of the lesser modes of analysis within the general strategy, it is expected that making repeated observations will be a useful technique. The data may also determine the need to analyse embedded units.

There are four principles, independent of the specific analytic strategies chosen, that ensure the highest quality of the analysis in any case study. The analysis should show

- that it relied on all the relevant evidence,
- that it included all major rival interpretations,
- that it addressed the most significant aspect of the case study, and
- that the researcher brought his or her own prior, expert knowledge to the case study (Yin, 1994).

The analysis of this research will seek to ensure that all of these principles are implemented. Data will be collected by means of student and teacher interviews, student journals, teacher diaries, and participant observation.

An Initial Framework

As an initial framework for investigating the growth of understanding in the concepts of variable and function, the model that Pirie and Kieren (1989, 1991, 1992, 1994a, 1994b; Kieren & Pirie, 1992, 1994) have posited is an appropriate one because of its dynamical nature. This framework is necessary in order to guide observation and analysis as the research proceeds. In summary the essential features of this model are indicated in figure 1 below:

A Dynamical Model of the Growth of Mathematical Understanding

In this dynamical model of the growth of mathematical understanding, Pirie and Kieren (1992) have proposed eight potential levels of understanding for a *particular person with respect to a particular topic*. "It is, in essence, a theory and a model of personal growth" (p. 245). They believe that there is "no such thing as understanding in the abstract" (Pirie & Kieren, 1994a, p. 39). Mathematical understanding is seen as "a process, grounded within a person, within a topic, within a particular environment" (p. 39). A brief overview of these eight levels is given.

Pirie and Kieren (1992) use the term *primitive knowing* to describe the starting level in this process of coming to understand any mathematical topic. It simply indicates the starting point and does not imply low level mathematics. It is concerned with the assumptions that the teacher may make with regard to the prior knowledge that a particular

student has before beginning a new task. The observer is not able to know what this primitive knowing really is.

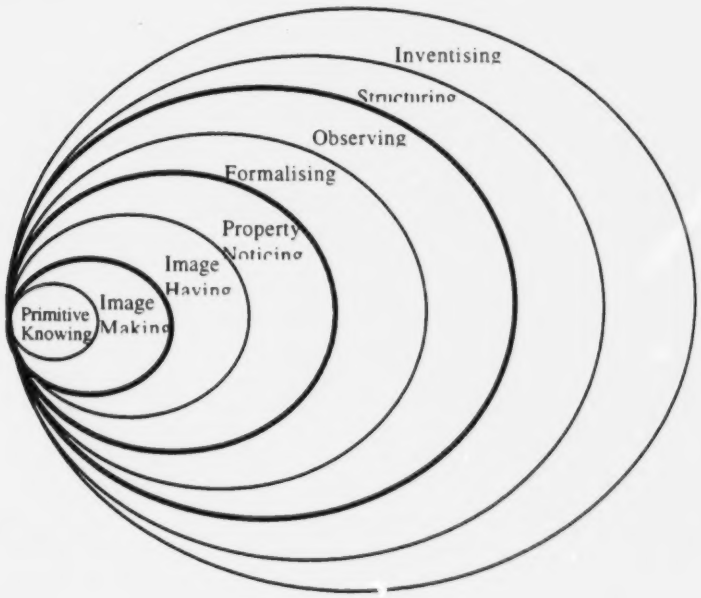

Figure 1: Model of a dynamical theory of the growth of mathematical understanding. (Kieren & Pirie, 1992, 1994; Pirie & Kieren, 1989, 1991, 1992, 1994a, 1994b)

The second level, called *image making*, is the level at which "the learner is asked to make distinctions in previous knowing and use it in new ways" (Pirie & Kieren, 1994b). It is through specific tasks that the student makes distinctions in previous abilities using them either to new ends or under new conditions. The term "image" here is used in a broad sense. It is not used merely for pictorial representation but for any kind of mental representation.

These singular, directed activities of the image making level are replaced by a mental object at the next level, *image having*. At this level, a student's mathematics is freed from the need to perform particular actions. "The image itself, as a mental object, can be used in mathematical knowing" (Pirie & Kieren, 1992).

A fourth mode of understanding occurs "when one can manipulate or combine aspects of ones images to construct specific, relevant properties" (Pirie & Kieren, 1994b). This involves making connections and noting distinctions between images. Combinations of images can occur. Because ones images are examined for specific properties in order to make these connections, distinctions and combinations, this fourth level is referred to as *property noticing*. It is at this level that relationships are recorded.

Formalising is the next level of understanding activity. "The person abstracts a method or common quality from the previous image dependent know how which characterised her [or his] noticed properties" (Pirie & Kieren, 1994b). This level of understanding entails thinking about the noted properties and informal patterns in the previous level and abstracting commonalities. Now the person has a mental object that is class-like and is not dependent on meaningful images.

Having established these formalisations, the students are in a position to reflect on them and reference their own formal thinking. Thus, the next level is called *observing* and involves noticing one's formalising and organising these observations. A deeper understanding is then gained at the next level where these formal observations are able to be explained in terms of a logical structure. Thus, this seventh level or mode of understanding is known as *structuring*. It means that the student is aware of his or her

assumptions and the sequence of the observations made at level six. Interrelationships between observations are established logically. "Structuring occurs when one attempts to think about ones [*sic*] formal observations as a theory" (Pirie & Kieren, 1994b, p. 171). The student is aware how the set of theorems is inter-related. Justification and verification of statements occurs at this level. Formal mathematical proofs are used in this mode of mathematical understanding. It may be thought of "as setting one's thinking in an axiomatic structure" (Pirie & Kieren, 1992, p. 247).

The outermost level in the model is called *inventising*, a word specially coined by Pirie and Kieren (1991, 1992, 1994a, 1994b; Kieren & Pirie, 1992, 1994). "In inventising, a person with a fully structured knowledge breaks away from all the preconceptions which brought about this complete understanding and creates totally new questions which might grow into a totally new concept" (Pirie & Kieren, 1992, p. 247). Initially they had used the term "inventing" (Pirie & Kieren, 1989, p. 9). This term was not intended to suggest that students did not or could not develop new ideas at other levels. To avoid any misinterpretation of the fact that students can invent at other inner levels in their model, Pirie and Kieren (1991, 1992, 1994a, 1994b; Kieren & Pirie, 1992, 1994) coined the word inventising and avoided their initial use of the word inventing (Pirie & Kieren, 1989).

Figure 1 is an attempt to represent Pirie and Kieren's (1991, 1992, 1994a, 1994b; Kieren & Pirie, 1992, 1994) ideas in a two-dimensional form. It is to be noted that growth of understanding is not seen as a monodirectional process. Thus, the model is presented as a sequence of nested circles. In this way it is seen "that each layer contains all previous layers and is embedded in all succeeding layers" (Pirie & Kieren, 1994b, p. 172). Any level of understanding is itself embedded in all outer levels while, at the same time, it has embedded in it all more inner levels. Growth in understanding is then viewed as a back and forth movement among layers. Understanding is characterised as a dynamic organising process. A critical feature of this theory is that of folding back. The non-unidirectional nature of this theory is highlighted by this feature. Pirie and Kieren (1992) argued that "one can fold back to any inner level of understanding activity in order to extend one's current, inadequate understanding" (p. 248).

Growth in mathematical understanding is aided by three basic types of intervention. These are "provocative, invocative and validatings interventions" (Pirie & Kieren, 1992). From a teacher's perspective these interventions are explained in the following way:

> *Validating* intervention is intended to allow the student to show where he [or she] is and why he [or she] is thinking that way, either to himself [or herself] or an observer. A *provocative* intervention, on the other hand, is an attempt to deliberately prompt the extension of a student's understanding to new cases or to an outer level. An *invocative* intervention is intended to pose or identify an obstacle to a student's understanding and prompt folding back to inner level activities, even possibly back to directed image-making activities. (p. 248)

It is important to note here, however, that even though the teacher may have such intentions, it is the *response of the student* that actually determines the nature of the intervention, namely whether it is provocative, invocative or validating.

This model reflects its comprehensive nature by including in it as a critical element of the theory, a mode of understanding which identifies a student who is able "to operate mentally or symbolically without reference to the meanings of basic concepts or images" (Pirie & Kieren, 1992, p. 248). This feature of the model is illustrated in figure 1 by the use of bold rings. Pirie and Kieren call these rings "'don't need' boundaries" (p. 249). Once students have an image (image having level of understanding) they do not need specific examples of the preceding level, image making. Their understanding is also independent of their primitive knowing. Formalisation at the fifth level indicates that their mathematical activity does not require a concrete meaning or an image. Similarly, a student with a mathematical structure at the seventh level does not need the meaning

brought to it by any of the inner levels. To be doing mathematics they do not need to perform formal algorithms and no longer need to have concrete meaning. Figure 1 indicates these "don't need" boundaries with bold circles which appear just inside image having, formalising and structuring.

An important aspect of this model that needs to be noted is the fact that not all mathematical understanding of all topics flows in a smooth connected manner even in the context of folding back. A student's understanding of a topic can be disjoint. Kieren and Pirie (1994) have termed this "disjoint understanding" (p. 51). While there is value in viewing understanding as a state that can be attained or achieved (Byers & Herscovics, 1977; Skemp, 1976, 1987), this dynamical model offers an alternative view of mathematical understanding. It is an "on-going, growing, process by which one responds to the problem of re-organising one's knowledge structures, possibly in the face of epistemological obstacles, by a process of revisiting one's existing understanding" (Kieren & Pirie, 1994). It is dynamic.

This model for growth of mathematical understanding provides a framework within which to investigate the thinking processes of the students in the current study as they attempt to make sense of the learning experiences of introductory algebra over time. It is recognised in this model that the first level, primitive knowing, is not intended to imply a low level of mathematics. There is no attempt to necessarily link better or high level mathematics with the outer levels. Using the language of levels and layers, however, Pirie and Kieren (1994b) indicated that there is some underlying hierarchy within the model.

Analysis

The relevant and significant aspects of the case study will be analysed in the first instance using this model for the growth of mathematical understanding developed by Pirie and Kieren (1989, 1991, 1992, 1994a, 1994b; Kieren & Pirie, 1992, 1994). In reference to their model Pirie and Kieren (1994b) noted

> analysis can only ever be based on what the teacher observes. This notion of mapping entails plotting as points on a diagram of the model, observable understanding acts and drawing continuous or discontinuous lines between these points, dependent on whether or not the student's understanding is perceived to grow in a continuous, connected fashion. (p. 182)

An example of such a diagram is given in figure 2. It is a map of the growth of a student's understanding during an interview reported by Pirie and Kieren (1992). Since this current research will monitor the students over two years, there will be a sequence of such diagrammatic representations over time for each student in the study. Such data will contribute to the generation of a theoretical framework to explain how students construct their initial algebraic knowledge. Besides applying their model to an eight year old child, Pirie and Kieren (1994b) and Kieren & Pirie (1994) have also applied this model to university students. This indicates the broad usage of their dynamical model to trace the growth of mathematical understanding at any level in any topic.

The analysis of data in this research will inform future teaching practice in the area. However, broad generalisation will not be possible. The analysis, which is fine-grained, is contextualised within the naturalistic setting of the classroom and as such may provide insights into how students respond to the act of teaching or instruction. Significant events in the growth of algebraic understanding for each subject will be mapped over the two year period which forms the timeframe for the research.

Questions in the domain of algebra relating to observable behaviours may be used as a particular focus in mapping a particular student's growth of mathematical understanding when applying the dynamical model of Pirie and Kieren (1989, 1991, 1992, 1994a, 1994b; Kieren & Pirie, 1992, 1994). If a student is able to demonstrate the ability to apply a general rule usefully within a particular mathematical topic it may suggest that for this topic this student is demonstrating the *image having* mode of understanding as a minimum.

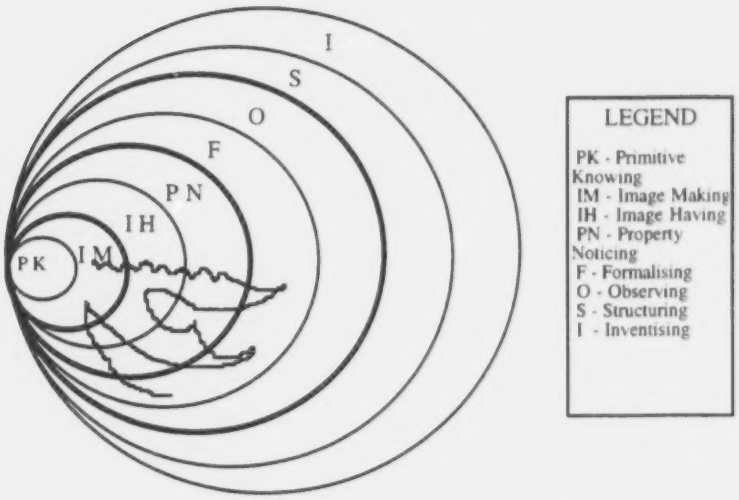

Figure 2: A map of the growth of a student's understanding during the interview. (Pirie & Kieren, 1992, p. 253).

Summary

This research aims to inform future teaching practice in the area of algebra by exploring how students construct their initial algebraic understandings of such concepts as variable and function. A case study approach is used for the study of a small group of students over two years of schooling which span the transition from primary school to secondary school. The dynamical model of Pirie and Kieren (1989, 1991, 1992, 1994a, 1994b; Kieren & Pirie, 1992, 1994) is used to develop the initial framework for this research. From this a theoretical framework may be generated to assist in explaining the constructions that students make in developing their initial algebraic understandings.

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Do they understand what we mean?: Assessment and communication.

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Communication is a vital part of assessment if student and assessor are to understand each other and not talk past each other. I consider why this misinterpretation should occur and what can be done to attempt to improve student's understanding of the intent of the question writer. Student answers on one question in particular are used to demonstrate some student misunderstandings.

Introduction

When considering how successful assessment tasks are in judging the process skills, it is necessary to consider whether the student reads (or reconstructs) the question as the writer intended. It is commonly assumed that the assessment process is clear, with questions asked in an unambiguous manner and all errors revealing students lack of knowledge of what is being tested. However much research suggests that this is not so and that children believed they were being asked a different question or thought they had to answer the implicit question which they perceived in the context rather than the explicit question asked (Donaldson 1978).

Words do have the power to communicate unambiguously if the listener has the cognitive structures necessary to link the ideas. However unless they are linked to real life examples there is a risk of concepts remaining as memorized items of use only in classroom mathematics. "...what students actually learn is not necessarily congruent with what their mathematics teachers want them to learn" (Ellerton and Clements 1991)

Macgregor found that phrases used for expressing comparisons in the vernacular were not standard and caused difficulty in constructing meaning for words such as 'than', 'as', 'more' and ' less'. Insignificant small words such as 'to', 'of' or 'by' were important to the sense, making skim reading which is stressed in other subjects, not sufficient for mathematical reading. There was a need to become fluent in formal English to be successful in understanding mathematical language. Some teachers have believed that the answer is to reduce the reading and writing component of mathematics lessons but this strategy is only successful in the short term. She also found that the order in which information was given did not always give the clue as to the order to attempt the problem. There is a need to actively teach strategies to students to aid in the selection and reorganisation of information in word problems. In addition she found that group work is only successful if students are able to understand the explanations shared (Macgregor 1993).

The research project Spoken Language and New Technology (SLANT) found the quality of talk in groups variable. Only exploratory talk was educationally valuable but the amount can be increased by practice on suitable tasks. Mercer sees a need for better methods for analysing continuous discourse and a willingness to bridge the gulf between qualitative and quantitative methodologies. He also stresses the importance of examining the language of classrooms in its cultural context and the analysis of genres (Mercer 1996).

Between 1988-92 there was a strong influence of process approach to maths teaching in many primary classes in Australia. (Ellerton and Clarkson 1992) ELIC (Exploring language in classrooms) and BLIPS (Basic learning in primary schools) may not have been enduring because "they were often used as add-on teacher-directed extension activities completed after the 'real' mathematics has been done". (Marks and Mousley 1990)

Language and cognition cannot be separated (Edelman, 1992). A child makes sense of things first and then of what is said. Language is used "to convey thoughts and feelings of individuals who already think independently of language". Therefore helping a student to make sense of the mathematics being learnt will help with prowess in using language.

Vygotsky regards learning as a social, culturally situated and guided activity and talk as a form of social action and social mode of thinking. Together people can construct

knowledge and understanding. Neo-Vygotskians such as Bruner and Wood, have described adults' support of children's learning as scaffolding. Adults or other students can challenge thinking or provide counter-examples which cause perturbations in the student's existing schema. Mathematics teaching begins by finding out where a student "is at", i.e. their existing schemas about the topic. (There are no clean slates to work on in students minds!) Starting from there, the teacher needs to provide activites which will develop new insights into the topic which can easily be assimilated into students existing schema or will perturb it sufficiently to cause a new schema to be produced which can accommodate the new knowledge. Vygotsky says that concept attainment is complete only when a student can frame the concept in appropriate words (Vygotsky 1986).

Initiation-Response-Feedback exchanges are not discrete loops but can be linked together into longer spirals. Teachers can guide students to see connections and continuities and increase the problem solving abilities of children. Students need guidance (scaffolding) to enter educated discourse (Mercer, 1996). According to working group 7's report to the 7th International Congress on Mathematics Education 17-23 August 1992 the introduction to mathematical knowledge is like an introduction to a new culture; students are initially speechless. In many classrooms communication is asymmetrical, based on technical terms defined by the teacher with little meaning, just rituals. "Different symbolic forms can carry a multiplicity of interpretations that to the mathematician are 'the same', but to the learner seem entirely unrelated." (Susan Pirie quoted in Sierpinska and Steinbring 1992). Symbols and diagrams often do not become thinking tools but stay as knowledge containers which are not helpful in solving non-routine problems (Sierpinska and Steinbring 1992)

Frid's study of tertiary students ideas of mathematics (Frid 1993) is relevent to this study as these were the students who had successfully accomplished school mathematics. Over 50% believed mathematics consisted of rote learning of absolute truths with only one method possible. They believed understanding was being able to do examples and get correct answers. Mathematics was different to other subjects in that it was foreign to real world understandings. Some saw symbols as meaningless because either they had no meaning or the meaning had never been explained to them. Words were ascribed meanings which related to everyday meanings of words which often lacked mathematical precision. Frid described the teachers role as that of facilitator (by providing counter examples to stimulate analysis of ideas) and mediator between the students language interpretations and those accepted by the mathematical community. Everyday language can be a help or hindrance depending on how well students can integrate everyday language with the technical language. It can be refined by confronting with situations when everyday meanings are inadequate. Language is an essential component of building mathematical meanings from experience. The value of students own language in explaining their thinking must be acknowledged and encouraged.

Most errors made by children were reading, comprehension or transformation errors amd occured before any application of algorithms (Newman 1977). The highest category was comprehension which is clearly language based. What can teachers do to improve a learner's comprehension of mathematics text or ability to transform a word problem into an appropriate mathematical process? One suggestion is the use of carefully designed, often open-ended questions (Ellerton and Clarkson 1992).

The teacher must intervene but when, how and with what aim? Constructivist style teaching does not mean lack of control over constructed meanings. The benefits of "consulting with someone who knows" are immense. The choice of the initial problem is important if students are to be motivated to communicate. Writing is a vehicle to understanding especially expressive writing but it is important to realise that pupils might not understand what they tell us they understand.

When problems are considered by a cooperative group the discussion between students will advance the social construction of knowledge. Some open questioning by the teacher can help students when they appear to be about to give up or on an approach likely to end in a cul-de-sac, but care must be taken not to stifle innovative thinking. Teachers need to be aware that there are often many ways of solving the same problem some of which may be new to that teacher, and that sometimes an apparent digression from the task in hand may lead to equally valid learning. Therefore the teacher should not

interfere too hastily. When conclusions are reached which are not generalisable then teachers can extend thinking by suggesting activities which will not fit that conclusion.

If teaching methods are to become more in tune with current research into how children learn, then assessment and instruction needs to be better integrated. (Driscoll 1995) describes an assessment feedback loop that can help align mathematical goals with teaching practice. Teacher questions can be not just assessment but also part of teaching. Assessment can occur while teaching is taking place where assessment is of new knowledge gained in the 'zone of proximial development' (Vygotsky 1978)

Method

As part of a larger study, I have examined in detail 306 student answer scripts to a question which aimed to test student abilities to use the mathematical processes of communication, problem solving and logic and reasoning. It was a question set on the 1996 moderation test for the internally assessed candidates sitting School Certificate that year. The question involved different ways to buy a video-recorder. The question was as follows:-

> An appliance store is selling new video recorders for $599.
> They offer two deals:
> - Trade in your old video recorder for $200, OR
> - take 15% off the price if you pay cash.
>
> Sulia's family is thinking about buying a new video, and her uncle has said he would buy their old recorder for $150.
> Which is the better deal for Sulia's family? Should they trade in the old video, or sell their old video to Uncle and use the cash to help pay for the new one?
> State what you are calculating at each step and show all your calculations.
> Clearly state your decision and say why you made this choice.

I have categorized their answers by a method similar to that described by Doig and Cheeseman. They say that "open assessment tasks can reveal conceptual understanding and mathematical strategies on a par with clinical interviews"(Doig and Cheeseman 1996). Though this question was not truly open, it had enough freedom of response for students to reveal their conceptual thinking.

Results

There was a rich diversity of answers which made categorizing difficult. Initially I went through all the papers and categorized each individual error, or in many cases number of errors. I found 55 different errors in total. Some of these occured frequently, some only occasionally and some were idiosyncratic. Only 8.8% had no errors of any kind. I went through the papers again and gathered similar errors into 6 main categories.

Category	Error	Percentage
1	Communication	29
2	Simplify to 3 options	14
3	Simplify to 2 options	20
4	Calculation	20
5	Muddled alternatives	5
6	No calculations	15

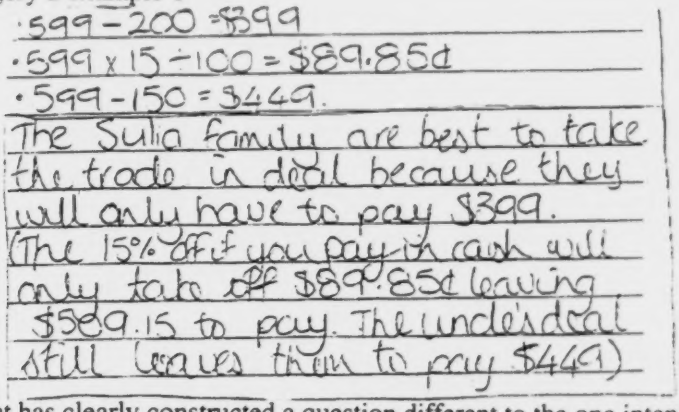
The first category were those with errors of communication. Many were able to correctly calculate the answers but failed to communicate what they were calculating at each stage. They did not explain why they made the decisions they did or how their calculations related to the problem or the decision reached. As this was an important process, clearly stated in the body of the question, this lost marks. 29% in total were in this category of inadequate or no communication. Of these 9.2% had no other errors so could have achieved full marks with more explanations of their reasoning. Most in this

category also had other errors. The lack of communication in their answers may reflect a lack of emphasis in some schools on explaining in written language the steps taken to reach a decision. It also probably reflects an 'answer only is important' teaching style by some teachers and children. It is of interest to wonder if more of the students would have made fewer errors if they had more clearly laid out their thinking.

The second and third categories were those who had problems in comprehending the question and as a result simplified the question. 34% of the sample worked out an answer by constructing a simpler question from the one expected by the examiner. 14% changed the question to comparing three different options instead of two (category two) and 20% compared only 2 of the options (category three). Whether this is a problem of reading (not constructing the same meaning as that intended by the writer) or of a developmental stage (cannot understand the intention of the question and so makes some sense of it at a lower level) will be part of the research which is still continuing.

Category 2 errors consisted of making the question have 3 alternatives instead of 2. These consisted of (i) a discount for cash making $509.15 to pay (ii) sell to uncle making $449 to pay and (iii) trading in making $399 to pay. With these three as the alternatives most could decide that the best alternative was to trade in. Was there a difficulty in combining two different calculations into one option? and was this difficulty caused by a lack of understanding of the meaning of the question or by the complexity of the problem?

Category 2 example 1


·599-200=$399
·599 x 15÷100=$89.85c
·599-150=$449.
The Sulia family are best to take the trade in deal because they will only have to pay $399.
(The 15% off if you pay in cash will only take off $89.85c leaving $509.15 to pay. The uncle's deal still leaves them to pay $449)

This student has clearly constructed a question different to the one intended by the writer. Has the student skim read the question looking only for the numbers and therefore read the question as having three options to choose between? Was the order of the last two questions important in deciding what the question was asking? Perhaps by phrasing the question slightly differently, or changing the order of the two questions at the end we could improve the number of students being successful on this question.

If that works it would prove that the problem is probably one of comprehension of the written word. However if that makes no difference to the number misconstructing this question in this way, then it would point to it being a developmental stage problem. These students would still be at a stage where this is the best interpretation that is comensurate with their existing understandings. The size of the group in category two and the consistency of their misconstructions makes it an interesting area to research further.

Category 3 consisted of three different means of simplifying the question to only two parts. (a) by a complete absence of the 15% discount. The question was simplified to trade or uncle's cash only. Perhaps the last sentence has been missed and not read so that the discount appears to be irrelevent to the question? However the calculations, the communications, the logic and reasoning are all without fault. (b) by a complete absence of uncle's cash. The question was simplified to trade or discount. Has the question reading been truncated and only the first few lines read? (c) a complete absence of the trade in. The question was simplified to discount or uncle's cash.

The fourth group consists of those students who made calculation errors and covers a wide range of errors. Some are possibly careless errors, such as miscopying a figure

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from the question or misreading a calculator answer, but most are misunderstandings as to how to calculate 15% of a figure or calculating 15% off the wrong amount. These included :-

- 15% as 100-15 =85
- 15% of $599 as 599/15=$39.93
- New price after 15% discount as 599/1.15=$520.87
- 15% of 599 as $599-15 =$584
- 15/599*100
- 599*100/15

All of these appear to be cases of misunderstanding or misremembering of taught algorithms. The next set are all cases of calculating the 15% off the wrong amount.

- 15% of $399
- 15% of $150
- subtracted the Uncle's $150 first and then calculated 15% of $449.

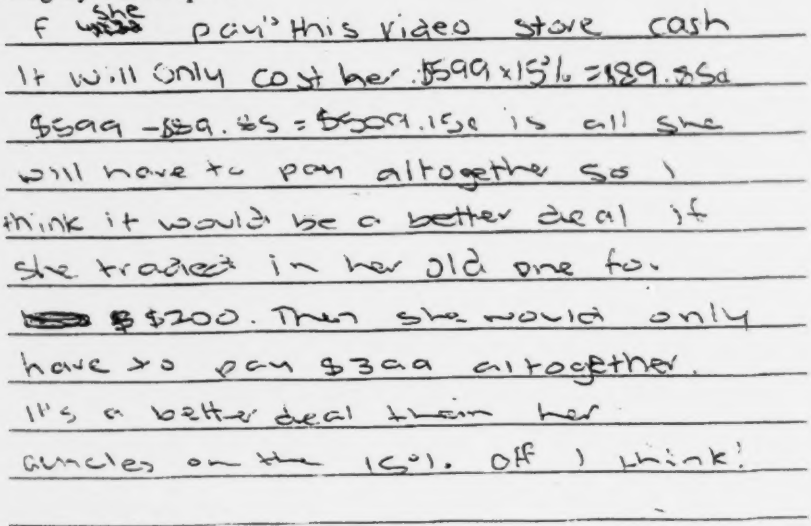
The last set are all idiosyncratic mistakes where no clear message is given as to method and it is not obvious how the answer is obtained. Extra questions such as could be given in an interview will be needed to ascertain the thinking behind these errors.

- 599-15%=$149.75
- 599-15%=23.25=575.75 with discount
- 15% discount =$1
- worked out discount by continual adding but made a mistake in addition.

20% of the candidates were in this fourth category.

A fifth category consists of those who muddled up the various alternatives. For example they mixed up trade and discount as if they were the same, or mixed up the discount and uncle's money, or believed they could both trade in and get the discount. In example 2 the uncle is seen as giving Sulia's family the 15% off instead of $150.

Category 5 Example 2


she
pay's this video store cash
It will only cost her $599 x 15% = $89.85
$599 - $89.85 = $509.15 is all she
will have to pay altogether so I
think it would be a better deal if
she traded in her old one for
$200. Then she would only
have to pay $399 altogether.
It's a better deal than her
uncles on the 15% off I think!

Category 5 Example 3

This example mixes up trade-in with the 15% off.

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Selling video to uncle

Selling = $150

$599 - $150 = $449 — $ 499 to pay

Trade In

599 - 15% = $89.85

$599 - $89.85 = $509.15 — $ 509.15 to pay

You should sell your friend video tracker to your uncle

The next example is extremely confused. The 15% discount for cash is calculated but instead of being an extra saving and added to the $150 gained by selling to uncle it is subtracted from that amount making a much smaller saving. Are the words 'take off' being read as a subtraction activity whatever the circumstances?

Category 5 Example 4

If tracked in $200 = 599 − 200 = $399

If sold for $150 = $\frac{599}{100} \times 15\% = 89.95$

$150 − 89.95 = leaves $60.05. Sulia's family would get a better deal if they sold their old video to her uncle.

One even believed that it was possible to trade in, get the discount and then sell to uncle and end up with a new video for $20. Another similar case said they should sell to Uncle and trade in (making only $50 to pay) and then get a discount on that. 5% were in this category.

The final category consists of those who made an attempt but produced no calculations. Sometimes they produced a statement of which was cheapest but with no justification. This was the most common in this category which comprised 15% of the candidates. Sometimes the statements did not fit at all with the problem. eg "the new one is best or cheaper than the old one" or "the Mitsubishi is the better deal"or "buy her uncle's and sell him her old one". One answer appeared to use knowledge gained in economics rather than using mathematics skills."Buy from the shop because if it doesn't work they feel guilty".

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Discussion and Conclusion

The mistakes made on this question are indicative of problems with the language used to express the question. The realistic context encouraged students to attempt the question (only 16.7 made no attempt at this question) and the everyday language in which the question was framed gave them confidence. However their answers revealed misunderstandings of what was being asked. The use of "or" and "and" appear to have caused difficulties for some students as found by Macgregor (Macgregor 1993) with small words being overlooked or misinterpretted. Skim reading may have occured which missed the point of the question.

The ambiguity of words may have caused some of the difficulties where more than one meaning may exist for the same word or phrase especially when this may be class related. For example the phrase "better deal" may have meanings other than monetary to some cultures. Lower socio-economic groups may consider it better to pay off in small amounts even though the total cost is higher. Some terms such as "trade-in" may be unfamiliar to some students. The context may have got in the way of the mathematical calculation being assessed. (Hipkins 1996)

Certain words have been found to trigger certain ideas about the calculation called for and this can be unfortunate when those calculations are not the ones required. "Take 15% off" in this question triggers a subtraction reflex but this needs to be **added** to the uncle's $150 to find the total savings. "The level of difficulty of a word problem is related not only to its mathematical content but also to its linguistic form and semantic structure" (Gibbs and Orton 1994)

It may be that the wording of the question caused some of the misinterpretations of the problem. By writing the same question in different ways, by changing the order of the sentences or the length of the sentences, the researcher will attempt to improve students' achievement on this question. By interviewing students who made particular mistakes especially in categories 2 and 3, the researcher will try to discover what part of the question caused them to miscontrue the problem.

The analysis reveals a number of errors that students have made in dealing with this question and this information should be able to inform teachers as to likely errors in understanding. This feedback on the structural understanding of students as a group rather than as individuals, can improve teacher understanding of how students may be thinking and help them adjust their teaching accordingly. Teaching strategies to use when dealing with word problems, and guiding students in the reorganisation of information may improve student's success at this kind of problem. It has been suggested that the use of questioning techniques by the teacher which draw the student's attention to the main points in the problem, leads to higher level discussion and a deeper comprehension of the problem especially when student generated questions are encouraged. (Bean 1985)

The use of cooperative group work with teacher guidance may also achieve improvement. Using everyday language to describe mathematical thinking may improve the preciseness of the language used and help refine the meaning of words as situations arise where the everyday meaning of words is inadequate. Having to explain your reasoning to someone else makes it much more explicit and can often improve understanding. Socio-cognitive conflict has been shown to change student's thinking as they interact with other students with different ideas and understandings. (Mercer 1995)

Ellerton and Clarkson (1992) have indicated that the use of open ended questions can improve students' ability to comprehend the meaning of a word problem and choose the appropriate mathematical process. The use of realistic contexts with emphasis on answers that make sense in the real world, can lead to an improvement in the thinking taking place when dealing with word problems.(Verschaffel and Corte In press)

This study of student's scripts gives interesting insights into their thinking and probable misinterpretations of the language used to frame the question. Further research in the form of interviews are now necessary to find out what the student's were

thinking when they misconstructed the question. Research will also continue into how assessment can be improved so that student and assessor do not talk past each other.

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Improving Understanding in Integration with the Computer

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Often students' tendency in approaching calculus is to follow an algorithm or manipulate symbols. Mathematics educators seek to provide a range of experiences that develop mathematical ideas in a cognitive manner so that the learner both knows and understands. In this paper I describe how students used spreadsheets and symbolic manipulators to investigate the processes and concepts of integration, using the modules of work to supplement traditional approaches to integration. They gained a significant improvement in proceptual understanding, especially with regard to misconceptions exhibited in the pre-test.

Introduction

Many students are gaining instrumental rather than a relational understanding (Skemp, 1976) in calculus, learning 'how to' rather than 'why'. They have experienced difficulties with calculus by relying on formalised rules, procedures and calculus algorithms, disregarding conceptual understanding. When the students are confronted with a complex definition, many conceptual problems may arise such as with infinite, logical and manipulative processes. The conceptual difficulties with limit, differentiation and integration have been explored (see e.g. Tall, 1986; Steen, 1988; Barnes, 1988; Li & Tall, 1993; Thompson, 1994) and considerable effort has also been put into finding ways to improve such understanding, often using computer software such as symbolic manipulators (e.g. Small & Horsack, 1986; Palmiter, 1991; Barnes, 1994; Hubbard, 1995). In a previous paper, Thomas & Hong (1996) have represented the sort of misconceptions which many students have in calculus. The study described here investigated student thinking and misconceptions when dealing with integration; how the students using spreadsheets and symbolic manipulators had significantly enhanced conceptual understanding.

Background

Process and Concepts

Robert & Schwarzenberger (1991) described abstraction as involving the recognition of objects and properties which not only apply to the objects from which a generalisation is made but also to any other objects which obey to the same properties. Estimating the area under a curve with sums and passing to a limit is a process. Dubinsky & Lewin (1986) describe the *encapsulation* of such a process as an object. Students who seem to understand the process of integration often have difficulty with the next step of varying, the upper limit of the integral to obtain a function. They lack the ability to encapsulate the entire area process into an object which could then vary as one of its parameters varies. The definite integral forms an important example that can be interpreted as encapsulation together with interiorisation (Dubinsky, 1991). Gray & Tall (1993) have introduced symbolism and define the notion of procept as an amalgam of three things - process, symbol and concept. Procepts are first met through a process, then a symbolism is introduced for the product of that process. It seems that much of the symbolism used in mathematics carries for the mathematician the dual role of process and concept. This distinction between the usage of symbolisation to stand for a process or a concept or conceptual structure depending on one's point of focus is clearly an important one mathematically.

The Role of the Computer in Calculus

It is possible that the computer experiences (Ayers *et al*, 1988) may help students. If a student implements a process on a computer, using software that does not introduce programming distractions then the students tend to interiorise the process. If the same process can be treated on the computer as an object on which operations can be performed, then the student is likely to encapsulate the process. So computer experiences

encouraging reflective abstraction may be constructed in a wide variety of areas of mathematics. Tall (1993) suggests that the computer relieves the learner of the difficulty of having to encapsulate the process before obtaining a sense of the properties of the object. By using software which carries out the process internally, it may become possible for the learner to explore the properties of the object brought forth by the process, before, or at the same time, as studying the process itself. This new adaptability in curriculum progression he has called the *principle of selective construction*. Tall & Thomas (1991) described a *versatile* mathematician as having the ability to think in terms of processes or concepts. It is our contention that using computer software in mathematics courses can encourage the students' understanding of processes, thus facilitating versatility. Monaghan (1986) emphasised how a process creates a *generic limit* concept in which the varying process is encapsulated as an indefinite *variables* object such as 0.9. Monaghan (1993) also studied the growth of 16/17 year old students' conceptualisation of real number, limit and infinity over one year. CAS (Computer Algebra System) students using the symbolic manipulator software *Derive* were better able to see the limits as objects than traditional students. Another link with procepts is that some students used x^n and nx^{n-1} as generic terms for polynomials and their derivatives and questioned the functional notation led by their teacher. Computer software can evaluate many limits, so the possibility arises that it may allow the curriculum to give a more balanced view of limit as concept and process by early focus on the limit as concept with the computer carrying out the process internally.

Method

We devised a test which aimed to measure the students' conceptual understanding connected with integration, and which comprised two sections. The 10 questions of section I were focussed on the process-oriented algorithms which any student of integration could be expected to know. For example, they were asked to integrate, using antidifferentiation techniques $\int f(x)dx$ =F(x)+C. For the questions of section II, a deeper understanding of concepts was introduced. The 13 questions of section II were aimed at concepts rather than processes. Some questions in sections I & II were linked to see if students had developed techniques of algorithms but did not have the corresponding concepts, or the ability to apply the techniques when solving problems. The questions in section II cannot simply be calculated without conceptual understanding but those of section I could be. The categories of section I & II applied were as follows: conservation of integral, the maximum values of an integral function, the definite integral of a function that crosses the x-axis and area, summation using sigma (Σ) & Riemann sum, Riemann integral, sketching the integral function. The test was tried twice, the pre-test was attempted after formal learning of integration in calculus. After that the computer tutorial was provided, the post-test was used to re-evaluate any improvement. The questionnaire was given to 36 (age 16~19 years) high school students in 1996, comprising 14 female and 22 male students. The same test questions were given in the pre-test and the post-test for the same students.

Using modules based on the Excel spreadsheet and Maple symbolic manipulator, the processes of integration were investigated for 6~7 one hour sessions. The Excel spreadsheet was used for four one hour sessions and the Maple symbolic manipulator for 2~3 one hour sessions. The students already had a basic knowledge of the Excel spreadsheet but it was only in operational, not mathematical aspects. The aim of these computer tutorials was to see if an improvement in the understanding of the concepts associated with integration was produced by giving the students direct experience of experimentation with the processes which lead to them. 17 of the students were individually interviewed to investigate further their understanding. They consisted of students who had: greatly improved; made some improvement; or others who showed no improvement in their understanding. Comparing their results on both tests, they were questioned in their interview about their view of integration and their practical experiences. The recorded tapes of the interviews were transcribed and used for the analysis of the students' understanding.

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Results

Misconceptions

Common misconceptions of students based on the results of the pre-test were categorised as; single value integral misconception, inversion misconception, general function misconception, and integration process disregarded misconception.

• Single Value Integral Misconception

In response to the question:

Section II. *Find the interval [a, b] for which the value of the integral* $\int_a^b (2 - x - x^2)dx$ *is a maximum.*

the single value integral misconception appeared as follows. The concept of maximising an area function is more difficult than calculating a single area, however, it does give students opportunity to demonstrate their ability to extend their algorithmic knowledge and apply it in problem contexts, since it involves seeing a, b as variables. Some students wrote the following in answer to this question:

$$\left[2x - \frac{1}{2}x^2 - \frac{1}{3}x^3\right]_a^b = 2b - \frac{1}{2}b^2 - \frac{1}{3}b^3 - (2a - \frac{1}{2}a^2 - \frac{1}{3}a^3).$$

It seems that such students are locked into a process-oriented view and unable to proceed beyond the algorithm they have instrumentally learned and are unable to recognise the interval [a, b] as representing a variable area. Rather they are only able to compute a single value. Hence we call this the *single value integral misconception.* They don't see $\int$ as a function, but as Dubinsky described (see above), they lack the ability to encapsulate the area process into an object. Another case of this misconception was that a and b were sometimes replaced by values such as 1, 2; $\int_1^2 (2x - \frac{1}{3}x^3)dx$, again showing that the students depended on a restricted algorithm.

• Inversion Misconception

Here students responded to the integral: $\int_a^b f(x)dx = c$ by writing $\int_b^a f(x)dx = \frac{1}{c}$ or $-\frac{1}{c}$.

This misconception applies where the limits were reversed under the changed variable, we call it the *inversion misconception.* It was manifested in response to the question,

Section II. Given that $\int_9^{16} \sqrt{x}dx = \frac{74}{3}$, what is $\int_{16}^9 \sqrt{t}dt$?

when the students inverted the answer given obtaining $\frac{3}{74}$, or gave $-\frac{3}{74}$. They seem to have applied an inappropriate algorithm obtained from *inversion of limits*, namely that they considered inversion of limits inverted the value.

• General function Misconception

Students were often unable to cope with the generalised function f(x) or f(t-1) since they could not apply a procedure. When considering the notation *f(t-1), f(x)+2*, some students misconceived the *t-1, x* as a variable to integrate or used *ft-f, ft-1, fx+2* as functions to integrate, with f as a constant. For example,

Section II.8. If $\int_a^5 f(x)dx = 63$, *and the graph of f(x) is shown below, what is a possible value of 'a'?*

Some students never used the given coordinates (5, 30) of the graph, instead they considered the function f(x) as x and obtained; $\left[\frac{x^2}{2}\right]_a^5 = 63$ i.e. $x \rightarrow \frac{x^2}{2}$

Another example was when dealing with:

Section II.9. *If* $\int_1^5 f(x)dx = 10$, *then write down the value of* $\int_1^5 (f(x) + 2)dx$.

One student regarded the function *f(x)+2* as *fx+2* so trying to find a constant f, he substituted *fx+2* and took the integral of it. Fortunately the answer was correct!

$$\left[\frac{f}{2}x^2\right]_1^5=10,\ f=\frac{5}{6},\ \left[\frac{5}{6}x^2\times 2+2x\right]_1^5=18$$

Section II.6. *If* $\int_1^3 f(t)dt = 8.6$, *then write down the value of* $\int_2^4 f(t-1)dt$.

The general function misconception appeared here as:

$$\left[\frac{f}{2}t^2-t\right]_1^3=8.6,\ f=2.15,\ \left[\frac{2.15}{6}t^2-2.15t\right]_2^4=8.6$$

Considering the notation f(t-1) the student misconceived the t-1 as a variable to integrate. i.e as the dependent variable rather than independent. Other examples were:

$$\left[\frac{1}{2}t^2-t\right]_2^4=4-0=4 \qquad \text{or} \qquad \int_2^4 f(t-1)dt=\int_1^3 f(t)dt\ -1=7.6$$

In every case above the students' known processes did not allow them to deal with functions such as f(t-1) and so they used not concepts, but incorrect procedures to deal with their lack.

- Integration Process Disregarded

$$\int_a^b f(x)dx \quad \Rightarrow \quad [f(x)]_a^b$$

Another way students coped with their inability to work with f(x) was to take the interval [a, b] and substitute it into the given function without integration; for example; $\int_1^4 (x-1)^2 dx=\left[x^2-2x+1\right]_1^4$.

The above examples of misconceptions show that students conceived integral as an instruction to carry out a process or algorithm not as the encapsulation of the limit of area sum. Also integral calculus was considered as a series of processes with related algorithms, without adaptable ideas.

Statistical comparison between pre- & post-test

Overall: A statistical analysis of the student' answers to the pre & post test was conducted to see if there was any significant difference in the aspects of process-oriented skills (section I) and conceptual understanding or proceptual thinking (section II) displayed by the students. The results of the t-test show that there is a significant difference for the results of the 36 students in proportions of correct answers of the pre and post test;

Section I: $\overline{X}_{pre}=0.44$, $\overline{X}_{post}=0.57$, $n_{pre}=n_{post}=36$, t=3.94, p< 0.001

Section II: $\overline{X}_{pre}=0.15$, $\overline{X}_{post}=0.32$, $n_{pre}=n_{post}=36$, t=5.19, p< 0.001

Although there is an improvement in both the process ability and the conceptual understanding of the students, the computer tutorial seems to have considerably improved the conceptual understanding as represented in the questions of section II compared with before the computer tutorial.

Gender Consideration: The results of the 36 students, 22 males and 14 females who were analysed for any gender influence in the computer work. A one-tailed t-test was used to investigate whether there was a significant improvement in the students' results after having computer tutorials. The males performed significantly better on the post-test than they did before the computer work (Section I: $\overline{X}_{pre}=0.43$, $\overline{X}_{post}=0.55$, $n_{pre}=n_{post}=22$, t=2.72, p< 0.01; Section II: $\overline{X}_{pre}=0.17$, $\overline{X}_{post}=0.30$, $n_{pre}=n_{post}=22$, t=3.23, p< 0.005). Also, the performance of the females was significantly higher than on the pre-test (Section I: $\overline{X}_{pre}=0.46$, $\overline{X}_{post}=0.61$, $n_{pre}=n_{post}=14$, t=2.86, p< 0.01; Section II: $\overline{X}_{pre}=0.11$, $\overline{X}_{post}=0.35$, $n_{pre}=n_{post}=14$, t=4.39, p< 0.001). This indicates that for both genders students' proceptual understanding was enhanced after having computer tutorials. However there was no significant difference in the results between the genders.

Individual question analysis: The individual question results were analysed on the basis of matched questions of section I & II, such as conservation of integral, the maximum values of an integral, transformation parallel to x, y-axis, as table 1 shows. Since it was hoped that there would be improvement, a one-tailed z-test was applied.

Table 1: Proportions of students giving correct answers

Concept and Section		$\hat{p}_{pre}$ (n=36)	$\hat{p}_{post}$ (n=36)	z	p
Conservation of Integral	Section I.1	0.69	0.72	0.26	n.s
	Section II.4	0.22	0.47	2.31	< 0.05
The maximum values of an integral	Section I.2	0.61	0.75	1.28	n. s
	Section II.3	0.00	0.17	2.68	< 0.005
Transformation parallel to y-axis	Section I.4 b)	0.72	0.89	1.83	< 0.05
	Section II. 9	0.11	0.44	3.40	< 0.001
Transformation parallel to x-axis	Section I.5	0.47	0.69	1.96	< 0.05
	Section II.6	0.11	0.50	3.95	< 0.001

We can see that the students who had used the computer performed considerably better afterward in the aspect of the conceptual questions indicated above. Moreover, student knowledge of the concept of transformation improved greatly in both sections. It means that the students had successfully achieved the bilateral aspects of process-oriented skills and conceptual understanding. One example, question 10 from section II, involving Riemann sums, was as follows:

> If f(x) is a strictly decreasing function, an approximation to $\int_1^4 f(x)dx$ using a Riemann sum (ie. of rectangles under the curve) with left end points and number of strips n=10 works out to be 7.615. Will an approximation with left end points and number of strips n=50 be:
>
> (a) more than 7.615 (b) less than 7.615
> (c) equal to 7.615 (d) less than or equal to 7.615
> (e) not possible to say (f) more than or equal to 7.615
>
> Explain your answer.

Analysing this revealed that the students improved in their understanding (section II.10: $\hat{p}_{pre}$=0.14, $\hat{p}_{post}$=0.50, n_{pre}= n_{post}=36, z=3.56, p<0.001). Many case of errors relating to Riemann integral involved the idea that left (upper) sum always increases as the number of strips increases regardless of whether it was a decreasing function. However the computer students obtained the idea that the value of the left (upper) sum decreases as the number of strips increases for a strictly decreasing function. Thus, regardless of gender, there were the significant improvements on students' understanding of process-oriented and conceptual integration for almost every question on the test.

Case study

Some of the students showed an excellent understanding in the post-test following the computer work. From now on I will discuss the work of one student who greatly improved in the aspect of conceptual understanding on section II. His proportion of correct answers in sections I & II were 8/10, 2/13 in pre-test, 10/10, 11/13 in post-test. For each of section I & II, the improved proportions of correct answer were as much as 20%, 71.6%. Improvements confirmed in the interviews after the tutorial included:

> Interviewer: Has this tutorial affected your understanding of integration? If so how?
> Student: Yes, I learnt more about finding integrals by summing rectangles.
> Interviewer: Which section was useful to you? How?
> Student: Riemann sum (leftsum, rightsum)

This shows that he thought he understood better than before. His solutions confirmed this. For example, one of the improved cases was the concept of maximum value where his work changed as follows:

The maximum value		Pre - Test	Post -Test
Section II	$\int_a^b (2-x-x^2)dx$	Single value integral misconception: He factorised, and substituted x=-2, 1, answered [1, $1\frac{1}{6}$]: $\int(2-x-x^2)dx = 2x-\frac{1}{2}x^2-\frac{1}{3}x^3$, $2(-2)-\frac{1}{2}(-2)^2-\frac{1}{3}(-2)^3=-3\frac{1}{3}$, $2(1)-\frac{1}{2}(1)^2-\frac{1}{3}(1)^3=1\frac{1}{6}$, so [1, $1\frac{1}{6}$]	He exactly solved by factorising: $2-x-x^2=0$, $(x+2)(x-1)=0$, x=-2 or x=1

His error in the pre-test involved the *single value integral misconception'*, the given function was factorised correctly, but the values x=-2, 1 were substituted for a, b to find the maximum value. He considered the maximum value as between x=1 and $x=1\frac{1}{6}$. However, he showed his revised understanding following the computer work as shown above, and solved correctly. For the question of transformations parallel to the x-axis he gave no attempt, in the pre-test. After the computer work, his improved concept that the translation of the graph leaves the area unchanged was shown as follows:

Transformation parallel to x-axis		Pre - Test	Post -Test
II.6	$\int_2^4 f(t-1)dt$	No attempt	He correctly made a relation as follows: $\int_2^4 f(t-1)dt = \int_1^3 f(y)dy$

In his interview, he demonstrated that he had an improved conception of transformation;

Interviewer: If we compare f(x)= x^2, with h(x)= $(x-3)^2$, what is the effect on the graph?

Student: It is moved right 3 units.

Interviewer: If we compare $\int_0^2 x^2 dx$ with, $\int_3^5 (x-3)^2 dx$, what's the change?

Student: Equal to $\int_0^2 x^2 dx$

Looking at his concept of Riemann sum, while he simply made a guess or no attempt before the computer tutorial, after he was certain of the concept. In the pre-test he selected a) by guessing, admitting this in the interview. In the post test he correctly selected, and his answer to the Riemann integral was improved as follows:

Riemann Integral	Pre - Test	Post -Test
II.5 Rightsum -Leftsum = 0	No attempt	He exactly selected.
II.10 Riemann sum	No attempt	He correctly answered: 'less than 7.615'
II.12 $\sum_{i=0}^{9} f(x_{i+1}) \cdot \frac{3}{10}$	He chosen the diagram of a) by guess.	He correctly selected the diagram b)

This improvement was confirmed in his interview, along with an understanding of limit:

Interviewer: For the number of equal sub intervals as $n \to \infty$, what's the value of the width? Will it be smaller, larger or 0? What about the limit of the width?

Student: 0, 0

. . . Interviewer: What's the meaning of the $\lim_{\Delta x \to 0} \sum_{i=1}^{n} f(x_{i+1})\,\Delta x$ and $\lim_{\Delta x \to 0} \sum_{i=1}^{n} f(x_i)\,\Delta x$?

How do they differ?

Student: Rightsum, leftsum

His interview showed a good understanding of the concept of antiderivative function too:

Interviewer: For example, x^2, x^2+3 and x^2-2 are all indefinite integrals of $2x$. Does it matter which of these you use when calculating $\int_a^b 2x\,dx$?

Student: No, they're all correct, but x^2 may be more correct because it could save confusion.

Interviewer: What's the meaning of 'antiderivative (indefinite) integral' and 'definite integral'?

Student: Antiderivative integral give a function giving area under the graph for any x-value, definite integral give area under the graph for a particular x value

He had a noticeable improvement in the concept of definite integral, antiderivative function, Riemann sum, Riemann integral, and transformation about the x-axis. Comparing the results of the pre-test and post-test, the computer work led to a great improvement, especially, in the aspect of conceptual understanding, and his interview showed his clearer thinking.

Conclusions

In the pre-test, students' attempts at solving section II problems where there is no clear process to execute exhibited gaps in their conceptual understanding. The cases of misconceptions and errors described here show students' understanding restricted to algorithms and process, lacking adaptable thinking. For example, $\int f(x)dx$ was seen as an instruction to carry out a process or algorithm not as the encapsulation of the limit of area sums. To improve the cognitive base for a flexible proceptual understanding of limits, areas and Riemann integral, suitable uses of computer software may be applied. The results of the post-test showed that the students' lack of understanding of concept was considerably improved. Evidence was provided by individual analysis, interview, and statistical analysis. The excel spreadsheet and symbolic manipulator allowed students to see what is included at each stage of the process, providing opportunities understanding of the concepts as well as the ability to calculate integrals. Specifically, the graphical aspects of concepts such as definite integral, transformations parallel to x, y-axis could be developed in a classroom, avoiding a passive style. The opportunity to study integration using the computer had given valuable perception into the processes of integration and this is in turn assisted the student's conceptual understanding. The experiment showed that when using the computer there was a significant development in conceptual understanding of concepts traditionally regarded to be of higher grade, and evidence of a more versatile form of thinking related to the computer experiences.

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Professionalisation and Change in Secondary Mathematics

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Secondary education in Victoria began in the 1850s, and the struggle for control of the mathematics curriculum dates back to this time. However this struggle was intensified with the emergence of public secondary education this century. By the 1960s University control was under attack, principally due to two main factors - the attempt to modify course content on the basis of a new mathematical paradigm, and the professionalisation of mathematics. The critical role that the "New Maths" played in these processes is examined.

It is difficult to realise that the first white settlement in Victoria only occurred in 1835. By 1853 the University of Melbourne had been founded thus signifying that education was to play a major role in the new colony. A challenge was immediately issued to the British traditions of University entrance - in 1855 attempts were made by two of the newly appointed professors, Hearn (History and Political Economy) and Wilson (Mathematics), to liberalise the entrance requirements. An air of utilitarianism spiced their comments. They pointed out that the compulsory study of the Classics or higher Mathematics was regarded by many prospective students as both a waste of time and a distraction from more 'practicable and profitable pursuits' (Clements, Grimison & Ellerton, 1989). Their objections also highlighted the consequent decline in standards occasioned by such mandatory requirements.

Their proposed reforms were rejected. A major area of concern was the maintenance of standards on a par with British institutions; but in seeking to allay this concern one of the consequences of the new University's first actions was to exercise an immediate influence on pre-university schooling. This may not have appeared a major concern at that time. In 1857 the role of the Universities with regard to schooling was outlined by the Vice-Chancellor of the University of Sydney.

> It is the privilege and the duty of the University ... to determine what previously acquired knowledge shall be demanded from those who seek admission to its teaching; and thus it virtually determines what shall be the general system in preparatory schools, private as well as public (quoted in Clements et al, 1989, 60).

The die had been cast. Despite the attempts by the Professor of Mathematics, and others, to loosen the constraints placed by Universities on preparatory schooling, and in so doing to broaden the appeal of university study to a larger cross-section of the community, the University of Melbourne introduced the requirement of a pass in Greek, Latin, Euclid, Algebra, Arithmetic, and English in order to gain admittance to the University (Ellerton & Clements, 1988). Even as questions were beginning to be asked of the central role of Euclid in mathematical education (Siddons, 1956), Euclid was firmly established as a topic in Victorian schools.

Indeed the events surrounding the questions of University entrance clearly illustrate that all professors, and not just those concerned with mathematics, were instrumental in influencing the type of mathematics taught at secondary level. In the period to 1900 Euclid was to feature in a number of other challenges to the nature of the content demanded of university matriculants. Two of these challenges, instigated by mathematicians, had mixed success, while the third, which the University was quick to accede to, was a request by the Faculty of Medicine for a change to the matriculation examination in geometry. The Faculty of Medicine was concerned that the examination with respect to Euclid was not rigorous enough to satisfy the British General Medical Council (Clements, 1989).

Throughout the 19th Century the government addressed the problems of primary education and left the provision of secondary education to the private sector.

This sector emerged in the eighteen-fifties to prepare students for the University of Melbourne, and consequently principals and staff were recruited from the English 'public' school sector. Not surprisingly they brought with them traditions and practices with which they were familiar, and so the formative developments in secondary education were influenced by English traditions. However, the unique social conditions of Australia necessitated many modifications to these traditions and more relevant courses were soon to emerge (Connell, 1962), but the demands by the professions to maintain international status exerted considerable conservative influences.

The mathematics taught in the schools was determined in part by the university but also in part by what was happening in Britain. In England the Revised Code of 1862 and the Taunton Commission of 1868 had argued for mathematics in secondary schools to be determined by student origins. The differentiation that this entailed meant that English school mathematics courses were built around arithmetic (Cooper, 1994). In 1873 Victoria also made arithmetic a compulsory subject at both primary and secondary level (Kamens & Benavot, 1991).

With courses strongly biased towards arithmetic, and with Euclid as essential for university entrance, Victorian education entered the twentieth century. The content of courses was entirely a selection from Classical mathematics. The Classical emphasis was on number as measurement (Spengler, 1926), and such an emphasis was particularly appropriate to a community grappling with the hardship of living in a land bereft of any of the traces of the civilisation that they had left behind. Measurement, primarily in the form of arithmetic, provided the technology to buttress commerce and ownership and not surprisingly was seen as an essential subject for all children to master.

Although the Western conception of mathematics had started to emerge from the mid-seventeenth Century little of its impact was to be felt within Victoria at the turn of this century. The Western paradigm emerged from the work of Descartes, Leibniz and others from the 17th Century onwards (Restivo, 1992). Its major achievement was the liberation of mathematical thought from the boundedness of sensory perceptions. Number, and shape, was abstracted from reality and new conceptions of number - complex numbers, quaternions, matrices - emerged. The infinite, banned from Classical thought, underpinned the calculus, and analysis, and the notion of function, became central to the comprehension of this paradigm. It was not until 1930 that calculus made its appearance upon the educational scene. In the ensuing years the secondary schools were at odds with the University over the purposes of schooling. To some extent this antipathy represented attempts by the Education Department to assert control over the secondary system, but it also represented the feeling within the private school sector that responsibility for determination of curriculum was the preserve of the schools (Phillips, 1962). This situation was resolved in 1945 by transferring responsibility for examinations to the Professorial Board. A sixth year of secondary schooling was introduced and three mathematics subjects were proposed, Pure Mathematics, Calculus and Applied Mathematics, and General Mathematics. Pure Mathematics, like its predecessor, was seen as defining the essential nature of school mathematics, and in this subject the emphasis remained the Classical mathematics paradigm. A prominent textbook of the time, that was used in classrooms from the nineteen twenties through the nineteen sixties, gave notional assent to the idea of function. Apart from a two-line definition and one and half pages of examples neither the word function nor the functional notation was used again in the remaining 500 pages of the book (Baker & Bourne, 1925).

A paradigm shift was yet to emerge. In fact, in the development of mathematics curriculum, major formative influences with an Australian bias were also lacking. In a country with an underdeveloped community infrastructure, with a passion from the mid-eighteen fifties for the quest for gold, and with vast rural expanses demanding proficient navigational expertise one might have expected a curriculum that at least gave some semblance of recognition to these conditions. Courses that included work on

probability, on astronomy and other navigational aids, on surveying, or the like might have been expected to be found. But sadly such courses were lacking. The original subject matter, and most of the later changes effected in mathematics courses up to this stage, came as a result of overseas trends or in response to overseas trends. What was very clear is that the teacher as educator had little impact on the nature of the content of courses. Why was this so?

Other influences on courses and their control were beginning to be felt. The emergence of a public secondary system might have been expected to introduce new influences onto the educational scene. After all Tate, the first Director-General of Education, ranked teachers very highly, and his ascendancy to the premier administrative position in education in this State might have been expected to elevate the role of the classroom teacher to curriculum developer (Selleck, 1982). And in the early stages new courses were introduced, but from the top down. One such course was a new arithmetic course that was introduced into the primary school in order to lay a good foundation for secondary education (Clements, 1989).

However, the initial fervour and support soon yielded to the demands of institutional control as the centralised system and growing bureaucracy within the Education Department marginalised the classroom teacher. The use of inspectors for both curriculum innovation and teacher assessment inevitably led to conflict in their role and subsequent diminishment of their effectiveness (Cramer, 1967). As many commentators were to note, the public secondary system by the early nineteen fifties was in a state of low morale and despondent malaise (Butts, 1955; McLaren, 1968). Neither in its secondary teachers nor within the ranks of the inspectors were the sources of innovation and reform, so desperately needed within the mathematics education fraternity, to be found.

Clearly the creation of the institutional infrastructure necessary to maintain secondary education had effects. But these effects were not all that might have been envisaged. The combination of factors arising from British influence, uniformity and centralisation had led to a situation wherein public educators, whether they be teachers or officials such as inspectors and administrators, had only minimal control over the determination of content within the education system. Other forces resistant to this aim were obviously in place. These forces related to the University professoriate and the Examination Boards.

Melbourne University saw the determination of syllabi in the preparatory years to university entrance, and as a consequence in all the years of secondary schooling, as within its mandate. As a consequence the responsibility for determining the nature of the education that students received rested with university professionals - the Council and members of the university professoriate. In 1912 the Government institutionalised this responsibility through the creation of the Melbourne University Schools Board as a sub-committee of the University Council. The Schools Board was the Examination Board responsible for the conduct of Junior and Senior, later to be called Intermediate and Leaving, examinations. Passes in these examinations were required for government employment. These examinations were also important for students wishing to matriculate.

What the Schools Board inherited was a tradition of university determination of the content of secondary schooling. This content was essentially derived from the conditions that prevailed in Britain. The Schools Board was so constituted that this tradition was not going to change. But derived courses tend to lack relevance and immediacy. Not surprisingly the form tended to dominate the substance - increasingly the education system came to be dominated by examinations. Education in Australia was seen by one prominent Mathematician in the following terms:

> The success or failure of their pupils in the examination has been regarded as the chief test of the value of the school. They have so dominated secondary education in Australia that even at the present time the most satisfactory way of describing the extent of school work in mathematics, or in any other

> subject, would be to give copious extracts from the handbooks for these examinations where the details of the subjects of study are to be found (Carslaw, 1914, p.11).

Examinations very early on began to dominate educational discussion. The combination of a developing public secondary system with minimal demand among its pupils for university education, combined with a registered school sector with a high regard for university status, and with special relationships and privileges with the university resulted in little or no resistance. But the situation was changing.

The era of the university as guardian of preparatory educational responsibility was coming to an end. The wider social and ideological stage had been set for a challenge to the hegemony of university mathematicians and teachers in the schools were only too happy to join the fray. The charisma of the university, its acceptance and respect within society, had provided the basis of its unchallengeable right to control secondary education. By the nineteen sixties, a new professional grouping within society was claiming secondary education as their concern. The professional educator had emerged as a major participant in the struggle for control of secondary curriculum. Their role was enhanced by changed social circumstances which witnessed an increasing demand for mass secondary education. The university responded by shifting its focus from wider educational concerns to the more narrow concerns of the career academic.

But it was not only the wider educational concerns that motivated the impending challenge. Mathematics itself had changed. While the developers of school courses were struggling to come to grips with the transition from a Classical paradigm to a Western paradigm, another paradigm was emerging. The name Bourbaki began to resonate in education corridors. The Structural paradigm had arrived. From number as measurement of the Classical to number as function of the Western had been added number as structure.

The "New Maths" provided one last opportunity for University mathematicians to retain their ascendancy in the area of curriculum formation. The motivation of Bourbaki, the systematisation and unification of mathematics using a set of fundamental underlying concepts - in part, operators, functions, topology, logic, sets and structures - struck a chord with those mathematicians looking for a renewal of the curriculum. The "New Maths" also reflected the desire of educators to teach a curriculum relevant to the practice of modern mathematicians. The curriculum was prepared by a group of experts and reinforced elitist notions of mathematics. In short, the "New Maths" offered a solution to the ever-increasing demands for a renewal of the mathematics curriculum in mathematical terms. The changes ran counter to the changing educational environment.

However, the "New Maths" was a significant attempt to bring about a paradigmatic change in the mathematics curriculum. It represented the embodiment of the Structural paradigm and offered the unification and codification of mathematics in terms of underlying structural concepts. It was at one and the same time seen as a simplification of the complexity and diversity of current mathematics research and the embracing of a new, and modern, paradigm. The attempt was an abysmal failure. The structural concept represented the work of mathematicians. It was unsuccessful because it failed to resonate with the concerns of the students and of the majority of teachers (Cribb, 1986). But one of its consequences was the transformation of the secondary curriculum. An unsuccessful attempt had been made to go forward but there was no turning back (Elkins, 1977). The Classical paradigm was also a victim of the failure of the Structural. Secondary mathematics curriculum moved on to embrace the Western paradigm. Loosely, we can now view the primary curriculum as being essentially underpinned by the Classical paradigm, the secondary by the Western, and the tertiary by the Structural. This attempt at reform represents the last *major* attempt to change content within the mathematics curriculum in Victoria. Looked at in terms of the underlying paradigms, we can see why. There is essentially nowhere else to go.

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"New Maths" is symbolic of the end of one era and the start of another. The era from colonialism until the nineteen sixties is characterised by its emphasis on subject matter - viz mathematics, by the emergence of secondary education, and by the fact that the major intellectual influences are to be found in the form of the University professoriate and of University mathematicians. The educational ideology of the era centred on education as training of the mind and the development of a cultural elite. The educational emphasis was on content and memorisation (Connell, 1962). To the extent that psychology was invented to socialise people to their society's views and values it is interesting to note that the major psychological theories that gained ready acceptance and dominated educational thought throughout this era, the connectionism theories of Thorndike and the rote learning theories of the radical behaviourist Skinner, reflected this emphasis on the mastery of content and character training through memorisation. Thorndike offered a classic formulation of his theory in the

> law of effect, which held that any act that produced satisfaction became associated with the circumstances under which it arose and was more likely to recur if those circumstances recurred. In other words, purposive behaviour was a compounding of learned reactions. Habits took the part in the mind formerly reserved for will, and character-training became a question of habituation" (Hale, 1980, 82).

Other psychological theories, although available, were not taken up (McQualter, 1974).

The underlying "cognitive base" of this era was mathematical and education was discussed in terms of mastery of content. This era represented a period of control of the secondary curriculum by the University. The institutional structures in place throughout this era seek to confirm this analysis. Firstly the Schools Board, legally gazetted as responsible for curriculum determination, sought to involve, under University guidance and co-operation, the participation of practising teachers and other bodies vitally concerned with the consequences of schooling. But when the concerns of teachers began to place demands on the secondary curriculum that were unacceptable to the University professoriate, control shifted quite forcibly to the Professorial Board. A strict hierarchical structure, together with a regime of examinations ensured that University mathematicians were firmly in control of the content of secondary mathematics curriculum well into the nineteen sixties.

In the nineteen sixties a new examination board was put into place. This examination board reflected the shift in emphasis to a system with professional education as its core. The era of control of the University Professoriate had begun to wane. Why did the influence of University mathematicians wane? Many explanations are possible - for one a new "cognitive base" was emerging. However there were also conditions that saw the University mathematicians contributing to their own decline.

Professionalism emerges from the institutionalisation of expertise by modern societies (Larson, 1977). Among the many characteristics that identify professionalism are the professional association, cognitive base, institutionalised training, as well as work autonomy and colleague 'control'. Are University mathematicians professionals? And, if so, what are the consequences of this answer for their interest and involvement in the definition of secondary school mathematics curriculum?

Professionalisation of mathematicians is essentially a consequence of events of the second half of the nineteenth century which culminated in the twentieth century (Schubring, 1981). University mathematicians, by the nature of their employment, obviously enjoy work autonomy and colleague 'control'. In 1947 the first PhD candidate graduated from an Australian University. In effect a well-recognised credential had become available to provide the category of institutionalised training.

The concept of "cognitive base" delineates the problems a profession must address, and thus enables a profession to assert supremacy over these problems. Such a cognitive base was essentially lacking from the conception of mathematics in terms of the Classical paradigm (Restivo, 1992). By its general nature and its pervasiveness of all tertiary educational courses this conception of mathematics did not provide the

exclusiveness needed to provide a unique jurisdiction to the mathematician. Many mathematical problems were solved by gifted amateurs (Davis & Hersh, 1981). However the Western paradigm provided the basis for a unique, and exclusive, cognitive base and thus created the conditions for the emergence of the professional mathematician (Schneider, 1981). With the advent of the Structural paradigm, which allowed for greater specialisation and hence greater abstraction, the conditions for the spawning of a profession within Australia were present.

All that was now needed was a professional association. This was provided with the formation in 1956 of the Australian Mathematics Society. Australian mathematicians could now be recognised as professionals in their own right. Professional mathematicians applied themselves to the domain of their own expertise - a domain that was becoming increasingly distanced from other disciplines and from secondary teachers. Kline (1980) observed that the work of mathematicians turned from applications to a preoccupation with mathematics itself. In Kline's terms, mathematics had become inward looking. But in terms of the professional phenomenon mathematicians were seeking intellectual opportunities that would enable them to make a research contribution. With professionalism comes increasing specialisation. Secondary school mathematics now only interested a few mathematicians who chose to specialise in that area. It was no longer viewed as an essential concern of the University in general.

To some extent the episode of the "New Maths" highlights one of the social changes that saw the universities lose control of the curriculum. For the focus of university academics had shifted from broader educational concerns to concern with the demands of the professional. The characteristic of professionalism, the institutionalisation of expertise, was also to lead to the alienation of the teaching fraternity. As the cognitive base of the academic professional became more abstract, so did the distance between the academic and the teachers. And with this distancing the gulf between what Universities were demanding at matriculation and what teachers were expecting from their students was widening. This loss of confidence in the University, allied with other social factors in evidence, effectively undermined the University's legitimacy in the field of curriculum determination.

Mathematicians had developed new passions, and consequently their grasp on secondary education curricula was weakened. Other groups with other educational philosophies would emerge to claim this responsibility. Prominent among these groups would be mathematics educators, teachers and others.

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Secondary Mathematics Teacher Beliefs About the Learning and Teaching of Mathematics[1]

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This paper continues an ongoing research investigation of teacher beliefs towards the learning and teaching of mathematics. The focus is the espoused beliefs of a group of secondary teachers. Analysis of returned questionnaires allows secondary teachers to be profiled according to correlations of their espoused beliefs about mathematics, mathematics learning and mathematics teaching, along a continuum of learning and teaching approaches.

Introduction

Though the investigation of teachers' beliefs is a relative new area of research (Thompson, 1992), evidence suggests that such beliefs play a critical role in determining how teachers teach (Pajares, 1992; Weissglass, 1992). This report continues the research agenda presented at MERGA 17 (Perry & Howard, 1994) which set the basis for the reported investigation of primary teacher's use of manipulatives (Howard, Perry & Conroy, 1995), primary teacher's espoused beliefs about the learning and teaching of mathematics (Perry, Howard & Conroy, 1996) and secondary teachers' use of manipulatives (Howard, Perry & Lindsay, 1996).

This investigation of the espoused beliefs of 249 secondary mathematics teachers, working during 1996 in government and Catholic high schools in the south western suburbs of Sydney, adds to the literature and provides some basis from which decisions concerning mathematics syllabus and pedagogy might be made. Further, it provides a foundation on which interviews and classroom observations may be based in order for comparisons to be made between secondary mathematics teachers' espoused and enacted beliefs.

Teacher Beliefs about the Learning and Teaching of Mathematics

The development of one's beliefs about the nature of mathematics and how one does mathematics "*are important not only because they influence how one thinks about, approaches, and follows through on mathematical tasks but also because they influence how one studies mathematics and how and when one attends to mathematics instruction*" (Garofalo 1989, p. 502). It is recognised that a student's prime source of mathematical experiences is the classroom (Frank, 1988) and what occurs in the mathematics classroom influences student beliefs. Critical to the classroom implementation of the learning and teaching of mathematics is the teacher and, in particular, the beliefs of the teacher. All teachers hold beliefs towards the learning and teaching of mathematics. These beliefs influence and guide teachers in their decision making and implementation of teaching strategies (Baroody, 1987). Indeed, it has been suggested that the investigation of peoples' beliefs about learning and teaching may well be the most critical factor in educational research (Pajares, 1992).

The beliefs espoused by teachers of mathematics fall within a continuum ranging from a traditional view of mathematics being taught to students through a teacher transmission of mathematical skills and knowledge to one where students are actively involved with mathematics through "*constructing their own meaning as they are confronted with learning experiences which build on and challenge existing knowledge*" (Anderson 1996, p. 31).

One model for categorising beliefs about the teaching of mathematics presented by Kuhs and Ball (1986) suggests that teachers can hold views that fall into four broad categories: learner focused; content focused with an emphasis on conceptual knowledge; content focus with an emphasis on performance; classroom focused. A further model, categorising teachers'

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espoused beliefs on the continuum of transmission to child-centred, has been used in an attempt to profile the mathematical beliefs held by primary school teachers (Perry, Howard & Conroy, 1996). In this approach, conflicting beliefs may become apparent when attempting to cluster and categorise espoused beliefs, partly because teachers will hold beliefs along the continuum and across identified views. Thompson (1992) has reported that teachers' conceptions of mathematics appear to be related to differences in their views about teaching mathematics. Teachers' beliefs seem to evolve from their teaching experience rather than formal study and there appears to be a strong relationship between teachers' conceptions of teaching and their conceptions of students' mathematical knowledge. This study focuses on identifying and categorising the espoused beliefs of secondary mathematics teachers towards the nature of mathematics, mathematics learning and mathematics teaching.

Methodology

The data for this investigation were collected using a specifically designed questionnaire consisting of both multiple choice and open-ended questions covering the following areas:

1. subject demographics such as gender, age, position in school, nature of teacher training, length of teaching experience, class(es) currently taught, class size, classes taught over the last ten years;
2. use of manipulatives in mathematics learning and teaching such as which are used, why and how they are used, and the areas of mathematics in which they are used;
3. beliefs about mathematics, mathematics learning and mathematics teaching.

The questionnaire relies on the self reporting of the teachers and parallels much of the work reported in Hatfield (1994). It is substantially the same questionnaire which was used to gather data from primary school teachers (Perry, Howard & Conroy, 1996).

In September, 1996, the questionnaire was posted, with reply paid envelopes, to 52 secondary schools in the South Western suburbs of Sydney. Fifteen schools were part of the Catholic Secondary Schools System while 37 were NSW Department of School Education (government) schools. Schools were contacted by telephone to gain the Principals' initial approval to undertake the survey in the schools and to ascertain the number of mathematics teachers in each school. Sufficient numbers of the questionnaire were posted to cover all mathematics teachers at each school. Two hundred and forty-nine responses were received by mid-October, 1996. Data from these responses were analysed using the SPSS-X program to provide descriptive statistics for the demographic data, the use of manipulatives and teacher beliefs about mathematics, mathematics learning and mathematics teaching. As well, inferential statistics dealing with relationships between these variables were generated.

Results

This paper reports on some of the baseline data concerning secondary mathematics teachers' demographics and their beliefs about the nature of mathematics, mathematics learning and mathematics teaching.

Demographic data

Of the 249 respondents, 112 (45%) were female. The respondents appear to be a relatively well experienced group, with 64 (26%) of the respondents having in excess of 20 years teaching experience, while 83 (33%), 51 (21%), 42 (17%) had from 11 to 20 years, 6 to 10 years, 1 to 5 years teaching experience respectively. Only 8 respondents (3%) had less than one year of teaching experience.

Fifty-four (22%) of the respondents had been at their current school for more than 10 years, while 89 (36%) and 74 (30%) had taught in their current school for 6 to 10 years or 1 to 5 years respectively. Only 29 (12%) had been in their current school for less than 1 year. The school positions held by the respondents are reported in Table 1.

Table 1 School positions held n=249

Current position	Number	Percentage
Principal	3	1
Leading Teacher	7	3
Head Teacher (Mathematics)	40	16
Classroom Teacher (Mathematics)	193	78
Other	5	2

Two (1%) of the respondents described themselves as two year trained teachers, 13 (5%) as three year trained, 57 (23%) as four year trained with a Bachelor of Education, 154 (62%) as four year trained with a Bachelors degree followed by a Diploma of Education and 21 (8%) as teachers with more than four years of training.

An attempt was made to ascertain where the respondents saw their major teaching focus. They were asked in which of the Year ranges 7 and 8, 9 and 10, 11 and 12, they had mostly taught in the past 10 years. Sixty-two respondents (25%) saw their major emphasis in Years 7 to 10 while 40 (16%) saw the major emphasis of their teaching being in the senior secondary school (Years 11 and 12). Of the remainder, 127 (51%) reported that they had taught across all Years in the last ten years and data were missing for the remaining 20 respondents.

Statements of beliefs - Frequency of teacher responses

Table 2 reports the overall frequency of teacher responses to the 20 statements of beliefs concerning the nature of mathematics and mathematics learning and teaching. Teachers were asked to respond to each belief statement by indicating whether they disagreed with the statement, were undecided about it, or agreed.

Table 2
Statements of belief - Frequency of responses (Percentage of respondents) n=249

Statement of belief	Disagree	Undecided	Agree
Mathematics			
1. Mathematics is computation	88 (35)	40 (16)	111 (45)
2. Mathematics problems given to students should be quickly solvable in a few steps	153 (61)	50 (20)	43 (17)
3. Mathematics is the dynamic searching for order and pattern in the learner's environment	22 (9)	43 (17)	180 (72)
4. Mathematics is no more sequential a subject than any other	171 (69)	36(15)	39 (16)
5. Mathematics is a beautiful, creative and useful human endeavour that is both a way of knowing and a way of thinking	12 (5)	31 (12)	201 (81)
6. Right answers are much more important in mathematics than the ways in which you get them	215 (86)	15 (6)	15 (6)
Mathematics learning			
7. Mathematics knowledge is the result of the learner interpreting and organising the information gained from experiences	12 (5)	30(12)	201 (81)
8. Students are rational decision makers capable of determining for themselves what is right and wrong	90 (36)	95 (38)	60 (24)
9. Mathematics learning is being able to get the right answers quickly	201 (81)	28 (11)	17 (7)
10. Periods of uncertainty, conflict, confusion, surprise are a significant part of the mathematics learning process	18 (7)	26 (10)	202 (81)
11. Students are capable of much higher levels of mathematical thought than has been suggested traditionally	51 (21)	107 (43)	87 (35)
12. Being able to memorise facts is critical in mathematics learning	72 (29)	37 (15)	137 (55)
13. Mathematics learning is enhanced by activities which build upon and respect students' experiences	5 (2)	31 (12)	209 (84)
14. Mathematics learning is enhanced by challenge within a supportive environment	2 (1)	13 (5)	231 (93)
Mathematics teaching			
15. Teachers should provide instructional activities which result in problematic situations for learners	6 (2)	38 (15)	202 (81)
16. Teachers or the textbook - not the student - are the authorities for what is right or wrong	149 (60)	49 (20)	46 (19)
17. The role of the mathematics teacher is to transmit mathematical knowledge and to verify that learners have received this knowledge	60 (24)	45 (18)	140 (56)
18. Teachers should recognise that what seem like errors and confusions from an adult point of view are students' expressions of their current understanding	13 (5)	51 (21)	180 (72)
19. Teachers should negotiate social norms with the students in order to develop a cooperative learning environment in which students can construct their knowledge	30 (12)	58 (23)	157 (63)
20. It is unnecessary, even damaging, for teachers to tell students if their answers are correct or incorrect	206 (83)	35 (14)	5 (2)

As with questionnaire research in general, a limitation to this study is that all results must be considered in the context that responses to the belief statements are dependent on the interpretations given to them by the respondents.

Beliefs about mathematics

Forty-five percent of the respondents indicated a belief that *mathematics is computation*. In spite of this, 86% disagreed with *right answers are much more important in mathematics than the ways in which you get them*. As well, 72% believe that *mathematics is the dynamic searching for order and pattern in the learner's environment* and 81% believe that *mathematics is a beautiful, creative and useful human endeavour*, perhaps reflecting the fact that most of the

respondents are university trained mathematicians. However, 69% of respondents disagree with the statement that *mathematics is no more sequential a subject than any other*, suggesting a belief about the lock-step nature of so much secondary mathematics teaching and learning.

Beliefs about mathematics learning

Eighty-one per cent of respondents believe that *mathematics knowledge is the result of the learner interpreting and organising the information gained from experiences*. This is supported by 84% of respondents who believe that *mathematics learning is enhanced by activities which build upon and respect students' experiences* and *by challenge within a supportive environment*. Further, 81% of the respondents saw that *periods of uncertainty, conflict, confusion, surprise are a significant part of the mathematics learning process*.

Fifty-five per cent of the respondents agree that *being able to memorise facts is critical in mathematics learning*. Only 35% agreed that *students are capable of much higher levels of mathematical thought than has been suggested traditionally*. The large number (43%) of respondents who were undecided about this statement suggests that either the statement itself was not clear or that the teachers were reflecting on the obligatory syllabuses in secondary schools and suggesting that there was not much scope for students to go beyond this. Eighty-one per cent of respondents disagreed that *mathematics learning is being able to get the right answers quickly*.

Of particular interest is the belief that *students are rational decision makers capable of determining for themselves what is right and wrong* where only 24% of respondents agreed with the statement. This would suggest that teachers do not have a high opinion of their students' independent thinking skills in mathematics.

Beliefs about mathematics teaching

The belief that *teachers should provide instructional activities which result in problematic situations for learners* was agreed to by 81% of the respondents with 60% disagreeing that *teachers or textbooks - not the student - are the authorities for what is right or wrong*. The majority (63%) of respondents agreed that *teachers should negotiate social norms with the students in order to develop a cooperative learning environment in which students can construct their knowledge*. Neither the nature of these norms nor their source was investigated in the survey but both are of importance in determining the nature of the mathematics classroom.

The belief that *the role of the mathematics teacher is to transmit mathematical knowledge and to verify that learners have received this knowledge* was agreed to by 56% of the respondents, while 83% of the respondents disagreed with the statement that *it is unnecessary, even damaging, for teachers to tell students if their answers are correct or incorrect*. Even though 72% of respondents agreed that *teachers should recognise that what seem like errors and confusions from an adult point of view are students' expressions of their current understanding*, the combination of these results suggests the continuation of the common, although stereotypical, view of secondary mathematics teachers as content oriented, transmission teachers who reluctantly accept that there may be ways to teach mathematics beyond those they have experienced as students.

Linking beliefs

It could reasonably be expected that teachers' beliefs about mathematics, mathematics learning and mathematics teaching would be linked in various ways. To investigate this, a correlational analysis was applied to the beliefs data supplied by the secondary mathematics teachers. From the 180 correlation coefficients generated in this analysis, 53 were significant to at least $p<.05$. Clearly, it is impossible to discuss all of these in this paper. Instead, we concentrate on two apparently contrasting belief statements about mathematics teaching:

17. The role of the mathematics teacher is to transmit mathematical knowledge and to verify that learners have received this knowledge.

19. Teachers should negotiate social norms with the students in order to develop a cooperative learning environment in which students can construct their knowledge.

These two items have been chosen for two reasons. Firstly, they are among the items which have the highest number of significant correlations with other items and some of the highest correlation coefficients. Secondly, they reflect the two ends of the transmission / constructivist continuum referred to earlier.

Significant positive correlations were identified between the first of these statements and the following:

2. *Mathematics problems given to students should be quickly solvable in a few steps* (r=.19, p<.005)
9. *Mathematics learning is being able to get the right answers quickly* (r=.19, p<.005)
12. *Being able to memorise facts is critical in mathematics learning* (r=.19, p<.005)
16. *Teachers or the textbook - not the student - are the authorities for what is right or wrong* (r=.15, p<.05)

Significant negative correlations were identified between Statement 17 and the following:

11. *Students are capable of much higher levels of mathematical thought than has been suggested traditionally* (r=-.14, p<.05)
18. *Teachers should recognise that what seem like errors and confusions from an adult point of view are students' expressions of their current understanding* (r=.-19, p<.005)
19. *Teachers should negotiate social norms with the students in order to develop a cooperative learning environment in which students can construct their knowledge* (r=-.18, p<.01)

Significant positive correlations were identified between Statement 19 and the following:

5. *Mathematics is a beautiful, creative and useful human endeavour that is both a way of knowing and a way of thinking* (r=.20, p<.005)
7. *Mathematics knowledge is the result of the learner interpreting and organising the information gained from experiences* (r=.15, p<.05)
8. *Students are rational decision makers capable of determining for themselves what is right and wrong* (r=.19, p<.005)
10. *Periods of uncertainty, conflict, confusion, surprise are a significant part of the mathematics learning process* (r=.14, p<.05)
11. *Students are capable of much higher levels of mathematical thought than has been suggested traditionally* (r=.18, p<.005)
13. *Mathematics learning is enhanced by activities which build upon and respect students' experiences* (r=.22, p<.001)
14. *Mathematics learning is enhanced by challenge within a supportive environment* (r=.22, p<.0005)
15. *Teachers should provide instructional activities which result in problematic situations for learners* (r=.26, p<.0001)
18. *Teachers should recognise that what seem like errors and confusions from an adult point of view are students' expressions of their current understanding* (r=.30, p<.0001)

As well, there were significant negative correlations between Statement 19 and the following:

9. *Mathematics learning is being able to get the right answers quickly* (r=-.13, p<.05)
16. *Teachers or the textbook - not the student - are the authorities for what is right or wrong* (r=-.27, p<.0001)
17. *The role of the mathematics teacher is to transmit mathematical knowledge and to verify that learners have received this knowledge* (r=-.18, p<.01)

While some of these correlation coefficients are relatively low, their levels of significance and the sample size suggest that they should be included in discussions of an overall profile of groups of secondary mathematics teachers.

Profiling teachers

The two groups of belief statements identified through the correlational analysis above give two very different profiles of secondary mathematics teachers. The first group might be categorised as a 'transmission' profile with the teachers believing that their role is to transmit and verify reception of knowledge by their students, and that mathematics learning is characterised by memorisation, quick and correct solutions. Further, this group of teachers is at variance with colleagues who espouse the likelihood that students are capable of more than has traditionally been thought and the value of recognising student errors as a part of the

learning process. In many ways, this is a profile of the stereotypical, traditional mathematics teacher.

The second group might be categorised as a 'constructivist' profile with the teachers believing that their role is negotiate the classroom climate so that students are able to construct their own knowledge and that mathematics learning is characterised by the beauty and relevance of mathematics and has its basis in the experiences of the students. 'Errors', challenge, problems, confusion and conflict all have roles in this learning according to this profile and students are seen as rational beings capable of more than they have been traditionally asked to do. Further, this group of teachers is at variance with colleagues who espouse that mathematics learning and teaching is about transmission and verification of knowledge by an outside authority and about getting right answers. This is a profile of a group of teachers, such as those described by Anderson (1996) and Lo, Wheatley, & Smith (1994), espousing 'constructivist' beliefs about mathematics learning and teaching.

Of course, the teachers who might be thought to comprise the two identified groups do not necessarily belong to one group exclusively. For example, 81 of the teachers (33% of the total sample) have indicated that they agree with both items 17 and 19, the key statements in the formation of these groups. This can be reconciled by recalling that any individual teacher will fall along the continuum of which the two identified groups form the end points.

Conclusion

Teachers hold beliefs towards the nature and the learning and teaching of mathematics. These beliefs do influence their teaching. There is evidence from this study that there are many teachers who espouse sets of beliefs which might be described as 'transmission' beliefs and many who espouse sets of beliefs which could be described as 'constructivist'. That teachers at these two ends of the continuum, and all points in between, can come to grips successfully with the current rigid syllabuses and examination systems in place in NSW secondary schools is amazing. How they do this needs to be investigated further through interviews and classroom observations. The researchers have commenced interviewing Head Teachers, Mathematics to seek some answers from the people who are probably most influential in determining the approaches to mathematics learning and teaching in individual schools. This work will be reported later.

As well as seeking data from interviews with teachers, there is support for the view that teachers' reflection on their classroom experiences can shape and influence their beliefs. Further investigation comparing the espoused beliefs of teachers with their enacted beliefs is required. This will require some detailed classroom observation. Particular emphasis needs to be placed on investigating the effect of classroom experiences on the evaluation and reorganisation of teacher beliefs and the effect of this reorganisation on what occurs in the mathematics classroom.

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The Role of Whole Number Knowledge in Rational Number Learning

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We present data from two children who participated in a teaching experiment to investigate fraction learning and the role whole number knowledge might play in it. A major source for the children's experiences was an operator-like computer program called CopyCat. How these children were constrained in their efforts to succeed with set tasks because of limited facility with whole number sequences, is demonstrated. Analyses of selected teaching episodes are discussed.

In a previous study (Hunting, Davis, & Pearn, 1996) we reported on the accomplishments of two 8-year-olds, Elliot and Shannon, as they solved fraction comparison problems using an operator-like computer program called CopyCat. We noted the interdependence of the development of rational number knowledge and whole number knowledge. In particular, we concluded that facility with whole number relationships enables students to solve fraction comparison problems. We also found that rational number tasks in operator settings can help stimulate and extend children's whole number knowledge. This study examines the behaviour of two other children, Narelle and Tanya, who participated in the same teaching experiment. These girls were judged to be similar enough in their understanding of whole numbers to be grouped together for the teaching sessions that we conducted. We will describe the fraction tasks these children were asked to solve in particular episodes, comment upon the quality of whole number knowledge exhibited through their efforts to solve these tasks, and consider implications for teaching and further research.

The aim of the teaching experiment was to investigate what basic fraction knowledge might be learned in a context where children's knowledge of whole number relationships, including numerical sequences, was needed. The whole number knowledge needed to succeed with the computer-based fraction tasks drew on pre-school and early school experiences foundational to formal symbolic manipulation. In that sense it was informal knowledge.

By the time fraction instruction commences, children have considerable knowledge of whole numbers and how they work. It has been observed that children's whole number schemes interfere with the acquisition of fraction knowledge (Behr, Wachsmuth, Post, & Lesh, 1984; Hunting 1986; Streefland, 1991). Recently attention has turned to investigating ways in which the whole numbers might assist, and indeed form a basis for, developing concepts of rational numbers (Carraher, 1993; Steffe & Olive, 1993). Fractions have many interpretations, including that of measure, quotient, ratio, and operator (Kieren, 1976; 1993). The interpretation embodied in the computer tool we have been using is that of fractions as operators.

Method

Ten children selected for the teaching experiment were all the Year 3 part of a mixed Year 2/Year 3 class in a state-funded primary school consisting of predominantly middle class children in Melbourne's eastern suburbs. Their ability range in mathematics was wide. The teaching experiment spanned two school years. The first teaching session commenced in May 1992; the last teaching session concluded in October 1993. We

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conducted teaching sessions twice weekly for periods of three weeks at a time. Each session lasted approximately 20-25 minutes. In a typical morning we would conduct three to four sessions; each session working with groups of two or three children. Tanya and Narelle were aged 7 years 10 months, and 8 years 3 months at the time of initial interviews.

The Computer Tool

The Copycat is an operator-like computer based learning tool developed in Hypertalk 2.0 for use on Apple Macintosh computers. Experiments performed may be directed at determining what fraction is responsible for observed numerical inputs and corresponding outputs, or at determining the numerical value of inputs or outputs for selected fractions governing the Copycat's behaviour.

On the Copycat itself are three "buttons;" each can be activated with a mouse-click when the cursor is positioned over it (see Figure 1). The arrow buttons control the number of counters placed on the "in-tray". Counters are added or subtracted one at a time. The Go button activates a script which determines what the Copycat will do. If the number of counters placed on the in-tray is divisible by the fraction denominator, **d**, and the Go button is clicked, an observer sees a group of **d** counters removed, one at a time, from the in-tray; simultaneously a distinctive sound is heard. Next, counters appear one at a time in the output tray until the number of counters in the tray equals the number in the numerator of the selected fraction. A new sound accompanies the appearing fraction. This process continues until all the counters in the in-tray have been used up. Finally, to the accompaniment of applause, all the counters reappear in the input tray; and the windows, above the words "In" and "Out," display the number of input and output counters. If the number of input counters is not divisible by **d**, the Copycat "explodes" to appropriate noises. Other buttons—visible and invisible—can be used to reset the machine after each experiment and to activate a screen to cover the selected fraction. The fraction can be selected by choosing a restricted set of numerators and denominators from the menu bar at the top of the display (not visible in Figure 1).

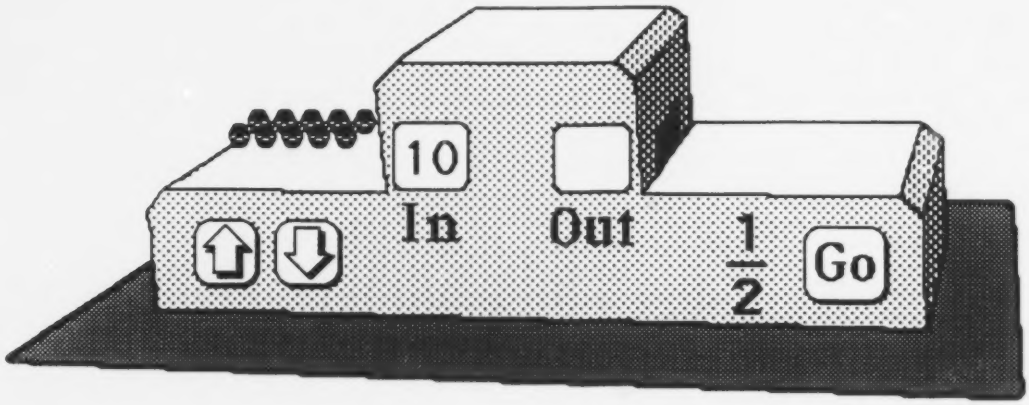

Figure 1: Graphical features of the Copycat

Results

We will discuss the behaviour of Narelle and Tanya from three teaching episodes; one near the beginning, one near the middle, and one near the end of the experiment.

June 5, 1992 episode

This session occurred early in the teaching experiment—on the third meeting after intial interviews had taken place to determine entering knowledge of whole numbers, sharing and fractions (Pearn, 1996). Gary Davis was the teacher on this occasion. In the group were four girls, Narelle, Tanya, Sarah, and Dianne. CopyCat was set to $\frac{1}{2}$, with the fraction visible. The general goal of each experiment was to predict the number of output items for each number of items input. Initial experiments were conducted with two, three, and four input items. The girls chose to call these items balls. CopyCat output one ball for two, exploded for three balls, and output two balls for four. Gary then input 20, followed by 12 balls. Tanya answered 10 balls for 20, and Dianne answered 6 balls for 12. The next three input numbers—24, 30, and 7—showed that numerical halving was indeed a restricted scheme for these girls, as the following excerpt demonstrates. Narelle first proposed 14 as the output from 24. Sarah agreed, and the other girls were easily convinced that this suggestion would be right. Similarly, for 30, one of the girls predicted an output of 20 balls, and the other girls agreed. For 7, Narelle suggested $5\frac{1}{2}$ balls, which indicated some understanding of the relation between odd and even numbers, and one-half. Seven was tested, and immediately after CopyCat exploded Tanya announced:

> You can't have odd numbers in it...
> Narelle: Because its half.
> Gary: Do you understand what she was saying Dianne? Tell us again Tanya what you mean by that.
> Narelle: (placing fingers down on table sequentially) 2, 4, 6, 8, ...
> Gary: Tell us what you mean, you can't have odd numbers. What does that mean?
> Tanya: It means that 2, 4, 6, 8, ... you can only have 2, 4, 6, 8, up to 100.
> Gary: Mm. Okay. But not like 1, 3, 5, 7,...
> Narelle: Yeah. Because that's an odd number, like if you have two, you have a partner and if you have three its one odd (holding up three fingers and touching one of them).

A few minutes later, Narelle chose to input 50 items, but Tanya objected:

> That's an odd number. You can't have it.
> Cath (observing at the side): Is it?
> Narelle: (turning round to face Cath): Its an odd number.
> Cath: Fifty's an odd number.
> Gary: Well press the (Go) button and find out.
> Narelle: Its an odd number.
> Gary: An odd number. You don't think there's partners in 50. Let's have a look.
> Tanya: Because there's five, and then there's (utterance indistinguishable).
> Narelle: (Presses Go button, which sets CopyCat processing).

Gary asked in turn if 10, 20, 30 were odd numbers. The girls hesitated for 10, then said no, said no for 20, but yes for 30. Gary asked one of the girls to input 30 items, and as this was being done, asked again if 30 could be made into partners. Each responded no.

Even though Tanya had the insight that CopyCat set to $\frac{1}{2}$ couldn't deal with odd numbers, and further, even numbers could be identified up to 100, these children's functionality with the concept of odd and even numbers was limited—on the evidence—to 20. Thirty was odd because it contained the odd digit 3; likewise in the case of 50. On the other hand, Tanya's observation was evidence of a mathematical generalization, although limited to her experience of even numbers up to 100. In fact, her conception of even numbers was limited to numbers less than 50, as her objection to 50 indicates. Narelle's reference to partners was an attempt to explain the meaning of even numbers at a physical level. The word partner may have also been a generalization for Narelle, since that word implies a pairing up of any number of items, but grounded in specific experience. Further

experiments with CopyCat, using inputs such as 30 and 50, would allow modification of these children's understanding of odd and even.

In this episode we witness the interrelation between whole number knowledge and fraction knowledge, in the context of these types of tasks using CopyCat set to operate as a half machine. To be able to succeed these children needed knowledge of specific number sequences, namely, ones and twos, and coordinations that would allow them to relate one to the other. It is clear that these children did have some basic knowledge. They knew half of 2 and 4. They knew 10 was double 5, 12 was double 6, and 20 was double 10. Both Tanya and Narelle began spontaneously uttering the verbal number name sequence for twos up to eight. We know from their initial interviews that both girls were able to count forwards by twos from 2 to 30, forward by twos from 1, and backwards by twos from 24. Later when Gary used Narelle's idea of partners, involving the six people in the room (three children and three researchers), they could visualise half of 6. But knowledge of 24, for example, being a number in a sequence of ones, and at the same time a number in a sequence of twos (which has correlates that are half, such as 12 in the case of 24), was not activated in this context. For a concept of even number to be functional in this context, we consider that the number sequence of ones has to be reconstitutable as composite units of two, where these units are a higher order than enactive or sensory motor units (Steffe, Cobb, & von Glasersfeld, 1988). A composite unit is a unit that is itself composed of units (Steffe, Cobb, & von Glasersfeld, 1988; Steffe, 1994). The CopyCat experiments allowed these children to extend their whole number schemes, just as their knowledge of number sequences limited their success with the experiments.

October 20, 1992

A different type of task was presented to the children on Friday October 16, and Tuesday October 20. An input number and corresponding output number were presented on a worksheet containing a representation of the CopyCat. There were no countable items on either the input tray or the output tray on the worksheet representation. The children were encouraged to experiment with the computer. The fraction was screened. The task was to discover what fraction was responsible for the input number and output result.

In this episode we observed Narelle counting balls by groups on the input tray as a way of checking if the predicted fraction would be correct.

In the first task a worksheet was shown with numerals 16 and 8 for input and output respectively. The computer CopyCat was blank, but set to $\frac{1}{2}$ with the fraction screened from view. Surprisingly, both children suggested $\frac{1}{3}$ as their first hypothesis. We say surprisingly, because just four days earlier both girls had readily proposed $\frac{1}{2}$ for an identical task. The CopyCat was set to operate as the fraction $\frac{1}{3}$, 16 balls were input. Of course the CopyCat exploded. Tanya then suggested $\frac{1}{5}$. The girls decided to test it, with negative result. Narelle then proposed $\frac{1}{4}$, set the CopyCat to that fraction, and input 16 balls.

> Robert: Now, how many do we want to come out here (pointing to Out tray)? Do you think a quarter will do it?
> Narelle: (Excited) hold on...(goes to screen counting under her breath while pointing her index finger at each of the first 12 balls displayed) one, two, three, four, (slight pause) one, two, three, four, (slight pause) one, two, three, four. (She did not count the last four balls.)
> Robert: Narelle what were you doing there? You were counting de, de, de, de,....
> Narelle: I don't think it will work.

> Robert: You don't think it will work. Why not? What do you think will come out?
> Narelle: Um, (points finger at input tray on screen and begins counting balls again) one, two, three, four (pause, then whispers) this four is one, (continues counting inaudibly, pausing after a count of four, counts sets of four subvocally). That,... four will come out.

Narelle had invented a way to check whether the CopyCat fraction would work, without having to test it. The numerical information given as 16 was insufficient for her. She needed sensory material upon which to act. It is most likely that her scheme was triggered by the proposed fraction denominator, in this case four. She counted segments of four from the display of 16 balls. Her provisional conclusion that the $\frac{1}{4}$ machine would not work—that is, would not output 8 balls—was confirmed when she focused on tallying how many lots of four were contained in 16. Later in this session, Narelle used the same strategy to reject $\frac{1}{3}$ as the fraction that would output 4 from 16 balls, confirmed that a 16-4 result would result from a $\frac{1}{4}$ machine, and voted down Tanya's proposal that the fraction $\frac{1}{10}$ might contribute to a 16-2 situation. Narelle's strategy was limited in that it eliminated fractions that would not work, rather than identify the fraction that would. There were some tasks for which Narelle and Tanya gave instant answers, without needing to measure out lots of balls. For example, immediately after the above task involving 16 balls in and 8 balls out, the girls said $\frac{1}{2}$ for 8 in, 4 out; 10 in, 5 out; and 16 in, 8 out again. But apart from such common relationships, tasks presented using a worksheet in which no countable items were available, were probably beyond Narelle's reach. Narelle did not know 16 as a composite unit of 4 fours, but she was able to construct this new knowledge, using the visual material on the screen. From an observer's perspective, the obvious way to deal with CopyCat tasks where the fraction is unknown, is to divide output by input. By virtue of there being a given output, for CopyCat, a fraction must necessarily exist. Such logic was not available to these children. But these children had yet to develop the formal relation of division as the reverse of multiplication.

August 25, 1993

On this occasion Narelle and Tanya were given the task of determining the CopyCat fraction, where that fraction was screened from view. In the first part of the session the fraction was set to $\frac{1}{3}$. To begin with, 18 balls appeared on the input tray and 6 balls appeared on the output tray.

We observed three instances where Narelle and Tanya used counting and grouping strategies; either by representing input numbers using tallies on paper, or by counting directly from the computer screen. Narelle began using a partitioning model, but this scheme was not well established, and broke down for her in the face of several CopyCat experiments. Tanya's scheme was more attuned to the operations of CopyCat. She counted out lots corresponding to the posited fraction denominator.

Narelle's initial response to the 18 in, 6 out, shown on the CopyCat, was to suggest that it was a half machine. While the teacher (Robert) was preparing a table to record experiments, with the headings In and Out, Narelle offered a counter-example to contradict her initial hypothesis, observing that if the Input number was 16, then the output wouldn't be 6. Tanya also remarked that half of 18 was 9. Narelle extended the fingers on both hands on front of her. Then on a sheet of paper provided by the teacher Narelle drew 18 small circular marks. She drew three larger closed curves around sets of six circular marks, looked up, and announced "I think it's a one-third machine". The teacher asked what would come out if 6 balls were input, if the CopyCat were a one-third

machine. Tanya said three would come out; Narelle hesitated but did not argue. The experiment was conducted, with two balls being output.

Twelve was proposed as the next input number to be tested. Narelle first suggested 6 as output, then reminded herself that such a result would indicate a half machine. Tanya said 4, reasoning thus:

'Cause if it was half of 6 it would have been three.

Narelle: Half of 12 would be 6.

Tanya: (Looks down at data appearing on sheet of paper) four.

Tanya may have extrapolated, reasoning that if 6 output 2, then 12 would have to output less than 6. She then changed her mind to 3. Narelle concluded 4 after drawing on her sheet of paper. She was not sure if the result would be 3 or 4. Just before Tanya began to test 12 using the computer, Narelle suddenly said:

Hold on! Don't touch it. (She began point-counting items on the screen) one, two, three, four, one, two, three, four, one, two, three, four. Oh, I'll have to do it again.

Narelle repeated the same counting procedure, except this time she kept a tally of the lots of four by extending fingers on her left hand. She and Tanya both concluded three would be output. But immediately Tanya went to the screen and began point-counting inaudibly. She changed her mind, confidently announcing "four's going to come out. Four's going to come out." Both girls counted three lots of four but came to different conclusions about what the output number would be. Narelle took the result of her tally of fours—three. Tanya put a different structure on the 12 items visible on the screen. We infer that she counted lots of three. Running the CopyCat confirmed Tanya was right. She did not express any sign of surprise at the result. Our inference is supported by a comment Tanya made to Narelle immediately after the conclusion of the 12-4 CopyCat result: "you counted by fours, not threes".

The teacher next asked what would happen if 24 balls were input. Narelle clicked 24 balls one by one on to the in-tray, commenced counting items, but became distracted when she noticed Tanya extending her fingers one by one. Tanya sequentially extended five fingers on one hand, then three fingers on the other, without speaking. She began again more deliberately, pausing after extending the seventh finger. "I think it's going to blow up" was her prediction. Tanya's counting sequence by three's was not robust enough to provide her with the solution. A mis-count likely led to a number other than 24 at the conclusion of her finger count. Tanya began counting screen items, asked Narelle for a pen, and drew a row of 24 small vertical marks on a sheet of paper. She encircled lots of three marks working from left to right, then announced 8 was going to come out. Narelle clicked on the Go button, and at the conclusion of the process, completed the data sheet, and said it was a one-third machine.

In the second part of the session the CopyCat was set to $\frac{2}{3}$, with the fraction screened once again. The initial data was 6 in 4 out. The teacher asked what would be output for 18 input. The children thought about it for 30 seconds, then Tanya suggested it might be a fourth machine. Eighteen items were input and Tanya knew before setting CopyCat to Go that $\frac{1}{4}$ was not right, because she was unable to divide 18 evenly by counting lots of four on the screen. Her next strategy was to draw 18 tallies on paper. She first confirmed her earlier decision that CopyCat was not a $\frac{1}{4}$ machine, by encircling groups of four tally marks. She discovered that 18 could be subdivided into 9 lots of 2, and concluded CopyCat was a $\frac{1}{2}$ machine. This hypothesis was subsequently disconfirmed when 12 items were output. The next suggestion—a one-third machine—was refuted by evidence from previous experiments. The fraction $\frac{2}{3}$ was more difficult for these children than was $\frac{1}{3}$.

In the August 25 teaching session, the CopyCat was set initially to $\frac{1}{3}$, and the task for the children was to discover what that fraction was since it was screened from view. The only way to find out was to investigate the results of inputting various numbers. If the input number was a multiple of three, the CopyCat would process that number; if not, it would explode. A record of successful input and output experiments was made on a sheet of paper to help the children remember experiments that worked. Narelle's scheme for working out what would be output for an input of 18 balls may have been based on a partitioning model, but how did she know the number of sub-units? Rather, it seems more plausible that Narelle took the output of 6 balls as a unit which was used to measure the 18. After observing the results of her work on paper, she noticed three equal groups of 6, hence her conclusion. Whereas in the case of 18 she associated the number of units of six with the needed fraction, in the case of 12, she knew there would be 3 lots of 4, but associated the number of lots of four with the output number, not the fraction. Thus she predicted 3 balls would be output. Tanya also used various devices to represent numbers in order to segment quantities. She used her fingers, but needed more permanent perceptual records on which to operate. She imposed a structure of threes on 24 tally marks. She had an implicit understanding of the operation of the CopyCat, enough for success. She did not understand one-third as an operation closely related to formal division. Rather, one-third was tied to counts of three. The degree of facility these girls had with numerical sequences, particularly in the case of one-third, sequences of threes, determined success in these tasks.

Discussion

In these three teaching episodes we have seen how Narelle's and Tanya's success with tasks involving the CopyCat was constrained by their whole number knowledge, particularly knowledge of number sequences, interrelationships between sequences having different units, and the development of numerical relationships we might call factors.

Two beliefs encouraged us to test an operator model of rational numbers in an experiment on fraction learning. First, we knew children bring to school a great wealth of informal knowledge about whole numbers, and that early childhood mathematics programs focus on developing and extending whole number skills and concepts. The CopyCat is an interpretation of rational numbers that draws on numerical relationships. We wanted to see how this approach to learning fractions was able to tap what children already knew. Second, we wanted to establish fraction knowledge beyond just associating symbols with restricted visual and physical quantities. We wanted children to understand, for example, that one-half is the relation that is common to 2 of 4, 3 of 6, 4 of 8, and so on. The CopyCat enabled the children to perform rapid experiments, which we hoped would engender deeper conceptions of fractions.

At the outset, we assessed the children's counting, partitioning, and fraction knowledge in clinical interviews. Tanya and Narelle were fluent with verbal counting forwards and backwards by twos, at least into the third decade. Neither child was successful counting forward by threes. It was no surprise then, that these girls constantly used one-half as a reference for checking possible results of CopyCat experiments. On the other hand, recognition of double/half relationships was limited to specific cases, such as 10-5, 12-6, 20-10, and so on. These children made the generalisation of numbers that could be halved to numbers that could have *partners*, and understood the significance of even and odd numbers to a limited extent. Thirty and 50 were odd because they contained a 3 and 5.

Not only do children need to be able to operate on number sequences, they need to be able to conceive of numbers as composite units—indeed as simultaneously comprising units of different size. Sixteen can be segmented into four units of 4, and also, through a mental shift of focus, into eight units of 2, or two units of 8. Narelle and Tanya had yet to develop the deeper understanding of number patterns incorporating different composite

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units. Their behaviour is consistent with children who Steffe, Cobb, and von Glasersfeld's (1988) would say have constructed the initial number sequence. They were yet to, in general, establish the first few whole numbers as *iterable* units (Steffe, 1994). Development of fraction knowledge in this setting was stimulating the creation of whole number knowledge which in turn enabled success with the fraction tasks.

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What Conflicts Help Students Learn About Decimals?

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Sixteen Form 1 and 2 students worked in pairs to solve problems involving decimal fractions. Their discussions were analysed to see what kind of conflicts led to learning about the meaning of decimals. Useful conflicts arose from those written into the problems, differences between calculator results and expectations, discussions with a peer, and differences between out-of-school experience and calculations. The most profitable conflicts could be classified as those that brought about what Piaget called beta reactions.

Introduction

Provoking cognitive conflict to help students understand areas of mathematics is often recommended. It is a technique that teachers used in a recent research project and reported to be very effective, especially when used as a task in which students were given a sheet with some incorrect answers and asked to "correct homework" (Britt, Irwin, Ellis and Ritchie, 1993). Students discussed reasons why an answer was right or wrong, usually correcting mistakes that they were likely to have made themselves. Another study that found conflict more effective than direct instruction was that of Swan (1983, 1993) who taught one of two parallel classes by having students talk about the differences between their original responses on a decimal problem to the answers that they worked out using number lines.

The hypothesis that learning happens as the result of a learner being faced with conflict was emphasised by Piaget (e.g. 1975/1985). He cites Inhelder, Sinclair and Bovet (1974) as having shown that the most influential factor in acquiring new knowledge structures was the perturbation, or discomfort, resulting from unresolved conflict. This discomfort motivates learners to seek a resolution that incorporates both their existing knowledge and the new information.

Not all conflict is useful in teaching, however. If a student meets a conflict that shows that they are wrong and they cannot find out why, they may be led to believe that mathematics does not make sense, is too hard for them, or that they don't like it. McLeod (1995) applies Mandler's theory of interruption of a schema to suggest that what happens when students meet a mathematical conflict that they cannot resolve. What they remember in this case is their negative emotional reaction, which in time may lead to a negative attitude toward mathematics.

Piaget (1975/1985) provided a framework for classifying reactions to perturbations, or the disequilibrium caused by a new fact that conflicts with an existing scheme. He does not refer to the educational value of different responses to perturbation. However, his categories provide a way of examining the educational usefulness of students' reactions to conflicts.

In brief, Piaget described three categories of reaction. The first of these, an alpha reaction, does not lead to change in the learner's scheme because either (i) it involves adding a fact to an existing category without any need for modification or (ii) the new fact is denied because the learner cannot see a way of incorporating it. The second category of reaction, a beta reaction, does involve a change in the learner's concepts. A learner's initial scheme is altered in some way and the new fact is seen in a different way so that it is transformed through reciprocal assimilation and accommodation. In the third reaction, a gamma reaction, new facts do not cause perturbation because the learner already has a scheme that is flexible enough to incorporate it. Piaget acknowledges the contradiction inherent in calling this a reaction to a perturbation, but the category is necessary to cover the range of reactions of a student to conflict.

From an educational point of view, the gamma reaction could be seen as the goal of learning in a particular mathematical domain. It involves a having a thorough grasp of a domain. Within the classroom, however, it is the beta reactions that seem appropriate for teachers to aim for in the short term. These reactions to conflict provide the conceptual

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building blocks that go to make up an overall flexible scheme. Alpha reactions, on the other hand, are less useful in learning. Adding of new facts to an existing scheme may help solidify concepts but after a certain point becomes repetition of knowledge that is already secure. Denial of a new fact is of use in maintaining a learner's self confidence but does not extend knowledge. It could be argued that when students respond with the first of these alpha reactions the work being presented to the learner is too easy, and when the learner responds with the second of these alpha reactions the work is too hard.

If conflict is as important as Piaget (1975/1985) indicates, and as useful as an educational technique as found by Swan (1983, 1993), then it could be expected that facing and discussing conflicting interpretations of decimal fractions would help students improve their understanding. Working in pairs should increase the potential for conflict, as two learners might have different views. In the study reported here many more conflicts were observed than had been expected. These conflicts were discussed by the students at length, and usually resolved in an appropriate manner. Some attempts involved clinging to an initial belief but many involved the construction of new understanding. Perret-Clermont, Perret and Bell (1990) as well as Piaget point to hearing the different ideas of others as an important factor in deciding whether or not one's own ideas needed to be changed.

The Study

This study of students' reaction to conflict was only one aspect of a larger study of ways of enhancing students' understanding of decimal fractions. In this phase of the study 11- and 12-year-old students worked on problems involving decimal fractions in pairs that included a more and less competent student, as ranked by the teacher. All students took a pretest on their knowledge of decimal fractions. Then they worked in these pairs on three sets of problems on three separate days and their discussions were tape-recorded for transcription. These problems covered the magnitude of numbers, addition and subtraction, and multiplication and division. All problems involved decimal fractions in a format which forced students to think about the meaning of place and the decimal point. There were two sets of problems: those which were based on context and those that used a purely numerical format. Both contextualized and numerical problems were set up to maximise the likelihood of conflicting ideas occurring. Some problems presented two conflicting answers and asked to students to decide which was right, and some used conflict between answers that it was expected that the students would give and calculator answers. In each case students were required to say why they thought an answer was right.

One contextualized problem that presented two conflicting answers was:

> The paper says that one New Zealand dollar = 0.9309 in Australian dollars.
> Susan said that was 93.09 cents Why did she say that? Is she right?
> Andrew said it would be 9309 dollars. Why did he say that? Is he right?
> What do you think?

An example in which it was expected that students answers would conflict with calculator answers was:

> $1 New Zealand exchanges for 1.5989 Samoan tala.
> How much would you get for $10 New Zealand?
> Try it on your calculator. Why did it give that answer?

Similar purely numerical problems used numbers that were somewhat simpler, but reflected confusions shown in other research. Examples were:

> Teri said that $93\frac{1}{4}$ was written as 93.04 in decimals.

Why did she say that? Is she right?

Peta said that $93\frac{1}{4}$ was written as 93.25 in decimals

Why did he say that? Was he right? What do you think it should be?

How much is $1\frac{1}{2} \times 10$?

Try it on your calculator. Why did it give that answer?

All students were given a posttest four days after their last problem-solving session and again one month later. Students given contextualized problems showed significantly more gain, with an interaction between the progress of the groups given the different types of problems and pretest-posttest scores ($F(1,12)=5.70$, $p=.03$). One of the reason for their greater gain appeared to be related to the conflicts that they discussed.

Evidence of learning from conflict was shown by students given both types of problems and by both more and less knowledgeable students. Conflict arose from what was written in the problems, from conflict between student expectations and calculator results, from discrepant results between a calculation and expectations based on outside experience, from conflict between different results from manual calculation, from differences between the peers, and from discrepancies between a new conclusion and previous views. Some conflicts involved both students in a pair offering suggestions and building on each other's understanding to resolve the conflict. Other conflicts involved an individual student attempting to resolve a contradiction through repeated attempts to calculate or understand an issue, bringing different aspects of knowledge to bear. Students also responded to these conflicts in a variety of ways, only some of which were educationally useful. These conflicts can be described using all three of Piaget's categories, with several subdivisions some of which relate to the writing of others.

Reactions to Conflict

Responses That Did Not Appear to Lead to Learning (Alpha Reactions)

Recognising an inconsistency as a problem to others but not to them: This appeared to be an example of Piaget's alpha (i) reaction, in which students were able to add on a fact without reorganising their own understanding. In solving these problems students acted like those who took part in the "correcting homework" exercises mentioned above. For example, in one problem a hypothetical students said that the next number after 3.56 was 4.57 (as some students in a previous part of this study had said) and the other hypothetical student said that the next number after 3.56 was 3.57. The students given these problems weren't drawn into the intended conflicts but they could see what the students named in the problem might have thought which led them to give a wrong answer.

Ignoring: Ignoring a conflicting element is an example of Piaget's alpha (ii) reaction. Vinner (1990) also wrote about inconsistencies that students ignored because they did not understand the contradiction. This reaction was displayed by K who responded to the question .9 + .1 by reading 'nine plus one', predicting the answer 'ten' and then reading the result of 1.0 on the calculator as 10. By ignoring the decimal point he did not have to deal with any conflict in this problem. The more common response was to read this question as "point nine plus point one" and predict "point ten" as the answer. Students who did this were surprised when the calculator gave "one point oh" as the answer and worked to resolve that conflict, in beta reactions.

Recognising and denying the importance of a conflict by removing the disturbing element: This was another example of Piaget's alpha (ii) reaction, but differed from the one given above in that students acknowledged the conflict but then chose to resolve it by denying some aspect of the new fact. They reacted by choosing to disregard it or in some way removed what was disturbing.

An example of removing what was disturbing was shown by G and A. This was an unplanned conflict between manual calculation and calculation with a calculator. In multiplying .355 x 6 G repeatedly got the answer 2.103 because he was convinced that in the first multiplication (from the right) of 6 x 5 you put down the 3 and carried the 0, writing:

$$\begin{array}{r} .^{3}3\,^{0}5\ 5 \\ \times \quad 6 \\ \hline 2.1\,0\,3 \end{array}$$

The calculator gave the answer 2.13. His partner finally resolved this conflict by crossing out the medial 0 in this calculation, making the answers the same. G did not appear to agree but gave in with a written answer that blamed the calculator, saying "Because it (the calculator) took out the zeros". He thereby maintained his confidence in his own calculation by blaming an electronic tool for making a mistake.

Responses That Could Lead To Learning (Beta Reactions)

These were the reactions in which learners integrated the perturbing element into their scheme, either by extending their existing scheme or changing their scheme to include the perturbing element, or both. In these students' discussions, the disturbance caused by the conflicting element was resolved by either treating the new fact as applying to a particular context (compartmentalising), or by integrating it into an existing scheme with the help of knowledge from a different domain, thus extending their initial scheme.

Compartmentalising new knowledge: Duffin and Simpson (1993) write about the reaction in which a new fact is treated as applying to a particular context as treating a new idea as 'foreign', or unrelated to any currently held mathematical ideas. It is common for mathematicians writing about conflict to give experiences from their own lives in which they learned a piece of mathematical knowledge in a compartmentalised manner and only years later saw its relevance to other areas of mathematics. (e.g. Duffin & Simpson, 1993; Mavshovitz & Hadar, 1993; Vinner, 1990). Contextualized or compartmentalised understanding is an important first step to more advanced understanding

There were several instances of compartmentalising knowledge that appeared to be a beta reaction. One instance was shown by a student who thought of halves and quarters in terms of time with 60 minutes in an hour, so that a quarter was written .15 and a half was written as .30. His partner was quick to grasp the source of this compartmentalised knowledge and say that it was true for time, but that decimal fractions were based on 100, so a quarter was .25 and a half was .50. The student who had held an understanding compartmentalised in telling time was happy to see the truth of his situated knowledge and at the same time accept the nature of decimal fractions a different situation in which a quarter and a half meant something different. In this case it was the quarter and half that were the wider concept, and both time and decimal fractions were specific situations to which the concept could be applied.

The result shown by a calculator was the source of many of the conflicts experienced by students. In each pair there was at least one student who knew that calculators dropped off the final zero(s) in a decimal fractions. However, this knowledge appeared to be compartmentalised, limited to the behaviour of calculators, rather than being seen as a general principle that could have been applied manual calculation as well. For example G explained to B why the calculator gives 0.3 as the answer to 1.45 - 1.15 by explaining, correctly:

"Cus on the calculator, when it does that it just doesn't put, add the zero."

The other main type of compartmentalisation was attention to the digits or decimal point without attending to meaning. M and N showed this attention to symbols rather than to meaning when trying to explain why the hypothetical student in the problem read $0.9309 as 93.09 cents. The dialogue suggested that they were struggling to understand

this novel representation of money. As it was the representation that they were trying to understand it is not surprising that they attended to this. Later in their discussion, when faced with another conflict, they make a decision based on meaning.

K That's how the paper records exchange rates.
N Zero point 9 3 0 9.
M Oh, cause she rounded it up.(Pause)
N Did she? It's still the same, she just took off the zero, put a dot....
M Oh no she just put the decimal point two backwards.
N Took out the zero.

Correction: In discussing different types of conflict, Vinner (1990) makes a distinction between superficial and deep conflicts. Superficial conflicts are those that can be corrected by redoing a calculation. These conflicts may not seem superficial for the students who rework a problem several times before they find and correct their errors. The conflict is resolved by changing the incoming fact, or calculation, to match expectations. Vinner classifies corrections as superficial because once the correct calculation is found the student immediately knows that it is right. Deep contradictions on the other hand require a greater reorganisation of existing understanding before they can be resolved.

There were many examples of corrected calculations in finding solutions to these problems. Students' first reaction to contradictory answers was often to rework a problem both manually and on their calculator It was these calculation conflicts that the lower-ranked students were helpful in identifying and resolving. On one occasion a lower-ranked peer told her partner that she should be adding not subtracting, thus breaking an impasse. In another example M had mentally calculated that 3000 - 500 = 800, presumable because she was adding the significant digits rather than subtracting. Her partner, N, quickly corrected her saying:

N No not eight hundred.
M Three thousand... five hundred.
N One..no..two thousand five \
M Two thousand three, no two thousand five hundred, yeah.
N Yeah, two thousand five hundred.

Addition of a new fact: There were also cases in which one student's confusion was overcome by the partner offering the needed piece of information as a correction. One example occurred when one student wrote $90\frac{9}{10}$ as 90.910. His partner then informed him that $\frac{9}{10}$ meant "nine tenTHS". This appeared to be new information for the first student who then added it to his store of facts.

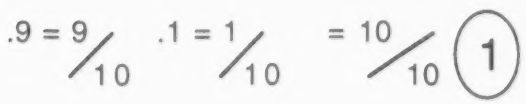
Conflicts That Were Solved by Changing the New Fact: Problems that were solved through seeing the relationship between measurement units fall in this category of changing an incoming fact. The problems involving both centimetres and millimetres, centimetres and metres, kilograms and grams, and kilometres and metres provoked discussion about the relationship of these units. The students asked each other how many of the smaller unit there were in the larger, and then changed the larger unit to a multiple of the smaller unit. This changing of the problem enabled students to fit it into their understanding without the need to deal with different units.

Conflicts between two contradictory facts both of which appear to fit the scheme: Several problems in this study were set up to provoke this type of conflict in the hope that students would see the error of the answer which involved a misconception and strengthen their understanding of the correct scheme. However, for some students this conflict was severe. Piaget would classify it as a conflict between two sub-procedures. Their reaction, which was to try to fit one or both facts into their larger scheme, would be

a beta reaction. One example was a discussion, that ran to five single-spaced pages of transcript, was a debate about whether $90\frac{1}{4}$ was written as 90.04 or 90.25. The answer 90.04, fits a fraction misconception in which the number of parts something is divided into is given as the decimal fraction. The second answer, 90.25, fits a belief that the decimal fraction for a 1/4 is .25 because there are four 25s in 100. The participants provided a variety of reasons for each of the answers being correct, but H insisted on continuing to work on the problem until she finally concluded that it must be 90.25 because 25 was one quarter of a hundred. The length of this dialogue and the number of twists and turns that it took provided evidence that this was not a minor conflict for them, yet they did feel that it was within their competence to solve.

Conflicts that involved change to both the incoming fact and the existing scheme: These reactions to conflicts involved changes that allowed the new fact to be understood in terms of the students' existing schemes because the existing scheme was changed or expanded in some manner and the new fact was understood in a different way. Students with this reaction appeared to be adding a missing fragment or piece of a puzzle that made their scheme make sense. A marked example of this was H's realisation that the reason why .9 + .1 = 1.0 was that .9 meant $\frac{9}{10}$ and .1 meant $\frac{1}{10}$ and $\frac{9}{10}$ and $\frac{1}{10}$ made one whole. In adding this missing fragment she both adapted her understanding of the presented facts and her wider scheme for understanding numerical relationships, so that both common fractions and decimal fractions were included in her understanding. The resolution of this conflict starts with an excited exclamation:

H ... OH!! I know why now! Ten tenths is one whole. (pause)
A Yeah!
H Ten tenths is one whole.
A Yeah.
H So obviously point nine and point one is ten tenths, that's one whole, where as, give me a piece of paper?
K You can write on that.
H [writes and says] Point 9 equals nine tenths, and point 1 equals 1 tenth which equals to ten tenths, 1 whole.



Responses to Conflict That Involve a Flexible System (Gamma Reactions)

This category is one in which there is no perturbation because the learner's overall scheme is already flexible enough to accommodate the new fact. The difference between this category and alpha (i) reactions appears to be in the flexibility of the learner's overall scheme. An example of a larger more flexible scheme that appeared to be developed within the time span of this study was provided by H, who talked with her partner a great deal. She had worked out that 2.30 x 10 would be 23 by adding 2.30 ten times, after which she noted the calculator answer of 23 and said:

"You take out that zero, you get 2.3, and if you timesed it by 10 you just put... 2.3 and then times it by 10, you put the dot one place up, one place over the three, instead of behind it, over the three."

Later she guessed that $1\frac{1}{2}$ x 10 would be 10.5 and when the calculator gave 15 as the answer she said,

"I get it now....You just take the decimal point from the 1 point five and just put it in front of the 5."

By the posttest she had solidified her knowledge into a rule, perhaps as a result of talking about it with an adult. If she did ask an adult about it, then this would be an example of a conflict providing the motivation for seeking out further knowledge. The question on the posttest and her response were:

> (Q) Give a rule for multiplying or dividing decimal fractions by 10. Give an example.
>
> (H) (writes) To multiply a decimal number(s) by 10 you bring the decimal point one place further to get the answer. i.e. 1.5 x 10 = 15. To divide a decimal number(s) by 10 you take the decimal point back one place in your answer. i.e. 15. ÷ 10 = 1.5

None of the other students was able to produce a rule like this.

Summary

In this study some students showed a higher proportion of beta and gamma reactions, or reactions that led to greater understanding. They were usually the students who had more knowledge initially. In Glaser's terms (1984) the more knowledgeable reasoned more knowledgeably. This was not only the higher-ranked students in the study. Lower-ranked students who dealt with contextualized problems had out-of-school knowledge to bring to the problem. This enabled them to also reason knowledgeably, in beta reactions, and thus expand their understanding. Nearly all the conflicts presented to pairs who worked on contextualized problems were resolved with either beta or gamma reactions, adding a fragment, compartmentalising, integrating, or seeing how new facts fitted into a flexible system.

This study suggest that teachers can both look for and engineer conflicts to help students construct their understanding. Having students work in pairs increases the likelihood of conflicts arising that they will be able to resolve. These conflicts are most productive if based on at least one aspect of knowledge that students know securely. They can then use this knowledge for solving the problem or resolving the conflict. At the same time a teacher needs to make certain that conflicts are within students' ability to resolve, so that they do not build up negative emotions about mathematics.

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Students' Thinking and Writing in the Context of Probability

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This study examined changes in students' probabilistic thinking and writing during instruction emphasizing writing to learn experiences. A class of fifth-grade students with no previous experiences in writing during mathematics made significant gains in probability reasoning and writing; however the correlation between probabilistic thinking and writing was not significant. Analysis of focus students revealed that their writing changed from narrative summaries to reasoned patterns and generalizations. However some used invented representations without interpretation and were reluctant to write in mathematical contexts.

Research in writing across the curriculum has recently focused on content specific writing and learning (Langer, 1992; Waywood, 1994). In particular, researchers have found that writing in mathematics classes provides opportunities for students to refine, clarify, and organize their mathematical thinking (Azzolino, 1990); communicate what they know (Drake & Amspaugh, 1994); and generally construct a better understanding of mathematical relationships (Abel & Abel, 1988; Shepard, 1993). Some researchers have also begun to investigate the reverse perspective, that is, the extent to which *learning to write* can be enhanced by content specific *writing to learn* experiences (Freedman, 1995). However, the extent to which writing experiences in mathematics influence students' writing patterns has not been widely reported.

The present investigation, undertaken with a group of elementary students, examined both writing to learn and learning to write in the context of an instructional program in probability. More specifically, the study examined: (a) changes in students' probabilistic thinking; and (b) changes in students' writing during instruction in probability that emphasized writing to learn experiences.

Theoretical Considerations

The study is based on two theoretical positions. The first is a cognitive framework that describes elementary school students' probabilistic thinking (Jones, Langrall, Thornton, & Mogill, 1997). The second is a model relating writing and phases of conceptual learning (Shepard, 1993).

Framework for Describing Students' Probabilistic Thinking

The Probabilistic Thinking Framework is based on the assumption that thinking in probability is multifaceted and develops over time. Based on previous research the framework (e.g., Acredolo, O'Connor, Banks, & Horobin, 1989; English, 1993; Falk, 1983; Fischbein, Nello & Marino, 1991; Piaget & Inhelder, 1975; Jones et al., 1997) delineated four levels of probabilistic thinking for each of four key constructs: sample space, probability of an event, probability comparisons, and conditional probability. The four levels of thinking evolved from observations of students' probabilistic thinking over a two-year period, and appear to be consistent with neo-Piagetian theories that postulate the existence of levels of thinking that recycle during developmental stages (Biggs & Collis, 1991).

Students at level 1 are narrowly and consistently bound to *subjective* judgments. For example, a level 1 student examining a gumball machine containing 6 red and 3 yellow gumballs may choose yellow as the most likely to come out because "It's my favorite color." Such students see no need to apply quantitative reasoning in probability situations. Level 2 students are *in transition* between subjective and naive quantitative thinking. For example, a level 2 student faced with the same gumball machine will sometimes choose red because there are more red, but will at other times revert to subjective judgments and choose either red or yellow. Level 3 students characteristically use *quantitative* judgments when dealing with probability tasks. For example, a student at level 3 will consistently choose red in the gumball example above, and will explain that there are six red versus three yellow. Students at Level 4 use

numerical reasoning and also express their probability thinking in terms of more precise probability measures. In the same gumball situation, a level 4 student will choose red because its chances are 6 out of 9, compared to only 3 out of 9 for yellow. The thinking levels of the Framework were used as the basis for constructing the instructional program and assessment instruments in this study.

Model of Writing and Phases of Conceptual Learning

Shepard's (1993) model focusing on writing and phases of conceptual learning in mathematics integrates three learning phases--initial, intermediate, and terminal (Shuell, 1990) with seven writing categories (Britton, Burgess, Martin, McLeod, & Rosen, 1975) that range from reporting to tautological writing. Shepard reduced the writing categories to six and generated six learning phases by splitting each of the original learning phases into an *early* and a *late* level. Shepard's learning-writing model was then produced by matching the six learning phases and the six writing categories. Because the final two learning-writing phases are typically not accessible to elementary school children, this study utilized only the first four: (a) initial early (record); (b) initial late (generalized narrative), (c) intermediate early (low-level analogic), and (d) intermediate late (analogic).

In the *initial early* learning-writing phase, individuals simply record or summarize direct experiences without making inferences. In the *initial late* phase, individuals largely report or summarize experiences, but they also begin to identify patterns and formulate generalizations associated with the experience. For example, in exploring the possible sums when two dice are rolled, students in the early phase are likely to record the sums they actually find, while those in the late phase explain that some sums are more likely than others. The *intermediate early* phase is characterized by writing that identifies true generalizations but does not recognize relationships between these generalizations. In the *intermediate late* phase, not only are generalizations identified and described, but they are also related in a cogent way. For example, in the exploration involving the two dice, an individual in the intermediate early phase might record and justify that a sum of 7 is the most likely event, but unlike an individual at the intermediate late phase, may not recognize that the probabilities of sums from 2 to 12 are hierarchically ordered: increasing from 2 to 7, then decreasing from 7 to 12.

Research Questions

This study sought to answer the following questions: (a) What changes occur in students' probabilistic thinking following an instructional program in probability that embraces writing to learn experiences? (b) What changes occur in students' writing within a probability context during an instructional program that emphasizes writing to learn experiences? and (c) Is there a relationship between learning-writing phases, as measured by Shepard's (1993) model, and levels of thinking in probability, as measured by the Probabilistic Framework (Jones et al., 1997).

Methodology

Subjects

The population for this study was fifth-grade students from a university laboratory school. Students in this school represent a broad spectrum of cultural and socioeconomic backgrounds. One of the two grade 5 intact classes from this school (n=24) was randomly chosen to participate in the Probability Writing Program of this study. Both classes had previously participated in probability programs during grades 3 and 4 (Jones, Thornton, Langrall, & Miller, submitted; Jones et al., 1997). Four students from this class--here named Marion, Steffi, Jacques, and Terrance--served as focus studies. These students were "purposefully" selected (Miles and Huberman, 1994) at the *completion* of the intervention: the first student, Marion, made gains in both probability thinking and writing; the second, Steffi, made no gains in either writing or probability; the third student, Jacques made gains in probability thinking but none in writing; and the fourth, Terrance, made no gain in probability thinking but a substantial gain in writing.

The Probability Writing Program

The Probability Writing Program (The Program) comprised ten 45-minute sessions. Two sessions were held each week over a period of five weeks. The teacher for The Program, was a final year undergraduate major in middle school education with specialization in mathematics. She had also served as a research and teaching assistant in the grade 3 and grade 4 probability projects. The Program, developed by the research team, consisted of probability and related writing tasks generated from the probabilistic framework (Jones et al., 1997). Each probability task was based on one or more of the key constructs of the Probabilistic Thinking Framework: 18% of the sessions focused on sample space, 27% on probability of an event, 42% on probability comparisons, and 13% on conditional probability. The tasks were chosen so that they would be accessible to students at all levels of the framework. Related writing prompts asked students to reflect on a probability task and to predict, explain, or justify their thinking. An example of a typical probability task and its associated writing prompts is presented in Figure 1.

An Instructional Writing Task

Q1: What are the different ways to mark "T" or "F" on a 4-question true-false test? List all of the outcomes and explain how you found them.

Q2: Given that the correct answer was TFTF, what was each student's chance of getting a perfect paper by guessing? Justify your thinking.

Q3: In our class, 2 students out of 25 earned a perfect paper. Does this surprise you? Write an explanation to support your response?

Figure 1. Sample probability task and writing prompts

The format for each of the two weekly sessions varied slightly. To begin the *first* session students responded to individual probability prompts in their journals. These prompts were generated by the teacher in response to students' journal entries from the previous week. This writing activity was followed by whole class discussion of a new probability task. Students made predictions, then broke into pairs and attempted to solve the new problem using both experimental and analytical approaches. That is, they first carried out a simulation of the problem, analyzed their results, and then tried to explain the situation using more formal probability reasoning. The students subsequently shared their thinking in a whole class discussion and completed a journal entry for the problem. The format of the *second* session of each week was identical to the first except that the initial writing activity was not undertaken. Each weekend the teacher assessed the students' journal entries, provided new prompts, and the cycle continued.

Instrumentation

The same Probability Thinking Protocol was administered in an individual interview setting at the beginning (September) and end (November) of the intervention. However, the Writing Protocols given at the beginning and end of the intervention were different. This was necessary to accommodate the fact that students had not previously undertaken writing during a mathematics class and to recognize their level of maturity in probability thinking prior to the intervention. The Probability Thinking Protocol was based on the Framework and comprised 20 items (See Jones et al., 1997). Five items were associated with sample space, four with probability of an event, seven with probability comparisons, and four with conditional probability. The items, linked to the four constructs of the framework, enabled researchers to explore students' thinking across each of the four probabilistic thinking levels.

A double coding procedure (Miles and Huberman, 1994) was used to assess students' thinking levels in relation to the Probability Thinking Protocol. Working independently, the first three authors coded pre and post intervention protocols to establish thinking levels for all students on each of the four constructs. Agreement was achieved on the coding of 93 % of the levels and variations were clarified until consensus was reached. Hence, each student's dominant (modal) level of probabilistic thinking was identified at the beginning and at the end of the intervention. The teacher

for The Program administered the writing protocols in a whole class setting. The Initial Writing Protocol (Figure 2) was set in the context of a card game while the Final Writing Protocol (Figure 3) was set in the context of a sailboat race. In each case the writing prompt invited students to respond to a probability question and provide full justification of their thinking. These prompts were sufficiently open to invite a range of written responses across the Shepard's (1993) four phases. In scoring the writing protocols, the researchers again used the double coding procedure in a similar manner to that discussed above. The only difference was that each student was assigned a dominant (modal) learning-writing phase (1- initial early, 2- initial late, 3- intermediate early, and 4- intermediate late) rather than a probability thinking level. . Agreement was achieved on the coding of 84% of the learning-writing phases.

Drawing the Cards

Q1: Draw 2 cards from a set of 4 cards consisting of an Ace, King, Queen, and Jack. How many different outcomes are there? List the outcomes and write an explanation of your thinking.

Q2: Jane says there are 6 ways to draw 2 cards from the set of 4 cards. John says there are 12 ways to draw 2 cards from the set of 4 cards. Explain how each of them could justify their answers.

Figure 2. Probabilistic writing protocol: Initial task

The Great Sailboat Race

You are going to play a game in which the sailboats are numbered 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, and 12. The rules for playing the game are:

- You roll two dice and find their SUM.
- If the sum is 2, sailboat #2 moves forward 1 space.
- If the sum is 3, sailboat #3 moves 1 space.

. . . and so on up to a sum of 12.

- The first sailboat to move 4 spaces wins the race.

Play the game several times and then respond to the writing prompt.

Writing Prompt: Which sailboat do you think will win the race?
Write a complete explanation to justify your response.

Figure 3. Probabilistic writing protocol: Final task

Data Sources

Data on students' thinking and writing in the context of probability were gathered from three sources: 1) assessments on probability thinking conducted at the beginning and end of the intervention; 2) assessments on probability writing conducted at the beginning and end of the intervention; 3) journals containing the writing of focus students throughout the intervention; and 4) researcher narratives of instructional sessions. Based on the data in 3) and 4), each of the first four authors generated a Researcher Writing Summary (RWS) on one focus student each session. These RWSs described students' writing and identified any diagrammatic or symbolic representations used by them. The RWSs also contained on-going researcher annotations which sought to relate students' writing to the four phases of Shepard's (1993) model.

Data Analysis

Both "within" and "cross-case displays" (Miles & Huberman, 1994, p. 90; p. 207) were used to guide the analysis of qualitative data on the four focus students. For each focus student, the 10 RWSs, one for each session, were assigned multiple codes largely arising from the writing levels associated with Shepard's (1993) model. Subsequently, a time-ordered matrix was generated for each focus student to display trends in writing over the 10 intervention sessions. These matrices were further analyzed to discern common patterns and relationships across the four focus students. Triangulation using both the researcher and the assessment data enabled alternative interpretations to be evaluated. Statistical analyses were also performed using assessment data on probability thinking and probability writing as dependent variables.

Because data were "ordinal," non-parametric statistical tests were used to compare probability thinking and writing levels prior to and following the intervention, and to examine relationships between these variables.

Results

The Effect of The Program: Quantitative Analysis

The students' thinking levels on the Probability Thinking Protocol were determined at two assessment points: prior to The Program (September), and following The Program (November). A Wilcoxin Signed Ranks Test (Siegel & Castellan, 1988) indicated that there was a significant difference between the pre and post assessment thinking levels in probability (pre assessment median = 2.5, post assessment median = 3, $T = 40.5$, $p < 0.05$). The student's learning-writing phases in the context of probability were also assessed prior to The Program (September) and following The Program (November). A Wilcoxin Signed Ranks Test (Siegel & Castellan, 1988) indicated that there was a significant difference between the pre and post assessment learning-writing phases (pre assessment median = 1, post assessment median = 2, $T = 139$, $p < 0.01$).

Spearman correlation coefficients were determined to assess degree of association between learning-writing phase and probability thinking level. The pre assessment measure was significant ($r = 0.41$, $p < 0.05$), but the post assessment measure was not significant ($r = 0.21$, $p > 0.20$).

The Effect of The Program: Qualitative Analysis

Two learning patterns with respect to writing in the context of probability were discerned by analyzing RWSs. Students (a) moved from writing narratives of probability simulations to writing about patterns and generalizations drawn from probabilistic situations and (b) often substituted symbolic representations, without interpretation, for explanations.

Narratives to Patterns

Marion, whose writing moved from phase 1 to phase 4, showed the greatest change toward writing that incorporated patterns and generalizations. Marion's response to the Initial Writing Protocol (Figure 2) was typical of many phase 1 students prior to the intervention. When asked to list all possible outcomes when two cards were drawn from a set of 4 cards consisting of an Ace, King, Queen, and Jack, Marion doubled up on two pairs (see boldface below), listing 14 outcomes rather than 12. More importantly, she made no attempt to respond to the writing prompt which asked her to interpret and justify her response. Her report merely listed her outcomes as shown below.

1. K, J 2. Q, A 3. K, Q 4. A, J **5. A, K** **6. K, A** 7. Q, J
8. J, K 9. A, Q 10. Q, K 11. J, A **12. K, A** **13. A, K** 14. J, Q

By the end of lesson four in the instructional program, Marion was attempting to look for patterns and provide more explanation. Consider, for example, Marion's response to writing prompt Q3 in Figure 1. Marion wrote, "It isn't surprising at all -- there are 16 ways but only one is right. A couple of people might have put TTTT. A couple of people might have put FFFF and a few more might have put FTTF." Marion correctly recognized that there were 16 outcomes and that only one was correct. Moreover, she also appeared to conjecture that two students might have written down each outcome. This suggests that she is using an approximation, that is, that 25 is roughly twice 16 and hence each outcome would come up twice. While Marion didn't provide sufficient written explanation to make her thinking clear, there was an attempt to describe a pattern and draw an inference from it.

By the post assessment, Marion's response to the Sailboat Race (Figure 3) exhibited phase 4 writing and showed that she was able to make true generalizations that were based on well reasoned arguments. She wrote, "Sailboat 7 would win because it had 6 outcomes (which she listed) and the others have less outcomes." Although her written response did not include numerical probabilities, she correctly ordered the sailboats according to their chance of winning, by making critical

connections between the number of outcomes having a particular sum and the probability of the corresponding sailboat number.

Terrance's growth in writing is more typical of the students in this study. In essence, Terrance moved from using a writing style that was essentially confined to narratives to a writing style that incorporated patterns and even generalizations. More specifically, our analysis showed that during the first four lessons Terrance regularly provided lists of outcomes for experiments and began to provide some justifications rather than mere descriptions of what was happening. For example in lesson 4, when asked whether it was surprising that 2 students out of a class of 25 correctly guessed all four answers (Figure 1), Terrance correctly listed all of the 16 possible outcomes and wrote, "Yes, because there are 1 every 16 and 2 for 25." His thinking was similar to Marion's but was numerically more explicit in that he displayed the approximate proportion that he had in mind. However, like Marion he did not clarify his thinking nor did he attempt to generalize by suggesting that with 32 students you'd expect 2, with 48 you'd expect 3, and so on.

Teacher comments in Terrance's journal consistently encouraged him to "explain his thinking more fully" and to give reasons "why" he drew particular conclusions. This encouragement appears to have paid dividends because in later lessons the teacher wrote "excellent job" on two pieces of Terrance's writing. By the written posttest, Terrance was rated at level 3 in writing and had become more consistent in justifying his thinking. For example, in the Sailboat Race (Figure 3), Terrance concluded that sailboat 7 was more likely to win and wrote, "7 will win because it has more chances than 4 [for example] when the dice are rolled. [Sailboat] 7 has 6 ways 5+1, 1+5, 2+4, 4+2, 3+4, 4+3. [Sailboat] 4 has 3 ways 1+3, 3+1, 2+2." Although he assumed that his justification that 7 had a better chance than 4 was sufficient to show that 7 had the best chance, his thinking process is generalizable beyond 7 and 4.

Invented Symbolic Representations

Steffi, who was a level 4 thinker in probability even prior to instruction, exemplified students who preferred to use symbolic representations rather than written descriptions and explanations. Like most students in the study, Steffi was rated at phase 1 on the Initial Writing Protocol (Figure 2). Although she listed all 12 outcomes using systematic pairings like AK and KA, she didn't provide written explanation of the pattern used. Nor did she attempt to explain why there were twelve outcomes rather than six, an equally valid response. She simply wrote beside her diagrammatic listing, "We wrote them down," implying that the diagram said it all.

In subsequent lessons, Steffi consistently listed outcomes correctly in various probability situations, but in every case remained mute to the need for explanations or interpretations of the patterns. This occurred in spite of the fact that the teacher consistently urged her to write justifications of her thinking. For example, in lesson 4, Steffi came up with her own invented model for listing the outcomes when a student guessed the answers to the four-item true/false test (Figure 1):

1. TTTT 2. FFFF 3. TFTF 4. FTFT 5. TFFF 6. FTTT 7. FTTF 8. TFFT
9. FFFT 10. TTTF 11. FFTT 12. TTFF 13. FFTF 14. TTFT 15. TFTT
16. FTFF

Even though she appeared to use a pattern that reversed the positions of T and F, she did not respond to the written prompt seeking explanation. In the same task, when asked whether it was surprising that 2 out of 25 students guessed all four items correctly, Steffi provided a number sentence as her written response. Having previously written that there was a 1 in 16 chance of getting a perfect paper, she responded to Q3 (Figure 1) as follows, " No! because 16+16 = 32." The mathematical reasoning is valid but covert, because Steffi did not feel any compunction to identify, describe, or interpret the generalizations she was clearly able to make through symbolic representations.

Given this predisposition for symbolic representations, it is not surprising that Steffi was still rated at phase 1 on the Final Writing Protocol (Figure 3). Moreover her probability thinking was completely out of character. She merely presented a symbolic listing of the sailboats that won when she played the games. Then she wrote, "Sailboat

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number 5 is most likely to win because 5 came up most often when I played the game." Further, she wrote that 5 was her lucky number because she liked its shape. Her preoccupation with symbols and lack of justification was typical of her written responses, but her regression to subjective judgments and her reliance on direct experience rather than analysis was not typical of her level 4 probability thinking.

The use of symbolism by Jacques, initially a level 2 thinker in probability and a phase 1 writer, is similarly interesting. While Jacques' tendency to rely on symbolic representations was not as pervasive as that of Steffi, he frequently used stand alone diagrams rather than explanations. Like Steffi, Jacques remained at level 1 in writing largely because of this tendency to provide symbolic representations without explanation. In summary, the proclivity of these students to use symbolic representations without written explanations appears to be associated with a feeling that such representations completely meet the requirements of mathematical justification.

Discussion

Although students made significant gains in probability reasoning it is unclear whether these gains resulted from further experiences in writing about probability tasks, additional probability instruction, or a combination of both. The lack of correlation between learning-writing phases and probability levels and the characteristics displayed by focus students like Steffi and Terrance does not support the position that writing helped the students to refine, clarify, and organize their probability thinking (Azzolino, 1990), or to construct a better understanding of probability relationships (Abel & Abel, 1988; Shepard, 1993).

One explanation of this situation may be that 71% of the students in this study began the intervention at the lowest writing level, that is, level 1. Given the initial fixation of these students with narration rather than explanation, it may be argued that their writing had not developed to a sufficient level of sophistication to enhance the generalizations required in probabilistic thinking. Alternatively, because these students had previous instruction in probability, their growth in probabilistic thinking was already substantial and they had less need for the scaffolding that writing could provide. Further research is needed to examine both of these alternative positions.

Student gains in writing, based on Shepard's (1993) phase model , were not only significant; they were larger than the gains in probability thinking. Although the correlation between writing and probability thinking was not significant at post assessment, 63% of the students made a gain of at least one level in writing. Marion's progression from phase 1 to 4 on the Shepard model was not typical, but the shift from merely narrating probability situations to describing patterns and generalizations certainly was. Hence, this study provides some evidence that mathematics can provide a contextual platform for improving writing.

Although this study was limited by virtue of the fact that only one group participated, some implications can be drawn for writing in the area of mathematics. There is evidence in this study that children like Steffi were reluctant to write in the context of mathematics and required more time to recognize that symbolic representations needed interpretation. This suggests that teachers need to adopt an early and systematic approach to integrating writing and mathematics. Further, given the improvement in writing shown in this study, there is evidence that regular feedback and individual follow-up prompts from the teacher enhance both the quality and quantity of students' writing.

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Towards an understanding of numerical workplace knowledge

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In this paper airline ticketing desk operations are studied in order to develop a greater understanding of numerical workplace knowledge as seen from a non-essentialist account of mathematical knowledge. In the first part of the paper alternative approaches to knowledge use are discussed and their epistemological groundings noted. Next, transcripts are analysed in order to ascertain the logic of ticket exchange task performance. Methods used for the study include those derived from linguistic philosophy; decisive use is also made of the Vygotsky inspired notion of artifact mediation within task performance. Finally, an attempt is made to find the place of numerical knowledge within the workplace knowledge of the episodes observed.

Background and aims of study

Doubt concerning the adequacy of essentialism as a theory of knowledge for the contexts of mathematics education has been expressed within the research literature over the previous fifteen years. Working from findings drawn from a variety of settings and theoretical and frameworks (anthropological studies: Gerdes (1985), Bishop (1988), Saxe (1991), Lave and Wenger, (1991), Nunes, Schlieman & Carraher (1993); sociological and linguistic theories: Bloor (1983), Lave (1988), Solomon (1989); micro-sociological analyses: Bauersfeld (1991), Voigt (1993); epistemological studies: von Glasersfeld (1995); discourse theories: Walkerdine (1988, 1994)) researchers have forged a widely held view that mathematics is a construction of some kind (as opposed to a pre-given entity) and one which at does not entirely (or at all) belong to the mental domain. Nevertheless, despite the solidity of these findings and their empirical base there currently remains a significant bias towards essentialist accounts in our thinking about mathematics within education circles. For instance, a recent pamphlet entitled 'Using mathematics ideas and techniques for practical purposes' (Australian Association of Mathematics Teachers, 1996), provides examples of mathematical ideas and techniques at work in practical situations. In writing about these, the pamphlet creates the strong impression of an essentialist bias in its description of mathematical knowledge. For instance, although it concedes that "required mathematical ideas and techniques will always be specific to a person's interaction with the context" it notes that "mathematical ideas and techniques are often not recognised as such". In making these statements, the pamphlet is not arguing that mathematical knowledge is somehow synthesised in the circumstances in which it is deployed, but that it is transferred to those sites. Indeed, a principal thrust of the publication hinges on the belief that a clear view of the problem situation will promote the effective use of mathematical knowledge precisely because in the presence of such clarity a recognition of "when and where mathematics could help" can be made. In short, mathematical knowledge is used precisely in those sites to which it can be transferred.

Given both the uncertain basis for essentialist thinking in mathematics education and, notwithstanding this, the presence of a strong bias towards it, analyses adopting non-essentialist perspectives are indicated. One such possible approach is reported in this paper. Workplace contexts - in this case from certain over the counter operations within the airline industry - were chosen as the site for investigation as these are frequently overlooked in studies relating to the construction and deployment of knowledge within the mathematics education research literature. Specific aims for the paper are, therefore, (i) to explore what a non-essentialist account of mathematics use in an actual workplace situation could look like, and (ii) to investigate how a task performance involving numerical knowledge within this site might be considered to operate. It is emphasised that as this paper is meant to be indicative of a possible line

for future investigation, findings, and indeed the form of the analysis itself, are to be regarded as preliminary.

Knowledge in use: transfer, construction, participation

When thinking about mathematics knowledge use, three traditions have had a strong influence on the research literature most recently. Of these, the approach drawing on the cognitive science and information processing literature is the one which has adopted an essentialist model for mathematical knowledge most completely. Characteristic of this approach is the concern for internal cognitive processes and their coordination with an external environment. Knowledge is coded representationally and held by the user; use, conceptualised as knowledge transfer, is determined by the capacity of this agent to elicit correspondence between the problem situation and internal representations (Singley & Anderson, 1989; Anderson, 1990). Correspondences can, for instance, be accessed by analogical reasoning and can be promoted by metacognitive activity (Brown, 1989; Reimann & Spada, 1996). In a second approach, knowledge in use is not considered to be transferred to a site of deployment, instead it is constructed by the agent who utilises internal cognitive operations to modify, extend, replace, attune prior knowledge in relation to the problem presented within the given situation (Confrey, 1990; Hiebert & Carpenter, 1992). Within this paradigm, knowledge is not given essentially as an abstract universal, rather it takes the form of a viable set of situationally attuned procedures within the terms of the problem space (Cobb, Yackel, Wood, 1992; Cobb, 1994). When considering the viability of subjective knowledge structures, criteria relating to constraints within the physical, conceptual and social domains are pertinent (von Glasersfeld, 1995). Within the third approach, knowledge in use does not lie within the head of the user so much as being distributed within a collective practice relating to a particular set of work or other culturally related tasks (Brown, Collins & Duguid, 1989; Rogoff, 1990). Knowledge use is seen as equivalent to participation within a designated "community of practice" rather than as the far transfer of specific content and procedural knowledge or the subjective construction by the individual of locally pertinent knowledge (Lave & Wenger, 1991). Critical in this approach, therefore, is the social domain - its habits, customs, values, conventions, together with its mediating artifacts, technologies of production and social organisation (Wertsch, 1985; Engeström, 1991; Engeström, Engeström, & Kärkkäinen, 1995).

In this paper the theoretical framework adopted has been influenced most strongly by the third of these approaches. Unlike much of this literature, however, less attention has been given to the nature and characteristics of the domain of practice related to the workplace site under observation, and correspondingly more attention has been given to the microgenetic formation of numeracy related task competencies. A discussion of what methods were engaged in this exploration can be found below.

An empirical inquiry

Data for the current study is drawn from a much larger exploration of over the counter operations within the airline sector of the transport industry. In the larger study data was collected within multiple sites. In the analysis reported here, however, specific attention is given to a body of data relating to operations on the ticketing desk within the Qantas domestic terminal at Sydney airport. Transactions among sales assistant staff (coded SA1, SA3), and other airline employees, customers, and researchers (coded RA) were recorded by a conveniently placed video camera over a period of about six (6) hours duration. Interactions recorded between airline staff and researchers were designed to prompt staff to elaborate discursively on workplace tasks as either they were in progress or as soon as was practically possible thereafter.

Review of the transcript corpus (supported by the corroboration of respondents) identified ticket exchange as being a regular feature of the work of site observed. Operations associated with this task are illustrated in the transcript below. The structure of the episode is as follows: a passenger (C45) approaches a sales assistant (SA3)

seeking to exchange a ticket for one currently held; this leads to the performance of a process which concludes with the completion of the transaction; a research assistant (RA) then prompts the sales assistant to discuss the context of the preceding interaction.

*SA3 to C45:
Hi
*C45 to SA3:
I just need to pick up a ticket for Harvey
*SA3 to C45:
Harvey, is it?
*C45 to SA3:
Yep.
*SA3 to C45:
Where're you going to?
*C45 to SA3:
Ah, Canberra. I'm on the eight-fifteen flight//
*SA3 to C45:
((inaudible))
*C45 to SA3:
Yeh, well they're going to use this ticket as a credit towards the/
*SA3 to C45:
Oh sure, no problems. ..What time are you flying to Canberra?
*C45 to SA3:
Eight fifteen
*SA3 to C45:
Eight fifteen
*C45 to SA3:
The booking's all changed now?
*SA3 to C45:
Yeh.... OK, so this ticket's valued at three eighty and your ticket's six hundred and eighteen dollars forty so you just want the difference to go to the Amex?
*C45 to SA3:
Yeh, you want the Amex?
*SA3 to C45:
No its' OK, the number's already in there... OK, it's two hundred and thirty eight forty the difference - and there we go - oh, sorry, actually if you can sign it please, might help ... just ah, down the bottom there, just anywhere there... great. There you go. It's all yours.
***RA to SA3:**
Can you just explain what happened in the last transaction there?
***SA3 to RA:**
This one here? He was umm, using a ticket that he had, that he wanted that money toward a new ticket. So we do an exchange ticket. So the ticket he had was worth three hundred and eighty. The new ticket was six hundred and eighty dollars. So he put the additional money through to Amex.
***RA to SA3:**
So each step that you went through? - like he paid ((inaudible))
***SA3 to RA:**
He wanted a ticket, yeh, but he already had a ticket (so thus I knew that) he either wanted to change his ticket - normally if they have this ((inaudible)) it can be just re-validated pay the additional money.

As indicated previously, a number of alternative accounts of knowledge in use can be given. As a point of departure in exploring aspects of these, it will be useful first to consider the episode above from a knowledge transfer perspective.

In a transfer approach then, workplace knowledge is interpreted essentially as a rendition of numerical knowledge applied to a practical situation. An existing ticket is valued at $380, the desired ticket costs $618.40, and the subtraction of one sum from the other leaves a total of $218.40 for the customer to pay. Thus, in identifying the application of numerical concepts of subtraction (or its inverse, addition), this episode can be interpreted as an instance of knowledge transfer in which certain general procedures relating to numerical operations have been applied to a particular situation in a real life context. Certainly, a strength of this interpretation is its apparent simplicity. In appealing as it does to notions of a unitary mathematico-logical structure, knowledge of the details relating to the particular instance of the interaction are down played

(Walkerdine, 1994). At most, consideration of workplace knowledge under transfer is concerned with the problem of how to validly select and deploy appropriate numerical operations. From an epistemological point of view, numerical knowledge is thus regarded as both pre-given and mentalistic.

How does such an account stack up against the full range of evidence provided in this episode? One way to make this assessment would be to look for the presence of good analogical representations of numerical operations such as might be expected from the application of cognitive transfer theory (Brown, 1989; Singley & Anderson, 1989; Reimann & Spada, 1996). Evidence to be advanced later in the paper will indicate, however, that no such analogical relationships appear to exist in this case. This finding echoes the work of Cobb (1988) and others, and tends to weaken the case for transfer of numerical knowledge in this instance.

Towards numerical workplace knowledge: the logic of task performance

In this section the aim will be to outline a view of ticket exchange which does not make essentialist assumptions concerning the nature of numerical knowledge. The analysis will constitute an approach to determining the logic of actual task competency, that is, how workplace knowledge was actually organised and implemented in the events observed. Identification and stratification of component tasks with respect to their significance for the ticket exchange operation is the first step in the analysis. Next, the linguistic elements of task performances are analysed in order to explore relationships among them and to evaluate the place of the numerical within the workplace knowledge displayed. Wittgenstein's concept of the language-game (Hunnings, 1988; Wittgenstein, 1991) proved useful in obtaining a focus on the form of the interactions observed rather than their putative content. This was important as a non-essentialist epistemological paradigm makes problematic the notion of communicational content as a pre-given.

Focus on 'exchange'

In commencing the analysis, the first step was to review the use of the sign 'exchange' and its cognates (eg 'exchanging') within the transcript corpus. When this was completed two families of usage corresponding to language-games employing these signs appeared. In the first, typical instances were the exchange of one ticket for another or the exchange of a ticket for credit (eg "exchange a ticket to upgrade"). In instances of this kind the sense of effecting an actual swap of ticket items or entitlements was characteristic. In the second language-game, "exchange" was found to take its meaning from items in the following list: a ticket or coupon (an "exchange coupon"), or stamp (an "exchange stamp"); a software protocol (coded 'in exchange') where this was associated with the workplace procedure used to manage the ticket exchange process; the workplace ticket exchange procedure itself (as in, "do an exchange ticket"). In comparing these language-games it becomes noticeable that in the first, primary activity is directed to the resolution of the customer's service consumption needs and preferences in relation to constraints imposed on service provision by the airline itself. Whereas in contrast, the focus of the second language-game revolves around the organisation and management of legitimate task performance within the domain of service production. Component tasks found within each of these are summarised below.

- *Exchange as swapping* (swapping tickets in order to satisfy consumer preferences and requirements): -
  - providing information to customers re transaction sums for customer and/or sales assistant generated ticket swapping scenarios
  - obtaining compliance with airline mandated ticket exchange regulations

- generating ticketing arrangement strategies in order to meet customer preferences and airline mandated compliance requirements
- making reasoned judgments relating to the appropriate application of airline mandated requirements.

- *Exchange as procedural management* (facilitating service transfer): -
  - warranting and validating ticket exchange
  - determining airline mandated ticket exchange regulations
  - processing sums relating to ticket exchange
  - managing and implementing the overall ticket exchange transaction process.

Thus far, the focus has been on what constituted ticket exchange task performance in the episodes observed. What a study of the logic of task performance now requires is an assessment of how these components were actually implemented. When approaching this question from a socio-cultural perspective, the mediating role of artifacts within the locality of the problem situation becomes decisive (Vygotsky, 1978). Paradigmatic examples of such artifacts are tools (physical and psychological) and signs (language and symbolic systems) associated with task performance. For later theorists (eg Leont'ev, 1981; Wertsch, 1981; Engeström, 1994) this list is also extended to include social elements of the problem space such as the organisation of tasks, rules and protocols within this situation, etc. Instances of mediating artifacts associated with the ticket exchange task listed were found to include:

- specific tools (such as concrete instruments eg exchange tickets, coupons and stamps)
- signs (such as particular words and symbols associated with exchange)
- language-games (including procedural rules for standardised software protocols, and authorised procedural protocols intended within specific and general levels of operation).

Given the limited scope of this report it will not be possible to treat each of these at any depth here. Analysis of the use of certain language-game will be attempted, however. In the following analyses, two complex language-games associated with the performance of 'exchange as swapping' (centred on the use of the word 'difference') and 'exchange as procedural management' (centred on the use of the work 'additional') will be explored. In a concluding section of the paper, this material will be used to move toward an understanding of how numerical knowledge becomes involved in the workplace activities observed.

Focus on 'difference' and 'additional'

Referring to the episode cited above, the sign "difference" at first performs the task of indicating a discrepancy between the value of the presented ticket and the value of the ticket sought by the customer. Later, this differential value is associated with an amount in dollars - "OK, it's two hundred and thirty eight forty, the difference" - and the denotation of this sum is understood to serve as a basis for a successful ticket exchange transaction.

Similar usage for the words 'difference' and 'different' can be found elsewhere within the corpus of the transcript. For instance, when a customer made an inquiry concerning the possibility of upgrading from economy class the following dialogue was observed.

*C61 to SA1:
Yeh, what's the difference between business and first class
*SA1 to C61:
OK - Business Class is eight hundred and fifty one, Economy is six hundred and fifty three, so what's that - a hundred and ninety eight dollars.
*C61 to SA1:
For?
*SA1 to C61:
And first class is nine hundred and seventy nine - that's about three hundred and ... twenty six dollars or something like that
*C61 o SA1:
Alright.. Business has the comfortable, more comfortable seats
*SA1 to C61:
Yeh, the seating's virtually the same, it's just a little bit different in the food presentation
*C61:
Oh, so seating's the same?
*SA1 to C61:
Yeh, virtually.. I think they're the same width and everything, but they could be a fraction longer, but..
*C61 to SA1:
OK
*SA1 to C61:
I think they've got velour on the first class seats and its just nice cloth on the other one
*C61 to SA1:
Oh right - if you can umm.. ((inaudible)) in first, `cause I just want some room and I want to be able to sleep
*SA1 to C61:
Oh OK. How would you like to pay the difference?

Now in this case the sign 'difference' was being used quite differently by customer and sales assistant. For the customer the matter is seen to turn on the size of the seats and the type of cloth used in their upholstery. However, for the sales assistant, the sign "difference" seemed to suggest a relationship between values, and between various numbers associated by protocol with these (viz ticket prices). Note that the customer's interpolation of "For?" in response to the sales assistant's expression of quantitative values was set aside by the sales assistant and a new set of numerical figures reiterated. This observation tends to confirm the view that the customer and the sales assistant were participating in quite different language-games. A second, related occurrence of the sign 'difference' was to indicate on what financial terms a transaction could in fact successfully transpire ("OK. How would you like to pay the difference?"). To summarise: in a variety of representative instances, 'difference' was used by the sales assistant (i) to express a relationship between abstract rather than concrete entities and (ii) to indicate an amount which customers are asked to pay for the service sought. It is concluded from this analysis that the language-game involving the sales assistant's use of the sign 'difference' could be regarded as a mediating artifact in the 'exchange as swapping' language-game.

Analysis will now turn to the sign 'additional'. Review of the transcript corpus shows this term is most often used by the sales assistant in the context of an offered explanation to the researcher regarding events associated with procedures relating to the issue of an exchange ticket (see transcript above). Like the sign 'difference' discussed previously, the sign 'additional' is used to refer to the differential value between the current and proposed ticket. But as a sign, 'additional' is used in quite a dissimilar way. In this case, the value amounts depicted are those which connote the accumulation of a certain stock - money. Later in the transcript, when the same sales assistant is recorded discussing software protocols for the automated regulation of procedures for ticket upgrading, the following pronouncement concerning procedure is recorded: "Upgrade to J or F is permitted at any time subject to payment of additional fare." Here again, use of the sign 'additional' connotes the accumulation of a stock whereas, in contrast, the sign 'difference' is used more abstractly to suggest a relationship between values. It is concluded firstly that this language-game functions as a mediating artifact within the

language-game of 'exchange as procedural facilitation'; and secondly, that the language-games associated with the signs "difference" and "additional" are quite dissimilar.

Towards an understanding of numerical workplace knowledge

In the analysis so far, progress has been made towards determining the structure of the logic of the ticket exchange task performance. First, ticket exchange was seen to be the conflation of two kinds of language-games, the first involved 'exchange' as swapping tickets and/or entitlements, the second involved 'exchange' as the procedural management of exchange operations. Next, a more detailed study of artifacts mediating the operationalisation of these ticketing tasks was undertaken. This lead to the identification of an additional range of language-games deployed in the service of task performance. For instance, a language-game revolving around the use of the sign 'difference' was seen to underpin the operation of exchange as swapping; whilst another quite different language-game, related to the use of the sign 'additional' was found to underpin the operation of exchange as procedural management. What then, by way of conclusion, do these findings suggest for the position of the numerical within the workplace knowledge observed?

From an essentialist perspective it would appear that this involvement is easy to state: the use of 'additional' and 'difference' is suggestive of an analogical representation of the inverse relation which exists between addition and subtraction within the domain of numerical knowledge. However, the evidence adduced in the preceding analysis against such an account is quite compelling. Based on the essentialist account one would expect to find 'additional' and 'difference' within the same language-game - however, on the empirical evidence available, this was not the case. What then are the implications for the place of numerical knowledge within the ticket exchange task performance? I will conclude the paper with the following suggestions.

First, the workplace situation itself has primacy within the organisation and implementation of workplace knowledge - thus any approach to workplace task competency should proceed from a detailed knowledge of the particular occupational situation, not from an abstract set of numerical knowledge statements. Second, numerical knowledge does not enter the workplace as a pre-given entity. Third, numerical knowledge is not used as a unitary mathematico-logical structure within workplace sites. Analysis based on the evidence advanced in this paper suggests that numerical workplace knowledge is better seen as a body of fragmented knowledge. Fourth, and lastly, the logic of this fragmentation is governed by the character of mediating artifacts engaged within task performance.

Note: It is acknowledged that data collected in this study was gathered in collaboration with other research associates at the Centre for Learning and Work Research, Griffith University. The study was made possible by a research grant from the Australian National Training Authority's Council for Research, 1996.

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Area integration rules

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This paper investigates the use of area integration rules. 36 children in Grades 4, 6 and 8 were given area judgement tasks, using rectangles of varying areas and perimeters. Information Integration Theory procedures revealed both additive and multiplicative judgement rules that determined the children's responses. It was found that judgement rules change intra-individually but there does not appear to be a relation between judgement rule and Grade level. The form of presentation of the rectangle was found to be important.

Although area is one of the essential concepts of mathematics instruction, it is a concept that textbooks make little attempt to define, and many (for example, Blane & Booth, 1989) discuss area with the apparent assumption that students already understand it. Textbooks aimed at the Year 8 level of schooling in Queensland appear to limit their approach to the concept of area to two ways: a combination of basic pre-formula exercises, statements of formulae and exercises using the formulae (e.g., Blane & Booth, 1989; Duffy & Murty, 1988); and directly into statements of formulae and exercises using the formulae (e.g., Clark, Clark, Burza & Conway, 1988; Priddle & Davies, 1989).

This apparent lack of definition and emphasis on formulae seems to be attributing, in part, to well documented misconceptions of the concepts of area in both primary and secondary school aged children (Kidman & Cooper, 1996a; Outhred & Mitchelmore, 1996; Clements & Ellerton, 1995; Bell, Costello & Kuchemann, 1983; Bell, Hughes, & Rogers, 1975). Research (e.g. Hirstein, 1981; Hirstein, Lamb and Osborne, 1978) has shown that one of the major misconceptions is confusion between area and perimeter. In particular, as Kidman and Cooper (1996b) found, students' have difficulty with the process of obtaining a shapes' measurements, determining which dimensions to consider and how to integrate these dimensions when calculating either area or perimeter. Research on student teachers (Baturo & Nason, 1996; Simon & Blume, 1994) has revealed possible inadequate knowledge of the concept of area. The student teachers are able to apply procedural formulae, but they confuse area and perimeter and use linear rather than square units. They show very little conceptual understanding of the relationship between area and side length.

As argued by Kidman & Cooper (1996a) and Wolf (1995), Information Integration Theory (IIT), a functional measurement technique (Anderson & Cuneo, 1978) used to identify the rules applied by children to integrate dimension information, and cognitive algebra offers an excellent opportunity to explain the process of area concept development in children. According to IIT, "all behaviours reflect a blend of stimuli, and a response is the consolidated resultant of multiple causal forces" (Kidman & Cooper, 1996a, p. 340). The methodological counterpart of IIT, called functional measurement, allows diagnosis in simple algebraic terms, "... of the rules which govern integration of information about perceived stimuli." (Wolf, 1995, p. 49-50).

At some stage between the age of 5 and 12, a child is expected to make the transition from an additive integration rule to the normative multiplicative integration rule (Kidman & Cooper, 1996a; Wolf, 1995; Schlottman & Anderson, 1994; Lautrey, Mullet & Paques, 1989; and Silverman & Paskewitz, 1988). The general consensus of these and many other recent studies is that students' judgements of area obeyed two-dimensional rules. It appears that 8 year old students were in a transitional stage between the additive and multiplicative rules. In general, students have been provided with different rectangular shapes and asked to place their area on a linear scale.

This paper describes an investigation to determine the judgement rules used by students in Grades 4, 6 and 8 in a private college in Queensland and reports on student responses to experiments to explore how the students integrated length and width dimensions to judge area of rectangular or near rectangular shapes. The purpose of the investigation was to:

- identify the way in which children integrate stimuli to determine area; and
- determine if integration rules change intra-individually.

The experiments was based on the body of literature and the functional measurement methodology stemming from the work of Anderson and Cuneo (1978).

The study

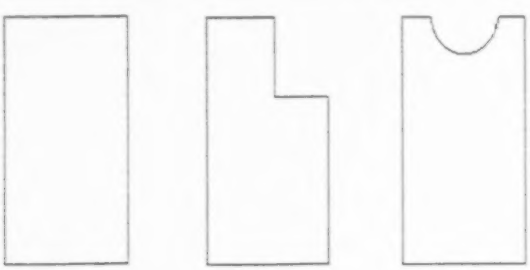
The study used a multi-method design where the quantitative methodology of functional measurement was combined with the qualitative methodology of semi-structured clinical interview.

Participants. The sample consisted of 36 students, 12 students from each of the three grade levels, and a range of mathematical abilities, one third each of below average, average, and above average, from each grade.

Instruments. The instruments used were three experiments and an interview. The first experiment contained 16 rectangular wooden pieces (Fig. 1a) painted to represent chocolate and with dimensions corresponding to the factorial combinations of 3, 6, 9, and 12 cm. The pieces were to be presented to students who would be asked to judge the area of the rectangular pieces in relation to two end anchors. To obtain measures of the children's area judgements, the children were provided with a 19 point scale with two end points. Two special pieces of dimensions 2.7 x 2.7 cm, and 15.8 x 15.8 cm were used as end anchors.

The second experiment used 16 rectangular pieces identical in dimensions to the first experiment, but with a rectangular corner 'bitten' off producing a figure of equal perimeter, but less area (fig. 1b). The dimensions of the 'bitten' off corner were all one third of the width and one third of the height of the rectangular stimulus. The third experiment again used 16 rectangular pieces identical to the first experiment, but this time they had a semi-circular 'bite' out of one side producing a figure with less area but greater perimeter (fig. 1c). The 'bite' was centred along one dimension with the radius of the 'bite' one third of the length of the shortest dimension.

The interview was short and semi-structured and asked each student to describe the method they used to rate each piece. They were quizzed as to whether they were aware of any changes they had made to their method over the course of the three interviews. Diagrams of identified methods were sought, from the students. At the conclusion of the interview, the students understanding of both area and perimeter was discussed, and the student was then asked to identify if he/she had employed either or both of these concepts to rate the chocolate pieces.



[a] Exp. 1 [b] Exp. 2 [c] Exp. 3

Figure 1. Experimental chocolate shapes

Procedure. The students were withdrawn from their class and the three experiments and the interview were administered in a separate room. The experiments and the interview were videotaped. The diagrams were sought at any time during the experiment, and the interview followed the third experiment. It took no longer than 30 minutes.

For each experiment, the children were first familiarised with the end anchors which were presented as corresponding to the end points of the scale. The scale had a smiling face at one end and a frowning face at the other. The small end anchor was presented as a piece of chocolate that the child would be unhappy to receive while the large end anchor was presented as a piece the child would be happy to receive. The students were then asked to judge how happy someone would be to receive each of the 16 pieces if they were chocolate to eat. The pieces, each of equal thickness, were presented individually, and judgement was expressed on a 19-point response scale (see Anderson & Cuneo, 1978, for more details). The presentation of the chocolates was randomised, and a practice phase preceded the test phase. The children judged three replications of the chocolate stimuli in each experiment.

Analysis. The experiments were analysed using Anderson and Cuneo's (1978) functional measurement methodology. This methodology uses algebraic rules as the base and frame for psychological scaling. These rules provide the breakdown of the observed response into its functional components, as represented by the scale values and weights of the various pieces of information (Anderson & Cuneo, 1978). This is used to identify the kind of rules underlying the judgements students make when provided with different rectangular shapes and asked to judge their areas on a rating scale. The scale positions for the rectangles (which are specific combinations of height and width) are represented graphically and then subjected to an analysis of variance. Conclusions regarding the kind of rule underlying the judgements are determined from the shape of the graphical plot, and the significance or nonsignificance of the main and interaction effects (Anderson, 1981).

The graphical plot of the responses is against the length of one of the dimensions of the rectangles. Thus, if the plot is an arrangement of parallel lines or parallel curves, the students' judgements are considered to be additively based, that is, they are tending to perceive area of a rectangle in terms of the *sum* of its dimensions. If the plot is fan shaped (expanding lines or curves), the students' judgements are considered to be multiplicative, that is, the students are tending to see area of a rectangle in terms of the *product* of its dimensions. Figure 2 presents hypothetical curves for these rules. If the plot lines or curves intersect, then an inference with regard to additivity or multiplicativity may not be possible.

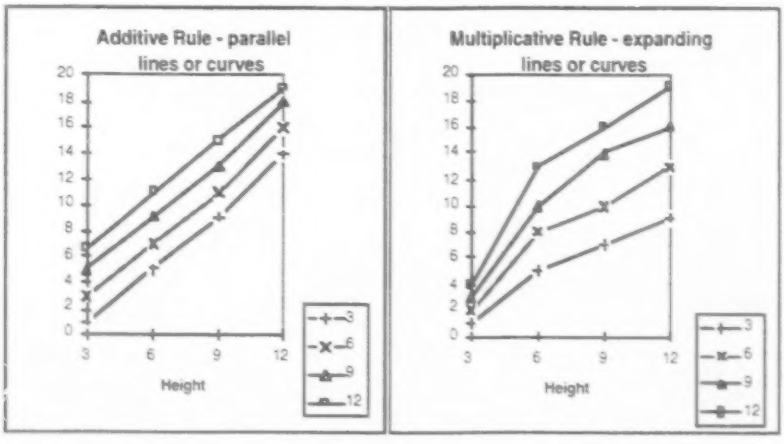


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Figure 2. Hypothetical plots for additive and multiplicative based judgements

Factorial plots were drawn for each student for each experiment, as well as a group plot for each of the three grade levels. The plots were then compared to the hypothetical plots shown in Figure 2. On the basis of this comparison, additive or multiplicative integration rules were assigned to that student for that experiment.

The interviews were transcribed into protocols and the students' statements compared with their experiment results in an endeavour to provide a second option for explaining students' responses.

Student responses to the experiments

Understanding the instrument. All students appeared to understand the judgement they were required to make in terms of being happy or unhappy with pieces of chocolate in relation to the end anchors. They appeared able to express their judgement unambiguously using the 19 point scale. The understanding of the response scale was checked by having the student point to specific sections of the scale (for example, a section depicting a little bit of sadness), as well as making a verbal statement about the section being pointed to in relation to the size of the piece of chocolate (for example, "I would be a little bit sad").

The most important procedural detail concerns the establishment of the frame of reference. "The rating of any one stimulus is always relative to what other stimuli are being rated" (Anderson, 1980, p. 9). The standard device for setting up the frame of reference is the stimulus end anchors which are just noticeably more extreme, higher or lower, than the rectangular experimental stimuli. These end anchors define the stimulus range, and helps the student set up a frame of reference for using the rating scale. The end anchors also tie down the end responses so that the responses to the chocolate stimuli come from the interior of the scale thus avoiding end effects.

Each students' scale positions were analysed with the functional measurement methodology. Table 1 shows the averaged integration rules for the three grade levels (Grade 4, Grade 6 and Grade 8) and the three experiments (E1, E2 and E3). The symbol **X** is used to denote the multiplicative rule, and **+** the additive rule. **Total** gives the number of multiplicative and additive students in each grade.

Table 1 Grouped integration rules for area

	Grade 4			Grade 6			Grade 8		
	E1	**E2**	**E3**	**E1**	**E2**	**E3**	**E1**	**E2**	**E3**
Ave	X	+	X	X	X	X	+	X	X
Total	X = 4 + = 7 ? = 1	X = 6 + = 6	X = 7 + = 5	X = 6 + = 6	X = 8 + = 4	X = 5 + = 7	X = 6 + = 6	X = 7 + = 7	X = 5 + = 7

In the majority of cases the resulting plots were obviously additive with clear plots of parallel lines or curves, or multiplicative with clear plots of expanding curves or lines. In cases where the curves intersected (for example, the curve for a width of 6 cm crossed the curve for a width of 9 cm), the general shape of the plot was recorded, but a '**?**' was also recorded indicating a 'questionable' rule usage. It was not possible to determine a judgement rule for Ben, a Grade 4 student, doing Experiment 1. This particular plot had four intersecting locations and no obvious parallel curves or diverging lines.

The differences between the grades was not as obvious as could be expected. The perception of area of rectangle being related to the sum of the rectangles' dimensions is fairly consistent across the grades. The group of Grade 8 students tested

do not seem to have advanced much beyond the Grade 4 or Grade 6 level. However, there were two interesting small changes. The first was the increase in multiplicativity in the Grade 4 results from Experiment 1 to Experiment 3. The second was the increase in multiplicativity from Experiment 1 to Experiment 2 across all Grades; and, except for the Grade 4 students, the decrease in multiplicativity from Experiments 1 and 2 to Experiment 3.

It is evident that judgement rules do change intra-individually. Ten students used an additive rule initially in experiment 1, but had altered this to a multiplicative rule by the conclusion of experiment 3. Surprisingly, 9 students did the reverse. They started using a multiplicative rule but changed to an additive rule in either the second or third experiment. Similar to the distribution of additivity and multiplicativity across the Grades, the changes in integration rule intra-individually within students) was also fairly constant across the grades with the number of students constant in their rule in each Grade remaining between 4 and 6 across the three Grades.

A preliminary analysis of the interview data reveals two interesting points. Firstly, of the ten children who had intra-individual rule changes from additive to multiplicative, seven showed a clear preference for a vertical alignment of the chocolate pieces rather than a horizontal alignment (See Figure 3a). Secondly, there was a strong tendency (88%) for children altering their judgement rule from multiplicative to additive to want to remove the part of the chocolate piece that had been altered (See Figure 3b).

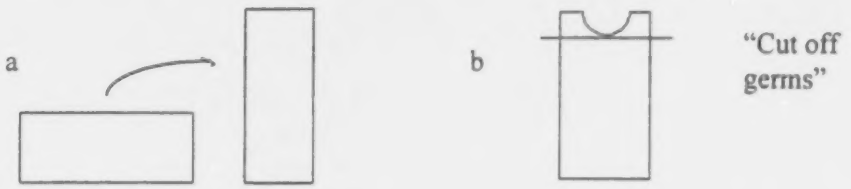

Figure 3. Alignment preferences and section removal

Discussion and conclusions

If the plot of the students' scale positions approximates parallel lines or curves, this reflects a perception of area where doubling the lengths of the sides of the rectangle can be seen as doubling the area. This reflects a perception of an additive relation between area of a rectangle and the dimensions of the rectangle. Thus, this perception can be considered as a confusion between area and perimeter for rectangles. This particular integration rule has been denoted as Area = Height + Width. By comparison, a plot which approximates a fan shape is reflecting a perception that doubling the sides more than doubles the area of the rectangle. This is seen as a correct perception and denoted as the Area = Height X Width integration rule. Plots with lines crossing indicate a very poor conception of area as this means that the student has judged a rectangle with smaller dimensions as having a larger area than a rectangle with larger dimensions.

This study has, therefore, supported the findings of Hirstein (1981) and Hirstein, Lamb and Osborne (1978) that there is confusion between area and perimeter. Around 50% of students from all Grades and in all experiments exhibited judgements that showed they were using the perimeter rule to determine area.

Experiments 2 and 3 presented modifications to the rectangles. The removal of a rectangular corner was found to reduce students use of the perimeter rule while the removal of a semicircular piece from an edge did not. The question is why? It could be argued that a rectangular piece out of a corner of a rectangle gives the effect of adding two more sides and thus the student tends to look at the amount of surface rather than

the length and width. It could also be argued that the removal of the semicircular piece from a side has a lesser effect on how the rectangle is perceived than the removal of a corner, and that the addition of a semicircle to the factors that have to be taken into account in making area judgements adds weight to an additive focus on length. However, the reasons for students use of perimeter in the three experiments will have to wait until the interviews are fully analysed in relation to these experiments.

Over 50% of the students (22 out of 36) changed their integration rules across experiments. Once again the question is why? There appears to be no pattern in the changes: from additive to multiplicative, from multiplicative to additive, and sometimes in both directions. There appears to be no relation to Grade level.

The conversion to vertical alignment seems to be important to some children as it was adopted by those who made additive to multiplicative intra-individual rule changes. The children were allowed to handle the chocolate pieces, and they rotated pieces prior to making their judgements. One student explained that chocolate bars are stacked vertically on shop shelves, so you need them displayed here in the same way. Having rectangles represented in a familiar way may have some influence on a child's ability to correctly judge area.

The removal of a section from either the corner or side of a chocolate piece seems to confuse some children. A quarter of the children could correctly judge the area of rectangles, however after removing a section, they used an additive judgement rule. The interviews showed the majority of these children visualizing the chocolate pieces without the missing section, but shorter in one dimension. All such children claimed, either verbally or diagramatically, to cut off the altered section. One child explained it was to get rid of the germs where someone took a bite out of it. Why the imaginary act of cutting a piece down to size causes a rule change is not yet evident.

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Problem Solving in Algebra: Ability Grouping, Affective Factors And Gender Differences

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Delineation of the quantum of influence of *extra-cognitive auxiliary inputs* like affective factors may help to explain gender differences by illuminating the mechanics of cognitive functioning in early adolescents engaged in mathematics problem solving. Part of the data from a semi-longitudinal study is analysed in this paper to explore the relationship between algebraic problem solving ability and affective factors based on the differential problem solving abilities among junior high school students. The results show that there is a difference in this relationship for female and male students. A *transient refractive affective state* of high achieving girls could be the reason for compromising any advantage over their male counterparts.

Background

Recent research in problem solving has identified the affective domain as one of the important areas where gender differences are noticed (see Frost *et al*, 1994). This may have an important bearing on the search to understand the differences in the cognitive domain of the students, especially with regard to the often stated underachievement of girls. Delineation of such *'extra-cognitive auxiliary inputs'* would illuminate the mechanics of cognitive functioning. Variables from both cognitive and affective domains have been seen to influence the problem solving ability (Boekaeris et al, 1995) as well as intra-domain interactions (Skaalvik and Rankin, 1995). Ethington &Wolfle (1984) concluded from their longitudinal study that there is a complex interaction between gender, selected other variables and mathematics achievement. These other variables include cognitive and affective variables. We have previously attempted to describe problem solving as the resultant of the interaction between these sets of variables (Thomas and Kota, 1996) and the variables in each set. Maker (1982, p. 30-31) stated that 'there is a cognitive component in every affective objective and affective component in every cognitive objective'. McLeod (1988, p. 134) sees this relation with problem solving, ' if the students obtain a solution to a problem, they typically express feelings of satisfaction , even joy' and these are seen to be important for problem solving performance. The National Council of Teachers of Mathematics (NCTM, 1989) and National Research Council (NRC, 1989) have encouraged mathematics educators to incorporate affective factors and cognitive factors such as algorithmic procedures and problem solving (Schoenfeld, 1985), in their mathematics teaching and learning. To do this efficiently and effectively, knowledge of the relationship between affective factors and problem solving ability and the changes in this relationship with age and academic level, would be very helpful for mathematics educators. This knowledge could help to narrow the gap between the genders in enrolment and achievement in mathematical sciences in their further education.

Affective factors

A body of research records the analysis of the relationship between affective factors and achievement in mathematics. Enemark and Wise (1981, p. 22) demonstrated that '...the attitudinal variables are a significant indicator of mathematics achievement'. Reyes (1984) examined a numbers of affective variables influencing the learning of mathematics. These variables included self-concept, anxiety and perceived usefulness and others considered enjoyment of mathematics, self-perception, motivation and interest (Skaalvik & Rankin, 1995; Mitchell, 1993; Watson, 1983). Kloosterman (1991) and Minato & Yanase (1984) tried to examine the relationship between mathematical attitudes and achievement which Anderson (1981) inferred through mathematics participation.

Other researchers have tried to establish a relationship between these affective factors and various other components such as class-room environment, participation and achievement and problem solving in the process of mathematics learning and teaching. McLeod (1992) proposed a framework in which beliefs, attitudes and emotions reflect the range of affective reactions involved in mathematics learning and teaching. Forgasz and Leder (1996) established a link between classroom environment and affective factors and stated those would provide vital clues about the factors shaping student beliefs which exhibit gender differences at high school.

Affective factors and problem solving

After the announcement of the agenda by National Council of Teachers of Mathematics (NCTM, 1980), 'problem solving should be the main focus of school mathematics'; additional research has gone into finding the role of affective factors in mathematics problem solving. Lester and Kroll (1989, p. 57) stated that 'the affective domain...is an important contributor to problem solving behaviour' and 'student performance in solving a problem may be very much influenced by affective factors, sometimes to the point of dominating the students thinking and actions.' Some researchers established a positive influence of these variables while others noted the negative affects of lack of some of these factors (see Wong, 1992, and Davis, 1973).

Affective factors, problem solving and gender

Evidence is accumulating that establishes gender as an important influence on affective factors as well as performance in mathematics. Hyde *et al* (1990) performed a meta-analysis on the magnitude of gender differences in the mathematics performance. They noted problem solving as one of the areas where gender differences exist and reported that junior high school years are important to notice favourable or unfavourable changes in attitudes towards mathematics (see also Aiken 1986 & Callehan, 1971). Aiken (1976) suggested a separate analysis by sex in prediction studies involving affective factors. From the meta-analysis by Xin Ma & Kishore (1997), Frost *et al* (1994) and Hyde *et al* (1990), the following points emerge about the relationship between gender, mathematics attitudes and achievement of students: a) junior high school appears to be the more suitable age group since they are not too young to lack the ability of expression nor too old to have fixed attitudes precluding interventions, b) grouping on the basis of differential abilities seem more appropriate, c) area specific tests such as algebra, geometry and the like may be more revealing, & d) high stake tests like problem solving could be better representatives of mathematical ability. The inference from these observations is that studies incorporating the above ideas would provide a clear view of the various relationships in the quest for bringing appropriate changes in line with directives of learned bodies (National Council of Teachers of Mathematics, 1989 and National Research Council, 1989).

The present study overcomes the majority of the disadvantages of the earlier research studies as implied in the above cited meta-analytical studies. When trying to establish a relationship between problem solving, affective factors and gender differences, a study based on students differential abilities, has not to the best of my knowledge been undertaken. Here, I have analysed the initial data of our semi-longitudinal study to note the gender differences in this relationship on the basis of ability grouping using correlational analysis. The results presented show that the extent of relationship between the problem solving ability and affective factors could be one of important reasons affecting students cognition and developing gender differences in algebraic problem solving abilities.

Method

Subjects

The data for the present study was collected from 345 form 3 and 4 students of ages about between 13 - 15 years, from 8 secondary schools in the Auckland region of New Zealand. Both single-sex and co-educational schools were represented in the study. Each

school reported that they were following the 1992 New Zealand curriculum published by the Ministry of Education. One class of average ability students from each school was randomly selected to form the subject group. The students were taught by specialised mathematics teachers.

Instruments

Each individual student in the study was given two questionnaires during regular mathematics classes in the first half of the academic year 1996, under the supervision of the mathematics teacher, before the algebra syllabus was taught for the year. To ensure the uniformity of administration the teachers were provided with clear instructions and directions. These questionnaires were repeated at the end of the academic year. The first measured seven affective factors using self-descriptive questions in a 5-point Likert format. The second measured algebraic problem solving ability. The details of the tests used are:

Self-concept: The mathematical self-concept scale of 27 items developed by Gourgey (1982) was used to measure mathematical self-concept. The internal consistency reliability of the scale is 0.96 and Gourgey stated that the analysis of the scale provided support for its validity and reliability.

Interest: The scale developed by Mitchell (1993) and used in his subsequent research was used to measure the level of interest in mathematics. The internal consistency coefficient for the independent subscales used ranges from 0.77 to 0.93.

Anxiety: The anxiety component of the Skaalvik and Rankin (1995) scale was used to measure the students' mathematical anxiety. There are 8 items in this scale and the Cronbach alpha for this scale was reported as 0.90 for a similar subject group.

Self-perceptions: The self-perception constructs were measured by using Skaalvik and Rankin's (1995) three-item scale for each of self-perceived ability and self-perceived aptitude. The internal consistency reliability coefficients are 0.8 for 12/13 years and 0.83 for 15/16 years.

Usefulness of mathematics: The usefulness of mathematics scale of Fennema & Sherman (1976) served to measure the individual student attribute about the usefulness of mathematics.

Mathematics Intrinsic motivation: Of the two types of motivation, intrinsic and extrinsic, we considered that the intrinsic factor was likely to be more consistent and reliable since it originates from the subject. This factor was measured using the English translation of the instrument developed by Skaalvik and Rankin (1995).

Enjoyment of Mathematics: Students enjoyment levels were measured using the instrument from Aiken's attitude scale as modified by Watson (1983). The internal consistency reliability coefficient was found by Aiken to be 0.95, using secondary school students.

Problem Solving ability: An algebra test consisting of five basic word problems was constructed from the problems used in the three algebra tests which we had piloted (Thomas and Kota, 1996). The problems were constructed based on the objectives for levels 2 to 5 of the New Zealand mathematics curriculum guidelines, published by the Ministry of Education. It was intended that factors like word order, situation, language and difficulty in carrying out numerical operations would not be obstacles in the process of solving the problems successfully. The type of problems used in the test were:

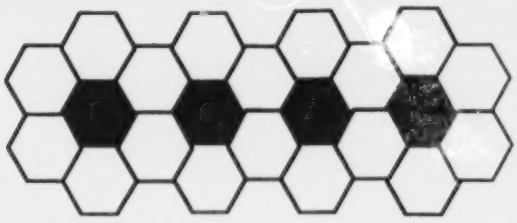
1. The number of girls in a school is 41 less than the number of boys. The total number of students in the school is 1539. How many girls are there in the school?

2.To hire the Pizza House for birthday party costs a basic rate of $70.00 plus $3.50 per person. If the total bill is $346.50, how many people attended the party?

3. Tickets for the school play cost $2 for children and $4 for adults. 500 tickets were sold for $1,640. How many children's tickets were sold and how many adult tickets were sold?

4. The Auckland city council wishes to create flower beds, surrounding them with hexagonal paving slabs according to the pattern shown below.

Complete the table below to find the number of paving slabs needed for 8 and 100 flower beds. Write an algebraic equation to find the number of paving slabs, N, needed to surround F flower beds.



Number of flower beds	1	2	3	4	8	100	F
Number of paving stones	6	10	14	18			

5. A wooden fence is made by placing 3 planks between 2 posts as shown in the figure. (A picture was given). Complete the table to find the number of planks needed for fences of 5 posts; 100 posts and S posts. Write an algebraic equation to find the number of planks L in a fence S posts long.

Number of posts	2	3	4	5	100	S
Number of planks	3	6	9			

Results

The total of 345 students, 216 girls and 129 boys, (boys and girls together, 177 form 3 and 168 form 4) participated in the study. The scores of these students are analysed using correlational analysis to look into the differences in the extent of the linear relationship between algebraic problem solving and affective factors. Whilst such an analysis does not permit definite inferences it is capable of giving some indication of the relationship. When we examined the difference in the mean scores on algebraic problem solving ability for female and male students, we found a significant difference. Girls performed better than boys in form 3 in algebra problem solving (for girls, mean = 13.702, for boys, mean = 10.171, t = 4.325, df=175, t critical = 3.98, $p<0.0001$). However, at form 4 there is no significant difference in algebra performance between girls and boys in (for girls, mean = 17.017, for boys, mean = 16.037, t = 1.073, df=166, t critical =1.974).

The student scores on each of the affective factors show a fairly high positive correlation with algebraic problem solving ability for form 3 girls (as described in table 1). But for form 4 girls, the correlation is positive and present only for self-concept and anxiety.

Table 1: Correlations between problem Solving Ability and Affective Factors in form 3 and 4 girls

Algebra Scores	*SC*	*Int*	*Anx*	*SP*	*Use*	*Mot*	*Enj*
Form 3 (n=101)	0.5133	0.3963	0.3853	0.398	0.2235	0.3893	0.3186
Form 4 (n=115)	0.2453	0.0120	0.162	0.0456	-0.014	-098	-0.130

Sc= Self-concept; Int = Interest ; Anx = Anxiety; Sp = Self- perception,
Use = Usefulness; Mot = Motivation; Enj = Enjoyment

We also see that there are much higher positive correlations between each of the factors and problem solving performance for form 3 female students. It appears that there is a role played by age and academic level of the students in the relationship between affective factors and algebraic problem solving ability.

When the corresponding results were analysed for the 129 boys, 76 in form 3 and 53 in form 4, we obtained the figures shown in table 2. All the correlations for boys in form 4 are positive and are fairly high for self-concept, self-perception, enjoyment and interest. But none of the correlations is comparatively high for boys in form 3 except for anxiety. In fact self-perception and enjoyment show a slightly negative correlation. Here

we see that a situation opposite of girls is true. As table 1 shows, the correlations between algebra performance and affective factors based on individual scores, are higher for girls in form 3 than in form 4. From these results it appears that the relationship between algebraic problem solving ability and affective factors is different for female and male students at secondary school level. There is an increase in the influence over this year for boys except for anxiety and decrease for girls.

Table 2: Correlations between problem solving ability and affective factors for form 3 and form 4 boys

Algebra Scores	*SC*	*Int*	*Anx*	*SP*	*Use*	*Mot*	*Enj*
Form 3 (n=76)	0.081	0.043	0.316	-0.010	0.130	0.0302	-0.006
Form 4 (n=53)	0.343	0.212	0.071	0.421	0.152	0.147	0.218

When we divide the students scores into low and high achievers, students whose algebra scores are between 5 and 15 and students whose algebra scores are between 16-25 (minimum and maximum scores obtainable in the algebra test are 5 and 25 respectively), we see some very interesting results which get masked when the result are taken as a whole for girls and boys.

Table 3: Correlations between problem solving ability and affective factors for high achievers and low achievers in Form 3 & 4 Girls

Algebra Achievement level		*SC*	*Int*	*Anx*	*SP*	*Use*	*Mot*	*Enj*
Form 3	low	0.298	0.178	0.155	0.228	0.029	0.120	0.001
	high	0.179	0.245	0.120	0.238	0.267	0.301	0.215
Form 4	low	0.313	0.163	0.400	0.320	0.350	0.271	0.259
	high	-0.105	-0.122	-0.104	-0.199	-0.307	-0.200	-0.200

When these correlations are examined for girls on the basis of ability grouping we obtain the figures shown in table 3. In form 3, the correlations are higher for the high achievers group in all the affective factors except self-concept and anxiety but in form 4, the correlations for low achievers group are higher in all the factors and in fact they are negative for high achievers group. Thus for the younger girls the scores in all the affective factors correlate positively with algebra problem solving performance for both high and low achiever groups, and are higher for the high achiever group for all the factors except self-concept and anxiety. But within just one year this effect has disappeared, and the correlations are positive only for the low achievers group and turned negative for high achievers group. It appears that higher levels in affective factors seems to decrease their scores in problem solving for high achievers group in form 4. The correlations have increased among the low achievers groups from form 3 to 4. Thus for girls as they grow in chronological age and academic level, the effect of affective factors seems to increase for low achievers but for high achievers it drops of markedly. One of the possibilities is that, they reached a *transient refractive affective state* thereby deviating from a positive view of mathematics and hence interest in further study.

When the individual student scores in algebra performance and affective factors are analysed on the basis of ability grouping for boys (shown in Table 4), all these correlations are higher for high achiever groups than low achiever groups in form 3 as well as in form 4. These correlations have increased for self-concept and interest from form 3 to 4.

Table 4: Correlation between problem solving ability and affective factors for low achievers and high achievers among form 3 and form 4 boys

Algebra Achievement level		*SC*	*Int*	*Anx*	*SP*	*Use*	*Mot*	*Enj*
Form 3	low	0.208	0.160	0.301	0.1866	0.256	0.2038	0.087
	high	0.388	0.235	0.380	0.425	0.399	0.375	0.456
Form 4	low	-0.284	-0.117	-0.127	0.001	0.092	-0.027	-0.053
	high	0.407	0.338	0.245	0.402	0.191	0.2633	0.426

Thus for boys, the effects of affective factors is maintained for the high achievers group but is decreased for the low achievers group within one year from 13 to 14 years.

For girls there is a positive linear relationship between affective factors and algebraic problem solving ability for the low achievers group in form 3 and form 4. For boys, this relationship is positive for the high achievers group in form 3 and form 4. An important result to be noted here is that this relationship between algebraic problem solving and affective factors is turning negative for high achievers group in girls and low achievers group in boys within just one year from form 3 to 4. That means for high achievers group in form 4 girls, algebra score and affective factor score move inversely to each other. Similarly, for boys the influence of the affective factors becomes negative for the low achievers group within one year, from age 13 to 14 years. It could be possible that high achieving girls have reached *a transient cognitive set point* so that their high levels in affective factors could be refracting their cognition away from mathematics. Literature on mathematics achievement and gender has noted that girls achievement is higher or equal to that of boys up to junior high school or early adolescence and by the end of high school boys achievement is higher than that of girls (see Frost *et al*, 1994). It could be possible that because of this *transient refractive affective state* girls compromise any advantage in mathematics over their male counterparts. This could be one of the reasons for female students to deviate from mathematical sciences preferring other subjects like social sciences and arts by the end of secondary school education.

Conclusions

Summarising the results we may say that affective factors have a positive effect on algebraic problem solving ability of girls and boys in form 3. For low achieving girls this effect increases with age from 13 to14 years. The positive effect is present in form 4 only for high achieving boys, increased for self-concept and interest. There is a clear cut gender difference in the effect of affective factors and changes in both the ability groups in the opposite directions for both the genders with age and academic level and in behaviour pattern of girls and boys.

The behaviour pattern seen is that, in boys the correlations between affective factors matched with their performance, ie, high achieving boys showing high correlations and low achieving boys showing low correlations but for girls they move in opposite directions, ie, low achieving girls having high correlation and high achieving girls having low correlations. This reflects the more mature out look of girls in that low scorers in problem solving ability project their ideas in the form of positive statements and those who are doing well or satisfactorily play down on their affective factors. In effect, high achieving girls have reached the *transient cognitive set point* arising from *transient refractive affective state* where their cognition gets refracted when they face a cognitive challenge in mathematics. A stage where the stimulus for the particular type of cognition disappears. This often occurs when cognition reaches a set point. In other words cognition ceases being reward by itself for those types of cognitive challenges where the affective factors failed to respond to stimulus. To sustain their achievement levels, *extra-cognitive inputs* like affective factors may be helpful to sustain their problem solving ability for high achieving girls and improve it for low achieving boys as they

grow in chronological age and academic level. When the results of the second data set are analysed we may discern the magnitude and direction of changes in affective factors and algebra problem solving ability. It will be important to look for 1) the changes in the female and male students who are high in both problem solving and affective factors, 2) students who are high only in problem solving and only in affective factors, 3) student who are average on both, and 4) low in both problem solving and affective factors. It may be possible that the results noted in this study could then be seen with better clarity. Further research is needed to look into the other possible factors responsible for the changes occurring in the preferences of girls and boys after the adolescent period.

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Access to level of mathematics study in high school: social area and school differences

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Using detailed information from the certificate records of 5,825 Year 10 students in Tasmania, this study found that the level of mathematics students access -- advanced, intermediate, or low -- is strongly related to socioeconomic background and school attended. Differences across schools, measured using multilevel modelling techniques, are related to the type of school (government, Catholic or independent), enrolment size and social composition. Residential segregation, the uneven distribution of selective schools, and school-level policies are the mechanisms discussed in explaining the origins of social area and school differences.

Introduction

Social inequalities in many areas of school participation have been declining, but social differences in access to mathematics courses persist. A study of mathematics participation in the state of Victoria has shown that access to university preparatory mathematics in the senior years retained substantial social class differences over the last forty years despite general improvements in the take up of mathematics as rates of school completion grew (Teese, 1994). The social trends are not confined to Victoria. A recent national study found a strong relationship between the social background of Year 11 and Year 12 students and the rates of participation in mathematics-science type courses: students from the highest socioeconomic group were more than twice as likely to take a mathematics-science course type than students from the lowest group (Ainley, et al, 1994). Other work has also shown large social differences in rates of mathematics participation (Lamb, 1996; Ainley et al., 1990; Teese, 1994). The evidence is a strong reminder that at a time when there are weakening social trends on some broad indicators of educational participation, such as school retention rates, large social differences in the use of school continue.

This paper reports original information on the levels of mathematics study of Year 10 students to explore the interactions between social background, school and access to the mathematics curriculum. Studies on curriculum access in the past have tended to focus on students in the senior year-levels (Years 11 and 12) where differences are more apparent because it is at this level that the curriculum is most differentiated. However, mathematics in the junior years of secondary school often involves pupil grouping practices such as setting or even more formal streaming. It is important to examine differences in the more junior grades because it is these year levels which operate as a staging period where differences in access to level and type of mathematics study lay the groundwork for trends in subject selection that emerge in the senior years.

To look at the relationships among social background, school and mathematics the study draws on information gathered from a statewide study of secondary schools in Tasmania. It will be argued that social class, mediated through the processes of residential segregation and selective schooling, and school policy are major influences shaping patterns in levels of mathematics study. Analysis of socio-economic patterns shows that students from higher socioeconomic backgrounds are far more likely than their peers to continue to access the most profitable streams of mathematics study. After controlling for student background, there are important differences in patterns of course enrolments across schools. These differences reflect the influence of school organisation and local school policies.

Data and analysis

The information on mathematics participation used in this paper was collected as part of a broader study investigating patterns of curriculum access and achievement in Tasmanian secondary schools. For that work, a data base was generated from records held by the Tasmanian Schools Board. The data base contains information on the subject enrolments and subject results for all School Certificate (Year 10), Higher School Certificate (Years 11 and 12)

300

and, more recently, Tasmanian Certificate of Education (Years 9, 10, 11 and 12) students from 1970 to 1995.

In this paper we focus on the mathematics courses taken by students in Year 10 in 1994 (N=5825). In Tasmania at this year level compulsory mathematics is provided as four separate syllabuses designed for students of different levels of mathematics ability. The top-level is designed for students planning to continue the study of mathematics into the senior years of secondary school and university. It includes the study of algebra, geometry, trigonometry, and measurement. The other three levels represent a decreasing order of difficulty with increased attention given to business and consumer mathematics and mathematical life skills.

The aim of this paper is to examine patterns of social and school differences in enrolments across the different levels of mathematics study. To do this the analysis will build on the tradition of social area research in Australia used to examine socioeconomic and regional differences in education. Over the last two decades various researchers have studied the links between residential segregation, uneven school provision, and various measures of educational activity and achievement (Fisher et al, 1973; Claydon, 1975; Teese, 1991). This is commonly done by means of census information relating to occupations, incomes, housing and other characteristics of the population of each geographical area in which schools are located, or the populations which schools serve are located.

In the present study the Census Collection District was used as the geographical area. Since the home address for each student was available, it was possible to identify the collection district in which he or she lived using a digitised electronic street directory on desktop mapping software. A socioeconomic index score, calculated at the level of the collection district, was then assigned to the record for each student. The socioeconomic index scores were those derived by the Australian Bureau of Statistics (1994) and known as the Socioeconomic Indexes for Areas (SEIFA). The ABS provides five indexes. The one used in the present investigation was the index of education and occupation.

To measure social area differences in mathematics participation, the collection districts were divided into ten equal groups based on their order on the scale of socioeconomic status. The Year 10 students were then divided into the ten groups according to the social status of the census collection district in which they lived. It was then possible to examine the social distribution of mathematics participation.

To measure the relationships with school the multilevel modelling technique, *hierarchical linear modelling* (HLM), was used. The technique is a relatively recent development within regression analysis and permits researchers to model student-level outcomes within schools and then to identify and model any between-school differences that exist (Bryk and Raudenbush, 1992). The technique is now widely accepted among educational researchers as the most appropriate for looking at school effects. Because categorical (dummy) predictor variables were used (e.g. studying top-level mathematics or not), nonlinear HLM analysis was performed.

Patterns of mathematics participation

Figures 1, 2 and 3 present the mathematics enrolment rates of Year 10 students in Tasmanian secondary schools. The vertical axis in each figure records the percentages of males and females taking part in the level of mathematics study. The horizontal axis records the student population divided into ten groups based on the SES decile grouping of collection districts. At the low end of the scale (1) are the students who live in the collection districts of low social status. At the upper end of the scale (10) are the students living in districts of high social status. Three rather than four levels of mathematics study are presented because of very small numbers in the fourth (bottom) level. These students have been included in the third mathematics (Figure 3).

In 1994 social area differences in maths participation were a feature of junior secondary schooling in Tasmania. Access to the top level of mathematics varied according to the social status of a student's district (Figure 1). Children living in lower socioeconomic areas were far less often taking advanced level mathematics than children living in higher SES areas. For example, only 15% of boys in the lowest SES group were taking advanced-level mathematics in 1994 compared to 45% of boys in the highest SES group. The social gradient in participation is

steep and largely consistent. Apart from some small falls in the middle groups, rates of participation rise sharply moving from the bottom to the top of the SES scale.

Figure 1: Participation in advanced-level mathematics:Year 10 students

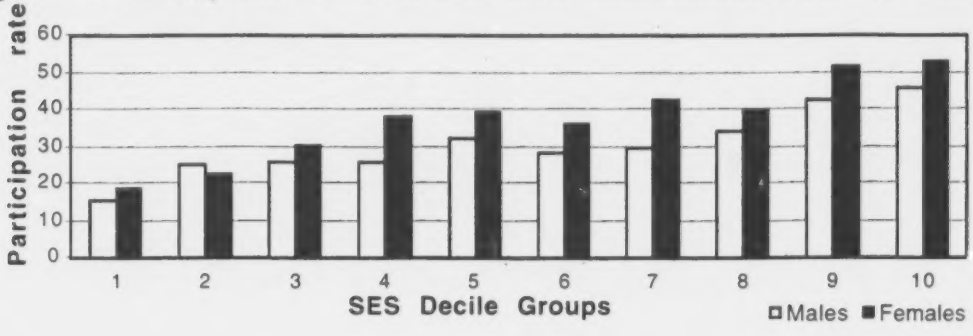


Figure 2: Participation in middle-level mathematics: Year 10 students.

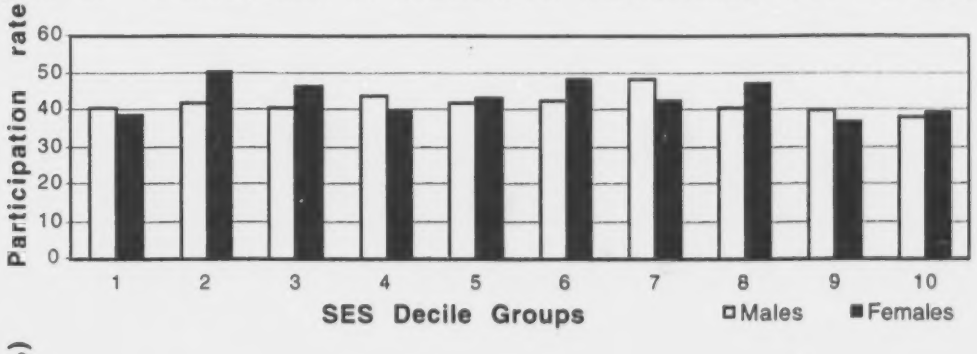
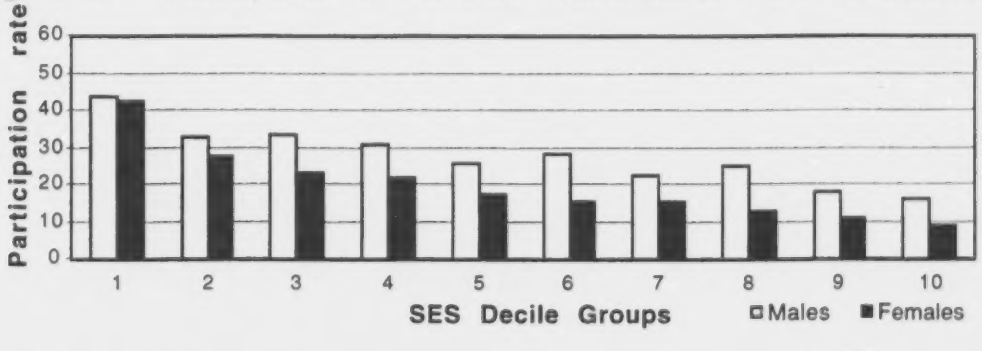


Figure 3: Participation in low-level mathematics: Year 10 students, 199



A point worth noting is that there are gender differences in the rates of participation in advanced maths, but these run against the patterns reported elsewhere (eg Lamb, 1996; Teese, 1991). At almost every point, as we ascend the SES scale, females are more often taking the top-level mathematics. The one exception is the second lowest SES group where the rate for males is slightly higher. The average gap between females and males is seven per cent. It would seem that in Year 10 mathematics girls are outperforming boys, if access to the most difficult level of mathematics is any guide. But the size of the gender gap does not alter the general social trends. For both males and females rates of participation increase as SES rises.

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The reverse social trends are apparent when considering participation in the bottom-level mathematics (Figure 3). Participation in this level of mathematics is dominated by children from low SES backgrounds. Over 40% of boys and girls from the lowest SES group are taking the least difficult mathematics. Alternatively, fewer than 10% of girls and 15% of boys from the highest SES group are enrolled in this level of mathematics. Again there is a sizeable gender gap. On average, across the groups, eight per cent more boys than girls are enrolled in the least difficult mathematics. Gender differences are weakest at the lower end of the SES scale, where the gap is negligible, and largest in the middle groups. However, despite these differences, for both males and females social differences are large, with steep falls in participation as the SES scale rises.

Enrolments in middle level mathematics do not vary much according to SES (Figure 2). However, it is important to view participation in the middle level mathematics in conjunction with enrolment rates in the other levels. Taken together the figures reveal that students from the lowest SES groups are predominantly studying either the middle-level or bottom-level mathematics. This is the case for over 80% of boys and girls from the lowest SES group. Alternatively, the vast majority of children from the upper end of the social status scale are studying either middle-level or the advanced-level mathematics (over 80% of males and females from the two highest SES groups).

It is clear from the results that there are social area differences in access to levels of mathematics study. But do these differences vary across schools? To address this question nonlinear HLM analysis is used to predict the likelihood of participation in mathematics across schools controlling for the effects of key background attributes of students. Gender and SES were the main student-level factors included in the analysis while school type (Catholic, independent, government), mean SES of the school, and enrolment size were the main school level factors. The results from the analyses are presented in Table 1. Estimates for middle-level mathematics are excluded because there were no significant school differences. Two models are reported for top-level mathematics and bottom-level mathematics. The first model for each analysis includes only the level one variables. The second model includes student-level and school-level variables.

The level one models show that SES and gender exert strong independent effects on the likelihood of studying top-level mathematics and bottom-level mathematics. Other things equal, students from higher SES backgrounds have a significantly higher likelihood of doing the most difficult level of mathematics compared to their lower SES peers, and significantly lower likelihood of being relegated to the least difficult level of mathematics study. Being male significantly increases the likelihood of studying the bottom-level mathematics and not accessing the top-level of mathematics study, all else equal.

After controlling for the level one factors there are strong school differences. Analyses using unconditional models predicting level of mathematics study show that over 30% of variance is due to differences across schools. Some of the factors contributing to this variation are included in the full models presented in Table 1.

Previous research has pointed to the importance of the social composition of schools as an influence on achievement and other outcomes (e.g. Coleman et al., 1966; Lee & Bryk, 1989; Rumberger, 1995). In the present study, the mean SES of schools was not related to the study of advanced-level mathematics. However, the social intake of schools did have an influence on rates of participation in the least difficult mathematics. Other things equal, schools with a higher mean SES intake significantly reduced the likelihood of students studying the bottom-level mathematics.

Table 1: Non-linear HLM estimates of level of mathematics study

	Advanced-level maths		*Low-level maths*	
	Level 1 model	Full model	Level 1 model	Full model
School-level Variables				
Base rate	-0.572***	-0.673***	-1.399***	-1.419***
Student composition				
Mean SES		-0.024		-0.097**
School type				
Catholic		0.667***		-0.759***
Independent		1.236***		-1.209***
School size				
Enrolments		0.005***		-0.003*
SES differential				
Base rate	0.108***	0.070***	-0.125***	-0.097***
Catholic		-0.046*		0.027
Independent		0.064*		0.044
Enrolments		0.001**		-0.000
Gender differential				
Base rate	-0.286***	-0.251**	0.432***	0.492***

N (students) = 5825; N (schools) = 72
*Significant at the .10 level; **Significant at the .05 level; ***Significant at the .01 level
NOTE: Only variables which are significant predictors have been reported.

School type has an important independent effect on level of mathematics study. Other things equal, attending a Catholic school significantly increased the likelihood of studying advanced-level mathematics and reduced the chances of relegation to the bottom-level of mathematics study. The effects for non-Catholic private schools were even stronger. The results suggest that attending a private school improves students' chances of accessing the advanced-level of mathematics. These results support other research showing strong independent effects of private schools on educational activity (Coleman, Hoffer and Kilgore, 1982; Williams and Carpenter, 1990; Lamb, 1989, 1994a).

School type also influences the relationships between SES and level of mathematics study. Research from the high school and beyond study in the United States has shown consistently that the social distribution of academic achievement is more equal in Catholic schools than in other schools (Lee and Bryk, 1989; Bryk, Lee and Holland, 1993). The findings in this study also show an effect of Catholic schools on the social distribution of level of mathematics study (at least in terms of accessing the top-level mathematics). The size of the effect for rates of participation in the advanced-level mathematics, presented in the figures on the SES differential, is small (significant at the .10 level rather than .05) but suggests that the social distribution of participation is weaker in Catholic schools than in other schools. This does not hold for participation in the bottom-level mathematics, however. Furthermore, the figures for non-Catholic private schools suggest that, in contrast to the Catholic sector, there is an increase in social differences in top-level mathematics participation. In other words, non-Catholic private schools increase rather than reduce social differences. This trend did not hold for patterns of participation in the least difficult level of mathematics.

Another school factor related to level of mathematics study is enrolment size. Other things equal, the larger the school the greater the likelihood of students studying advanced-level mathematics. This may be because in smaller schools separate classes of the advanced level mathematics are not possible due to small student numbers. Enrolment size also has an interactive effect with SES at level two. The figures suggest that the social distributions of participation in advanced-level mathematics are less equal in larger schools than in smaller

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secondary schools. This may reflect differences in the provision of classes related to student numbers. It suggests that in smaller schools where classes in advanced mathematics are made available they tend to involve a broader social mix of students.

Discussion: explaining social area and school differences

The origins of social area and school inequalities through the mathematics curriculum are partly based on the spatial groupings of the population along social class and cultural lines. Populations across the different suburbs and regions of Tasmania form concentrations differentiated by levels of income, types of occupations, educational qualifications, quality of housing, life styles and wealth. This process of residential segregation, producing unequal distributions of cultural capital and material wealth, means that the families which are the most likely to enjoy mathematics success (measured by level of mathematics study) are concentrated in particular residential areas. These families in high SES areas are able to promote success by exploiting the resources which they have at their disposal. Their knowledge of the school system, more richly developed language skills, high educational and occupational aspirations, aesthetic dispositions, literary and artistic experiences and tastes, and middle class life styles provide substantial advantages for their children in meeting the particular intellectual and behavioural demands associated with the selection, organisation, communication and evaluation of school knowledge (Lamb, 1990).

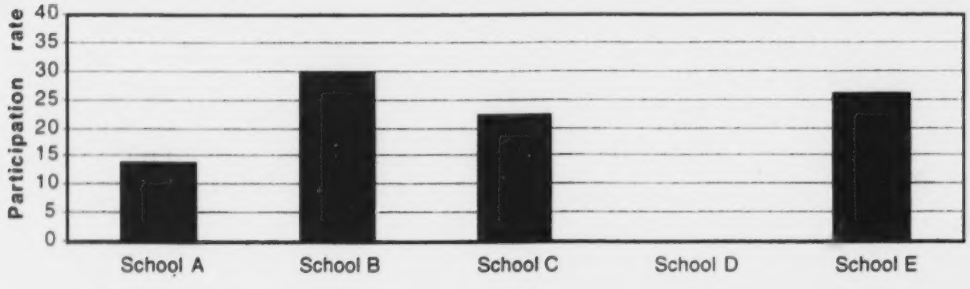
There are also indirect effects of residential segregation. The concentration of families with high levels of cultural capital, parental education, and income across particular geographical areas contributes to a community pooling of resources and avoidance of socially and culturally mixed environments. The aggregate levels of neighbourhood resources, and the establishment of family and friendship networks in this context, may well account for the neighbourhood or local community effects on mathematics achievement reported elsewhere (Entwistle, Alexander and Steffel-Olsen, 1994). Access to level of mathematics study, in other words, is sensitive to neighbourhood as well as family resources. The separation of populations residentially also means that the resources represented by parental education, cultural capital and employment are concentrated in school catchments. The effects of a pooling of family and neighbourhood resources in particular schools helps contribute to achievement and student progress. In the present study this may help explain why the chances of children being relegated to the lowest-level mathematics were significantly lower in middle class schools than in schools with a predominantly low SES intake.

Overlaying the mechanism of residential segregation is the uneven distribution of selective schools. The main forms of selective secondary schooling in Tasmania are Catholic and private non-catholic schools (in other states it would include selective-entry public high schools and declared language or music schools). There is an unven distribution of private schools across the different suburbs and regions of Tasmania. The uneven pattern of distribution builds on initial inequalities and intensifies effects related to residential segregation. The pooling of children from professional families (thanks to residential segregation, the imposition of fees, academic scholarships) provides private schools with a platform on which to erect elaborate frameworks of pupil management, (e.g. formal systems of pastoral care, more extensive extra-curricular activities in music, drama and art, and more clearly articulated policies on discipline and order), as well as pursue intensive curriculum orientation based on higher education course planning and achieved through a more narrowly focused academic curriculum, formal grouping practices, selective promotions and careful guidance in subject selection (Lamb, 1994b). These are some of the ways in which private schools are able to successfully exploit the advantages of their narrow social intake. The more rarefied teaching environments established in private schools help promote higher achievement in mathematics and, therefore, higher levels of promotion of students to advanced-level mathematics. Many government schools attempt to emulate the practices of private schools. But this must often be ineffectual because these schools cannot provide the same operating conditions thanks to their role in serving much broader populations.

A further layer of influence on social area and school differences is related to local school organisation and school policy (how individual schools manage students and deal with pupil

diversity). Figure 4 shows the rates of participation in advanced-level mathematics in five government high schools matched in terms of the social distribution of their student intake (predominantly low SES) and their enrolments (over 60 students in Year 10). It reveals that rates vary substantially across schools serving similar populations. School D had no students in 1994 doing advanced-level maths, whereas the other schools had up to almost a third doing so. Other work suggests that such differences are related to the academic focus of the school, promotion's policies, breadth of curriculum, organisation and attributes of teachers, and policies on entry requirements to levels of study (Lamb, 1996; Bryk, Lee & Holland, 1993; Teese, 1989). Clearly, there are school-level factors influencing access to level of mathematics study.

Figure 4: Participation in advanced-level mathematics: five schools, 19



In conclusion it is important to note that the interactions between the layers of influence on access to level of mathematics take place within the context of a hierarchically organised, centrally controlled curriculum. As long as the curriculum at the Year 9 and Year 10 level continues to be shaped around the needs of selection for senior schooling and university entry then it is likely that mathematics will continue to operate as a major source of social selection.

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An Evaluation of Two Coding Systems in Determining van Hiele Levels

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In the early 80s Mayberry (1981) developed a diagnostic instrument to be used in an interview situation to assess the van Hiele levels of pre-service primary teachers. At the University of New England, a detailed testing and interview program was undertaken, replicating the Mayberry study. The students' responses to the Mayberry items were assessed using two different methods, first by Mayberry's method, and second, using the method developed by Gutiérrez, Jaime and Fortuny. This paper presents an evaluatiuon of the two coding systems.

The ability to instruct students at their level of understanding is dependent, in part, on the teacher being able to assess the students' levels of understanding. In order to make this assessment, there needs to be available a reliable diagnostic instrument. In the early 80s Mayberry (1981) in her work with pre-service primary teachers, developed such a diagnostic instrument that could be used in an interview situation. Mayberry's test items and method of evaluation are based on the key assumption that the van Hiele levels are discontinuous (Mayberry 1981, p.22). This led Mayberry to design each item to test for understanding of a specific van Hiele level, the response being assessed on whether it reflects that level of thinking. There is no grading of the degree of difficulty of the Mayberry items within a level, nor of the depth of understanding of the level displayed in a response.

An alternative paradigm for the evaluation of the acquisition of van Hiele levels by students has been presented by Gutiérrez, Jaime and Fortuny (1991). In contrast to Mayberry, they have based their research on the idea that the van Hiele levels are not discrete, rather that they are of a more dynamic nature, that they are continuous rather than static (Pegg, 1992, p.25). Their theory (Gutiérrez, Jaime and Fortuny 1991, p.237) is based on observations that, when answering questions, although most students show a dominant level of thinking, a response frequently displays some reflection typical of another level. This paper presents an evaluation of the two coding systems in determining van Hiele levels displayed by the students in their responses to the Mayberry items.

Background

The van Hiele Theory

In the 1950s, Pierre van Hiele and Dina van Hiele-Geldof completed companion PhDs which had evolved from the difficulties they had experienced as teachers of Geometry in secondary schools. Whereas Dina van Hiele-Geldof explored the teaching phases necessary in order to assist students to move from one level of understanding to the next, Pierre van Hiele's work developed the theory involving five levels of insight. A brief description of the first four van Hiele levels, the ones commonly displayed by secondary students and most relevant to this study, is given:

Level 1 Perception is visual only. A figure is seen as a total entity and as a specific shape. Properties play no explicit part in the recognition of the shape.
Level 2 The figure is now identified by its geometric properties rather than by its overall shape. However, the properties are seen in isolation.
Level 3 The significance of the properties is seen. Properties are ordered logically and relationships between the properties are recognised.
Level 4 Logical reasoning is developed. Geometric proofs are constructed with meaning. Necessary and sufficient conditions are used with understanding.

The van Hieles saw their levels as forming a hierarchy of growth. A student can only achieve understanding at a level if he/she has mastered the previous level(s) They also saw (i) the levels as discontinuous, i.e., students do not move through the levels smoothly, (ii) the need for a student to reach a 'crisis of thinking' before proceeding to a

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new level, and (iii) students at different levels speaking a 'different language' and having a different mental organisation.

Mayberry's Research

JoAnne Mayberry's study (1981) investigated, in part, whether the van Hiele level, at which a student is functioning in geometry, can be discerned. To carry out this investigation, Mayberry created a diagnostic instrument consisting of 62 items (many of them containing separate question parts) designed to the operational definition of each of the five levels. The items covered seven geometric concepts, namely, square, right triangle, isosceles triangle, circle, parallel line, congruency, and similarity. These concepts all occur in the elementary curriculum in the USA. A matrix/grid was used to develop questions by level and concept so that the questions would have parallel forms. One or more questions were developed for each cell in the grid. Experts in the fields of mathematics and mathematics education, among them Pierre van Hiele, were asked to validate the items by judging whether the items satisfied certain criteria (Mayberry, 1981, p. 52). The final form of the diagnostic instrument was then used in an interview situation to investigate the understandings of 19 pre-service elementary education students at Georgia College, Milledgeville, Georgia. In her method of assessment, a credit point is given "if the subject's responses to questions . . . indicated that the subject was thinking on the given level" (p.60). However, in her study, Mayberry assessed the response to every question part equally, whether the question required a simple yes/no answer, or whether it required a complex explanation. A criterion was set for each concept and level (ranging from 50% to 100%). If a student has given sufficient correct answers to reach Mayberry's criterion, the student is credited with having mastered that van Hiele level.

The research of Gutiérrez, Jaime and Fortuny

The method of evaluation developed by Gutiérrez, Jaime and Fortuny, based on the premise that the levels are continuous, results in a qualitative assessment of a student's degree of reasoning in each of the four levels. Gutiérrez *et al* (1991, pp.238-239) maintain that initially students are not aware of the need to think at a level. They have no acquisition of that level. As they become aware of the new level, an attempt to work at the level is made and a low degree of acquisition is acquired. Continual growth in awareness is shown in an increasing degree of thinking by the students at this level, through an intermediate degree of acquisition, a high degree, until they have a complete acquisition of the thinking at that level.

Several steps are necessary in evaluating a student's van Hiele level(s) using the method of Gutiérrez *et al.* First, in considering a response, the highest level of reflection displayed in the response needs to be determined in order to give the student full credit for the understanding displayed. In making this decision, it can be necessary to consider the response in conjunction with the student's other answers (p.239). For example, a response which appears to be close to the necessary and sufficient conditions sought in Mayberry Item 24 can be an attempt at expressing minimum conditions (Level 4), or it can be a statement of the few properties known by the student for that topic (Level 2). Consideration of the student's other responses is necessary to determine which is the correct level.

Item 24
Circle the smallest combination of the following which guarantees a figure to be a square.

a. It is a parallelogram.
b. It is a rectangle.
c. It has right angles.
d. Opposite sides are parallel.
e. Adjacent sides are equal in length.
f. Opposite sides are equal in length.

It is not always necessary to consider other statements or responses. Many students in their responses consistently show a dominant level of thinking. With such students, their statements constantly re-confirm their most common level of reasoning. However, the assessor needs to be aware of students who are beginning to explore a

higher level for some aspect of a concept and may give a better than expected response if the question is focussing on that aspect or characteristic. Conversely, students who have been attempting to work at a new level of reasoning sometimes, in a response, revert to a lower level which is more familiar to them.

Having decided on a level, the response is now assigned one of eight types of answer. This categorisation depends on the degree of mathematical accuracy, and on how complete the solution to the question is. Gutiérrez *et al* (pp.239-240) explained "To determine which type an answer belongs to, it is necessary to consider it from the point of view of the van Hiele level it reflects, since an answer can be adequate according to the criteria of a given thinking level but not valid according to the criteria of a higher level."

These two steps result in an answer being assigned a vector (l, t), which shows the highest level (l) the answer reflects, together with the type (t) of answer according to its completeness and correctness. The responses are then quantified according to each vector, and the student's degree of acquisition of each van Hiele level determined by "calculating the arithmetic mean of the values of the student's answers to those items that could have been answered at that level" (Gutiérrez, Jaime, Shaughnessy and Burger 1991, p.109). If a response has been given at Level n when the question could have been answered at Level $(n+1)$, a zero score is given for Level $(n+1)$. *However*, if a response has been given at Level n and the question could also have been answered at Level $(n-1)$, a score of 100 is allocated to Level $(n-1)$, since a response at Level n implies complete acquistion of Level $(n-1)$ (p.246). Finally, the student is assigned a qualitative degree of acquisition when the arithmetic mean is converted to a subjective interpretation of No acquisition (0-15%), Low (15-40%), Intermediate (40-60%), High (60-85%) or Complete (85-100%) acquisition.

In formulating their alternative paradigm for the evaluation of the acquisition of van Hiele levels by students, Gutiérrez, Jaime and Fortuny (1991, p.239) started with some assumptions. These are:

- that it is more important to observe the students' type of reasoning than their ability to solve certain problems correctly in a set time,
- that a partially correct (or even a totally incorrect) answer may also afford information, and
- that an incorrect answer, when considered in conjunction with other answers, may give more than a negligible amount of information.

Design

In order to consider Mayberry's work in an Australian context, a detailed study of the geometric understanding of 61 first-year primary-teacher trainees was carried out at the University of New England. The study aimed, in part, to provide a written test based on the Mayberry interview schedule. Conversion of the Mayberry items to a written test involved some modification of the wording to ensure that the intention of each question was clear. A preliminary study validated the reliability of the written questions. Level 5 items were omitted, hence the written test assessed van Hiele Levels 1 to 4 (Mayberry items 1 to 57). Follow-up interviews were conducted with students to validate the levels of thinking as determined in the written test.

The responses were assessed initially using Mayberry's method. However, there were indications that not all questions measured the level for which they were designed (Lawrie 1993). This led to an amended version of the Mayberry test and marking scheme (Lawrie, in press). The students' responses were then re-assessed using the amended version of the marking scheme. Every endeavour was taken to replicate Mayberry's evaluation of responses. Her thesis was examined in depth to ascertain her expectations in the responses to the items. Following the amended Mayberry assessment, the students' responses were again re-assessed, this time using the method developed by Gutiérrez *et al*. For this assessment, all response attempts by students, whether wholly or partially correct or incorrect, contributed to the overall

picture of their depth of understanding of a topic. Also, to allow for the unequal expectations for some of the Mayberry questions, e.g. the difference between the expectations of a yes/no type of question compared to a question requiring a complex explanation, each item or group of question parts in the Mayberry test were assessed as one complete item. The two coding systems were than evaluated.

Results

Table 1 shows the number of students demonstrating reasoning for each van Hiele level and for each of the seven concepts for both methods of assessment.

Table 1
Number of Students Attaining Each van Hiele Level

Concept	van Hiele Level	Amended Mayberry Results	Gutiérrez et. al. Results Degree of Acquisition				
			None	Low	Intermed	High	Complete
Square	0	0					
n = 61	1	2	0	0	0	0	61
	2	51	0	2	7	24	28
	3	4	37	9	13	2	0
	4	4	59	2	0	0	0
Right Δ	0	1					
n = 31	1	6	0	1	0	1	29
	2	17	0	2	2	13	14
	3	6	14	10	2	4	1
	4	1	30	1	0	0	0
Isos Δ	0	2					
n = 30	1	8	0	0	2	0	28
	2	12	0	2	1	8	19
	3	5	15	6	5	4	0
	4	3	29	1	0	0	0
Circle	0	0					
n = 31	1	5	0	0	0	0	31
	2	13	0	4	0	11	16
	3	12	17	8	3	2	1
	4	1	29	2	0	0	0
Parl lines	0	0					
n = 30	1	7	0	0	1	1	28
	2	22	0	2	3	7	18
	3	1	23	6	1	0	0
	4	0	30	0	0	0	0
Congruenc	0	0					
n = 31	1	11	0	0	0	7	24
	2	14	2	2	5	10	12
	3	2	16	7	5	3	0
	4	4	27	4	0	0	0
Similarity	0	0					
n = 30	1	13	0	0	1	0	29
	2	12	0	5	4	3	18
	3	3	18	7	5	0	0
	4*	2	26	3	0	0	0

*Similarity Level 4, n = 29

The amended Mayberry results give the highest level achieved by a student for each concept, while the Gutiérrez *et al* results show the degree of acquisition reached by the students for each level. Both sets of results agree with the hierarchical structure of the van Hiele levels. The results from the Gutiérrez *et al* method of assessment indicate that the higher the level, the lower the degree of acquisition, and, in general, confirm the findings of the Mayberry assessment, that the majority of students have mastery of Levels 1 and 2, but little or no understanding of the higher levels.

A closer inspection of the two sets of results shows that a high or complete degree of acquisition in the Gutiérrez *et al* results corresponds with mastery of a level in

the Mayberry evaluation, while a low or no acquisition corresponds with failure to reach Mayberry's criterion. Results showing an intermediate degree of acquisition (40 to 60) tend to correspond with scores close to the Mayberry criteria, some failing to reach the criterion, others achieving mastery of the level. For example, for the concept square, eight students (4@L3 and 4@L4) have been assessed as demonstrating mastery of Level 3 or better in the Mayberry results, while the Gutiérrez *et al* results for Level 3 show only two students having a high or better degree of acquisition. The balance of six students showing Mayberry mastery are among the thirteen students registering an intermediate degree of acquisition. Again, for the triangle results, the Mayberry scoring shows one student failing to reach the criterion in the identification of the right triangle while being assessed as having a low degree of acquisition of Level 1 in the Gutiérrez *et al* assessment. Similarly, the two students failing to reach the criterion in identifying the isosceles triangle, are both shown as having an intermediate degree of acquisition for Level 1 for the later assessment. These results illustrate one difference between the two methods of assessment. With the Gutiérrez *et al* method of assessment, the taking into account all responses, whether complete and/or correct, makes the evaluation more realistic and gives a more accurate measure of a student's degree of understanding of geometry.

However, a comparison of the results for the two methods does not always show correspondingly similar results. In the assessments of the responses to the circle questions, the amended Mayberry evaluation credited thirteen students (12@L3 and 1@L4) with mastery of Level 3 or better. In contrast, the Gutiérrez *et al* method found only six students with a comparable degree of mastery, one having a complete degree of acquisition, two having a high degree and three having an intermediate degree of acquisition of the level. The remaining twenty-five were not credited with having better than a low degree of acquisition of the concept circle for Level 3. Inspection of the Level 3 items for the circle showed there were several question parts requiring only a yes/no type of answer. Answers to such questions can earn a score in the Mayberry assessment method without requiring demonstration of Level 3 reasoning. This supports the notion that the results of coding using the method of Gutiérrez et. al. are the more realistic.

In conclusion, the alternative paradigm described by Gutiérrez *et al*, because it measures a student's capacity to use each one of the van Hiele levels in every statement made, results in a more flexible interpretation of the reasoning of the student. In particular:

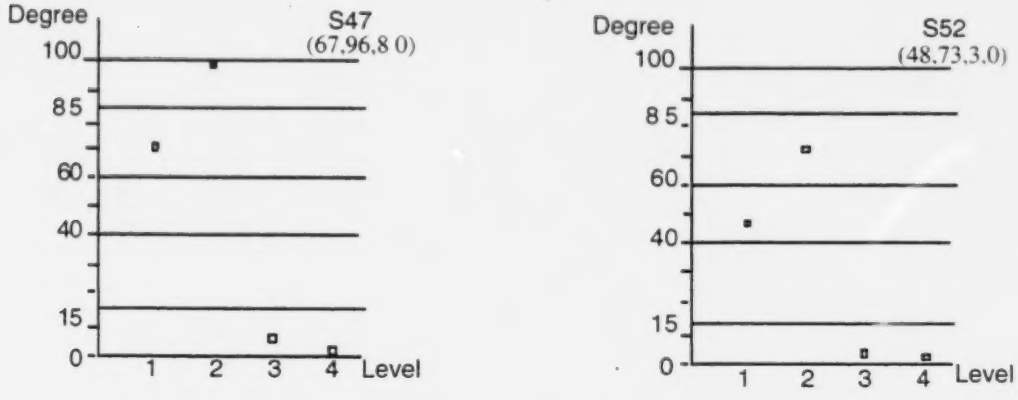
1. a student can be shown to be developing in two consecutive levels of reasoning at the same time;
2. the incorrect assignation of a level to a question is of minimal significance;
3. the effect of unequal distribution of questions across levels in minimised;
4. incorrect assessment resulting from 'lucky' guesses such as in true/false questions, from weak, and from misinterpreted questions is minimised; and
5. inequalities associated with success criteria are eliminated.

Unusual Behaviour Patterns

An inspection of the quantitative results obtained in the re-evaluation of the students' responses using the method of Gutiérrez *et al* reveals that not all results agreed with the hierarchical structure of the van Hiele levels. In twenty-six (11%) of the two hundred and forty-four assessments, the degree of acquisition of Level n is not less than the degree of acquisition of Level $(n-1)$. In every case, the non-hierarchical behaviour occurred between Levels 1 and 2, the degree of acquisition of Level 1 measuring below that indicated in responses at other levels. However, many of the pattern errors in these quantitative results are considered to be trivial, the value of Level 1 being less than 10 points below the value of Level 2. For example, one student (S05) scored 75 points (high) for Level 1, 83 points (high) for Level 2, 29 points (low) for Level 3 and 0 points (no degree of acquisition) for Level 4. The results of two students (S47 and S52) illustrate cases in which neither the qualitative nor the quantitative assessment fit the hierarchical pattern. S47 was assessed as showing a high degree of acquisition of Level 1, complete acquisition of Level 2 and no acquisition of Levels 3

and 4, while S52 was assessed as an intermediate degree of acquisition of Level 1, a high acquisition of Level 2 and no acquisition of Levels 3 and 4. The two patterns of unusual behaviour are graphed below.

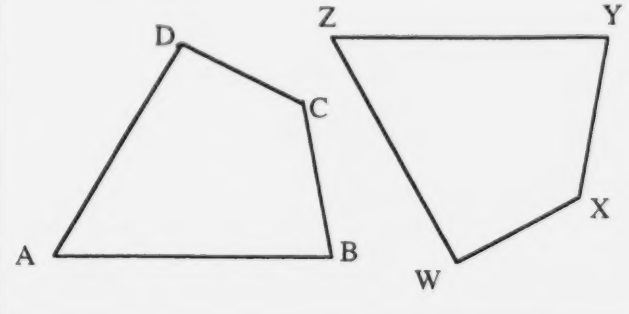
Graphs of Unusual Behaviour Patterns



The occurrence of these patterns in which the degree of acquisition of Level 1 is lower than it should be is suggestive of two factors; (a) that the Mayberry questions designed to measure Level 1 are not always clear in their intention, and/or (b) that the criteria adopted to measure the acquisition of levels using the alternative paradigm of Gutiérrez *et al* may be more suited to responses demonstrating reasoning as with Levels 2, 3 and 4 rather than to the visual identification of Level 1. A closer inspection of the responses of the twenty-six students whose results showed the degree of acquisition of Level *n* being greater than the degree of acquisition of Level *(n-1)* confirmed the under-evaluation of the students' Level 1 understanding. This supports the notion above, that the Mayberry test questions and the Gutiérrez *et al* evaluation method do not necessarily measure Level 1 adequately.

Although Gutiérrez *et al* (1991, p.248) found four of their fifty students recording a Level 2 degree of acquisition lower than for Level 3, this did not occur in the evaluations for this study. However, several students who showed comprehensive knowledge of properties in their responses to the Level 3 questions appeared not to understand the direction of the Level 2 questions, and hence, did not register fully their Level 2 understanding. For example, one student (S06), although failing to reach Mayberry's Level 2 criterion, demonstrated sufficient knowledge of the relationships between congruent figures to reach the Level 3 criterion. The Gutiérrez *et al* evaluation credited her with a high degree of acquisition of Level 2 and an intermediate degree of acquisition of Level 3. Her responses (below) to two of the congruence questions illustrate this feature.

Item 22 *(Level 2)*



These are congruent figures.

What is true about their sides?
AD = _____

What is true about their angles?
∠B = _____

S06 correctly named the corresponding side WZ and the corresponding angle, ∠Y. Her responses to the other question parts are correct but not acceptable for the Mayberry assessment:

What is true about their sides? *S06 they are unequal*

What is true about their angles? *S06 they add to 360°*

Item 43 *(Level 3)*

Δ ABC is similar to Δ DEF (in that order).
Are the following a) certain b) possible, or c) impossible?
Give reasons for your answers.

b) ∠A = ∠E *S06 (b) yes if equilateral*

d) AB = EF *S06 (b) only if equilateral*

As shown for Item 43, several of S06's responses to the Level 3 questions demonstrated an understanding of congruency relationships. Previous research has also found students who answer higher level items better than lower level ones (Mayberry 1981; Usiskin 1982).

Conclusion

Whereas the Mayberry coding system provides a more direct and quicker method of determing a student's van Hiele level of undertanding geometry, the method developed by Gutiérrez *et al*, in evaluating the degree to which understanding of each level is expressed in every response, measures more comprehensively, the van Hiele level of reasoning of a student. The latter provides a mechanism for measuring a student's degree of acquisition in two or more levels. It also provides insight into the quality of a question. It minimises many of the problems occurring in the Mayberry items and the associated method of assessment. However, the alternative paradigm, as developed by Gutiérrez *et al*, also needs further investigation and refinement.

1. Questions at van Hiele Level 1 are not always assessed accurately. Is there a limitation in the ability of the coding system of Gutiérrez *et al.* to evaluate visual recognition?

2. The automatic allocation of a credit of 100 for Level *(n-1)* for an attempt at Level *n* does not always seem justified and possibly presents a contradiction to other aspects of their coding system.

3. The coding system is very time-consuming. Whereas it gives a much more detailed and accurate picture of a student's ability to work in each van Hiele level, making it an excellent research tool, it is not suitable for use in the regular classroom in its present format.

Gutiérrez *et al* note that some students show a better acquisition of Level 3 than of Level 2. As with previous research (Mayberry, 1981; Usiskin, 1982), this study has found cases in which students demonstrating a high degree of acquisition of Level 3, do not show their full understanding of Level 2 in their responses to the lower level questions. This needs further investigation.

This research shows that there are many advantages in the coding system developed by Gutiérrez, Jaime and Fortuny, particularly when compared to Mayberry's coding system. However, it also demonstrates that further refinement is needed.

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Looking Back Towards the Future: A Case Study in Mathematics

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This paper focuses on the affective and cognitive behaviours of a small group of grade 7 students who worked for eight lessons on an extended mathematical task in the classroom. Data sources included videotapes, field notes, students' reactions, self-report affective measures, and interviews. By drawing on these data sets, inferences were made about cognitive and affective behaviours influencing learning and the students' longer term involvement with mathematics five years later, in grade 12.

Introduction

The main aims of the present study were to describe students' affective and cognitive behaviours as they worked, over an extended period of time, in a small group learning setting on a substantial and realistic mathematics task and thus to infer their likely long term commitment to mathematics. Our observations of students' behaviours and interactions served as important data sources, as did the students' responses on different affective measures and when interviewed. Possible gender differences were also of interest. Specifically, we aimed to answer the following question: are students' cognitive and affective behaviours in class predictive of their longer term involvement in mathematics?

Previous Research

Links between affect and cognition in the learning of mathematics have been examined in several studies. Various authors in McLeod and Adams (1989) described affective reactions experienced by students during mathematics problem-solving. Data gathering techniques included interviewing students about their problem-solving strategies, eliciting affective information as part of that process, and focusing on affect during interviews. Boekaerts, Seegers, and Vermeer (1995) used faces with very sad to very happy expressions to trace students' feelings as they worked on mathematical problems. Finding ways to infer attitudes from behaviours has been a challenge to researchers (McLeod, 1989).

In extensive reviews of gender issues in mathematics learning (Leder, 1992; Leder, Forgasz, & Solar, 1996) it was reported that, internationally, participation rates in elective and specialised mathematics courses persistently reveal gender differences favouring males. Mixed results were found for student achievement. Explanatory models to account for these observed gender differences share a number of common features:

> the emphasis on the social environment, the influence of other significant people in that environment, students' reactions to the cultural and more immediate context in which learning takes place, the cultural and personal values placed on that learning and the inclusion of learner-related affective, as well as cognitive, variables (Leder, 1992, p.609)

Included among the affective variables are: self-concept of ability, confidence, attributional style, expectations for success, sex-role congruency, perceived difficulty of maths, learned helplessness, societal influences and teacher attitudes. Cognitive variables include: general ability, spatial skills, mastery orientation, engagement in high level cognitive tasks, willingness to work independently. Several of these variables were incorporated in the research design of the present study.

Research Methods

Our research problem called for a combination of quantitative and qualitative research techniques. Observational methods, using videotaped records and a descriptive system similar to that used by Clements and Nastasi (1988) and interviews were supplemented with self-report data.

The mathematics task and setting

The group task was set by the teacher as part of the regular mathematics program. It involved a study of the feasibility of building a new tuckshop in the school. With anticipated growth in enrolments and pressure already being felt on existing facilities, an additional cafeteria was seen as an option worth considering. To allow the students to put forward their views was consistent with the philosophy of the school.

The teacher organized the 28 students into six small groups of four or five. Each group was required to investigate the feasibility of building the new tuckshop and to prepare a report supporting the case presented. Students had covered the topics of simple data analysis, presentation and interpretation (tallies, bar charts, pictographs etc.) and percentages prior to the period of monitored lessons. The problem set thus allowed them to use skills learnt and revised earlier and to work on a substantial mathematical task over a number of lessons. The students were also encouraged to draw on skills learnt in other subjects. For example, the survey ultimately used by the observed group was compiled during a 'computer' lesson (the same teacher taught both subjects).

The groups worked on the mathematics project for eight lessons, each of 45 minutes duration. Seven lessons were videotaped. One was missed due to a timetable clash (a second class in the school was also being monitored by the researchers).

The sample

The sample comprised one of the small groups consisting of five students: three females (Carol[1] [C], Cheryl [Ch], and Jenny [J]) and two males (Brian [B] and Mark [M]). The male teacher (DM) was asked to nominate articulate students for inclusion in the targeted group. Each was described by him as being very good or excellent at mathematics. For one lesson each week, a female teacher (BL) assisted DM. Due to absences and attendance at special lessons such as music, the target group did not have its full complement at all lessons. This is not unexpected in the reality of classrooms and the changed composition of the small group provided additional insights into the interpersonal dynamics of group members and the interplay of cognition and affect.

Procedures

Data gathering involved videotaping each lesson, subsequently transcribing the tapes, keeping field notes, monitoring students' reactions to the work done each day and their contributions to it, administering self-report instruments tapping the students' attitudes and beliefs about mathematics and themselves as learners of the subject, and interviewing the students. The data sources and their use are described below.

Data sources and modes of analysis

Lesson transcripts: Focussing on the targeted group, lessons were videotaped and subsequently transcribed. The students' behaviours and conversations were captured.

Field notes: Field notes were kept. Notes were made on general observations, critical incidents out of camera view, and comments by teachers to the observer.

Students' perceptions of the lessons: At the end of each lesson, students completed "Today's Maths Lesson" sheets (based on Clarke, 1992). There were seven items (e.g., "Circle the face which shows how you felt about your *understanding* of today's maths lesson" and "Explain briefly why you felt this way").

Interviews: The five targeted students were interviewed individually some time after the monitored period. They were asked questions which encouraged them to reflect on the "Tuckshop" project, on the year's mathematics activities, and on their futures.

Self-report measures: Several Likert and Semantic Differential scales, as well as open-ended items, were administered to each student. The data are summarised in Table 1. The first six instruments were completed before the observational period; the remainder towards the end of the school year.

Results and Discussion

The researchers' perspectives

General comments: The group worked well with minimal prompting from the teacher. DM spent less time than average with the monitored group. The highest number of

visits for any one lesson, eight, was recorded during Lesson 7 and totalled about eight minutes.

Table 1. Summary of self-report instruments

Instrument	Type and number of items	Items
1. *About you*	Semantic differential, 25	What kind of person are you? e.g., messy/neat; boring/interesting
2. *About you and maths I*	Open-ended items, 2	Do you like maths? Explain
3. *About you and maths II*	lines, 5: ⊢——+——⊣ Excellent Average Weak	Mark with a cross: • How good are you at maths? • How good do you think your teacher thinks you are?
4. *About maths lessons I*	Open-ended (drawing optional)	Write a description of the kind of maths lesson you usually have in grade 7
5. *Maths lesson activities*	28 activities: Frequency (1:never, 4:always) Enjoyment(1:dislike, 3:like)	• Teacher explains to whole class about a topic • Working alone from a textbook • Working in groups to discuss math problems
6. *About grading in maths*	Open-ended items, 5	How does your maths teacher find out what you know and have learnt?
7. *More about maths*	Likert scale, 25 items (Nicholls et al., 1990)	I like doing maths problems which make me think hard
8. *How good are you?*	Similar to *About you and maths II*	

The task was meaningful to the students as they had a personal stake in the eventual outcome. This appeared to provide some motivation for a genuine attempt at it. The level of mathematics to which some members of the group were engaged was unlikely to be so enthusiastically embraced by grade 7 students in a more traditional setting. The following episode from Lesson 6 reveals the level of complexity of a calculation performed by Brian and Mark using a calculator, their persistence with it and in interpreting the outcome, and their level of collaboration.

Lesson 6: Time - approx. 9 minutes into the lesson

M: 2630 divided by 2500
B: (inaudible comment)
M: Start again. 2, 6, 3, 0...
B: Yeh
M: Divided by..
B: Yep
M: 2500
B: 1.052
M: (incomprehensible comment)
B: No, it's not. The more students you have, the less it's going to be...
M: ...(unclear) How come we get .2 cents?
B: .052 cents.
M: That'd be about right
B: So that's a dollar and half a cent, um...

Individuals' levels of cognitive involvement: For the duration of the project, Mark, Brian and Jenny were very involved in mathematical tasks and discussions. Mark and Brian spent most of their time actively engaged in mathematical activities both at high and low cognitive levels. They plotted graphs, used calculators, consulted, and co-operated well. Jenny's cognitive involvement was mainly in discussing mathematical issues related to directions for further action, particularly in DM's presence. Her contributions were not always constructive and the cognitive level of many of her comments was questionable. Her remarks or assertions were often made without supporting mathematical reasoning. Throughout the monitored period, Mark and Brian spent most of their time engaged in higher order mathematical activities, Jenny and Cheryl took charge of the presentation of the project, while Carol was least involved in the group activities.

Overview of affective behaviours: Self report data after each lesson indicated that the girls were more confident than the boys that they had understood the work associated with the project. Since the girls' mathematical involvement was generally limited to discussions about what needed to be done, rather than actually undertaking any of the analyses or calculations, they may not have fully appreciated the complexity or levels of challenge and difficulty associated with the tasks completed by the boys. For example, at the beginning of Lesson 6, the following exchange took place:

Lesson 6: Time 0.10 - Brian gets up and leaves the table

J: (to Mark) Have you guys got all the facts and figures, sort of, or are you working on them? And have you got them or...
M: What do you mean by facts and figures?
J: Oh, just the sort of figures you guys were working on last week.
M: (picking up and looking at sheets) No, we've figured out that everybody spends...
J: We need the sort of profits and things like that
Ch: We need the costs and the profits
M: On average, everybody spends $1.17 every week...
J: Mmm

The teacher interacted differently with the students. For example, only some received positive feedback. Consistency in these interaction patterns may have longer term implications for individuals' perceptions of teacher support and of themselves as learners of mathematics.

Students' perspectives

Jenny's and Brian's *feelings about* and *understandings of* three representative mathematics lessons are summarised on Table 2.

Table 2. Jenny's and Brian's reactions (from "Today's maths lesson" sheets)

Student & lesson		*Feelings about lesson*: Point of lesson	Level of understanding
J	3	*Pleased*. Seeing whether we as a group could successfully establish a hypothesis based on our survey result.	Learnt to deduce facts from graphs. Easy to understand
	4	*Frustrating*. Sorting out what was messed up the previous lesson	Learnt nothing, but the work was easy
	8	*Pleased*. Finishing the project: frantic but fun. Learnt not to rush work	Work was easy: doesn't require much thought
B	3	*Pleased*. Graph work, how to cooperate	Understood what he had done
	4	*Neutral*. Graph work, working with figures	Not sure, more figures would have helped
	8	*Pleased*. Organisation and cooperation	Understood now they had finished. "It was a very good project which was difficult"

The different aims of the lessons identified by the students is intriguing. Jenny, for example, focussed more persistently on organizing and writing up the work. Brian included working with figures, with calculators, and graphing as important lesson goals. That all three girls were generally found to be more confident than the boys that they had understood the work was also noteworthy. Differences in the activities in which students engaged (see above) may account for their different perceptions.

Comparing the students' affective and cognitive reactions to the lessons with the affective and cognitive measures derived from the self-report data was informative. The profiles of the five students compiled from the self-report data are summarised in Table 3. The teacher also provided achievement levels for each student.

Table 3. Affective and cognitive profiles of the targeted students

Affective indicators	Cognitive indicators
Carol: Liked maths: important for future, useful, encouraged logic. HGM[1], CGM, TGM: 4.	TR: 4. Liked: talking to class about maths (not as part of a sma group), working in small groups. Pleased when: easy to get right answers, worked hard to solve problems, worked cooperatively.
Cheryl: liked maths: a challenge and important for real world. HGM, CGM, TGM: 5	TR: 5. Liked: talking about maths to friends and in small grou knowing more than others, challenging problems, working ha on them, alternative solutions. Pleased when: correct answers and no mistakes. Enjoyed lessons when: everyone understood worked co-operatively. Success from: understanding not just right answers, interest in learning, co-operation, persistence.
Jenny: liked maths: challenging, fairly simple, interesting, helped concentration, promoted logic, helpful later. HGM, CGM, TGM: 5.	TR: 5. Liked talking about maths in small groups. Pleased wh correct answers and no mistakes. Liked: challenging problems working hard to solve them, alternative solutions, answers tha made sense. Enjoyed lessons when: everyone understood, worked cooperatively. Success from: cooperation and persistence.
Brian: Liked maths: interesting, always different, never boring. HGM: 5, CGM, TGM: 3.	TR: 5. Did not like talking about maths. Liked: small groups tc discuss problems, hard work to solve problems, answers making sense, no mistakes. Success from: interest in learning, understanding more important than right answers.
Mark: Liked maths: challenge "to get it right", success meant "easier to find a good job". HGM, CGM: 5; TGM: 4.	TR: 4. Liked: talking about maths problems in small groups, challenging problems, solutions that made sense, alternative solutions, being only one to answer correctly. Enjoyed maths lessons when: everyone understood, co-operative. Success from: interest in learning, understanding not just right answers persistence, hard worked, setting work out neatly.

[1] HGM: self-rating of mathematics achievement, CGM: believed classmates rating, TGM: believed teacher rating, TR: teacher rating of achievement

Our predictions

Affective measures gathered from self-report data and from observed classroom behaviours did not always match. Greater consistency was evident for the boys than for the girls. Mark, for example, demonstrated his concern for both mathematical accuracy and neat presentation, and persisted at challenging tasks. Although there was some uncertainty about small group experiences evident on his self-report data, Brian indicated that he had enjoyed the group activity, displayed a preparedness to work hard and clearly tried to make sense of the mathematical solutions. As critical elements of the models postulating explanations for gender differences in participation rates in higher level mathematics courses, these observed behaviours and personal characteristics imply a likelihood for longer term involvement with mathematics.

On the other hand, there was little evidence that Cheryl discussed, persisted at or tried to find new ways to solve mathematical problems. While she showed task persistence, it was as reporter and recorder rather than as a user of mathematics.

Jenny's involvement in mathematical activities similarly did not go beyond discussion. She typically repeated strategies already discussed, did not initiate new approaches to solving the task at hand, and did not do any of the calculations required. At one stage Jenny argued for data selectivity. This behaviour seemed to belie her expressed concern for accuracy. The girls expended much effort on the task. They focussed, however, on the peripheral activities associated with reporting and presentation, rather than on mathematical activities. Jenny's leadership role, tendency to adapt others' "good" ideas masked her mathematically off task behaviours. She did, however, frequently engage in mathematical discussion, particularly in the presence of DM. Carol's behaviours were least consistent with her expressed beliefs. She was virtually uninvolved in the work at hand, simply observing or sent, by Jenny, to fulfil menial tasks. Her lack of involvement was exacerbated by the tendency for the others to work in same-sex pairs and marginalise her. DM was also implicated in that he did not encourage the others to include her nor did he pay much attention to her contributions when questioning the group on its progress. The signs did not appear to augur well for Carol's longer term commitment to mathematics.

Gender differences were also evident in both cognitive and affective engagement. Differential task engagement was obvious and might partially explain the gender differences in affective outcomes. The girls' greater confidence in the understanding of the work provides an intriguing anomaly. Since they had not actively engaged in the mathematical aspects of the project, the small group experience may have engendered apparently functional beliefs about themselves as learners of mathematics. In future, when required to tackle similarly challenging mathematics problems, their lack of experience may expose the fragility of their expressed 'confidence'.

What actually happened

Five years after the initial data were collected, the targeted students' mathematical paths through school were explored. Brian had left the school to attend another. The other four students, now in grade 12, were interviewed. The students were also asked to complete an information sheet on which they indicated the mathematics subjects they had taken at school, achievement in these subjects, the extra curricular mathematics activities they had participated in, and some background biographical data including number of siblings, place in the family and parents' occupations. A few months later, when published in the daily press, the tertiary places offered to the students were determined.

Our predictions about longer term involvement in mathematics were largely born out for the four students we were able to trace in grade 12.Their mathematics choices followed gender-stereotypic lines. In grade 12, Mark was the only one taking *Specialist mathematics*, the most demanding option offered in Victoria. His parents, both mathematics teachers, had strongly supported his selection of subjects. His decision to enter the Australian Mathematics Competition (AMC) was rewarded each year (1991-1995) with a Distinction. In grade 7 he said that he hoped he would be a Victorian Certificate of Education [VCE] mathematics student, although he was not sure what career he would follow. At the grade 12 interview, he intimated that he wanted to be an engineer. He was, in fact, offered a place in an engineering degree. In grade 7, Carol was less certain of being a grade 12 mathematics student but said that her mother would like her to do so. She, too, was unsure of her future career direction. Carol also gained a distinction in the AMC in grade 7, but not in other years. She took a less demanding grade 12 mathematics study, *Mathematical Methods*, and gave this as her least favourite subject. Nevertheless, she hoped to get into a combined arts/science course and indicated that history was a particular passion. As early as grade 7, Cheryl's uncertainty about VCE mathematics was evident. She said that she hoped she would continue but she had heard that VCE students were under a lot of pressure and felt that if she did not enjoy the mathematics or experienced a teacher who "doesn't really help much", she might not persist. At that time, she was enthusiastic about a career in chemistry. Like many other students at the school, she also sat for the AMC and consistently received a Credit certificate. Her grade 12 studies also included

Mathematical Methods. She had chosen it because "I think you need maths ... (and) I think they should make it a compulsory subject". After considering many career options she decided she would like to do accountancy and was in fact offered a place in a commerce course. Pointers to Jenny's grade 12 choices were also evident in grade 7. We recognised that she was a motivated, articulate and confident student. Yet, in grade 7 her efforts in mathematics during the monitored period were directed to peripheral, non-mathematical tasks, although she frequently told the boys (and the other girls) what they should do and checked that they had done it. On the self-report data, she had indicated that she found mathematics "fairly simple". Aware that her grade for the activity was linked to the group's output, she may have consciously directed her efforts to the report and its presentation. She gained a Distinction in the AMC in grades 7 and 8, a Credit in grade 10 and had not entered the Competition since then. Jenny's grade 12 interview revealed that she was an *International Baccalaureate* [IB] student. She said she preferred the IB because it was examination-focussed, she liked its structure and emphasis on working independently, and she felt she performed best under these conditions. Her IB course included mathematics, but not the "high level" mathematics subject. Her father, an ambulance para-medic, was her role-model, resulting in her desire to practice medicine. As a result of her grade 12 efforts, she was offered a place in a medical course.

Final Words

Clearly, there are many dangers in attempting to predict, through a snapshot of some eight consecutive lessons spread over two weeks, students' long-term motivations and achievements. Nevertheless, multiple source data including observations of students' cognitive and affective behaviours during mathematics lessons, gathered and analysed when the students were in grade 7, seemed remarkably prophetic of their grade 12 mathematics subject choices and tertiary destinations. While we do not wish to ignore or minimise the many other factors, inside and outside the classroom and school context, which can influence subject and career choice, it appears that intensive observations of students can yield early indictors of eventual pathways. What can we, as mathematics educators do with this knowledge, to ensure that *all* students realise their full potential?

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Notes

1. Fictitious names have been used for the students and the teacher

Young Children's Knowledge of Three-Dimensional Shapes: Four Case Studies

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This paper provides case studies of the attempts of four young children (a high achieving first-grader, a high achieving kindergarten student and two low achieving first-graders) to solve three tasks involving three-dimensional shapes and their two-dimensional representations. Detailed descriptions of students' solutions are provided; these highlight the appropriateness of challenging geometric tasks for young students. High achieving students spontaneously made more frequent use of formal geometric terms when explaining their actions.

Spatial skills and understandings are an essential component of the primary school mathematics curriculum. For example, authors of the NCTM's Curriculum and Evaluation Standards for School Mathematics (1989) claim that "[s]patial understandings are necessary for interpreting, understanding and appreciating our inherently geometric world" (p.48). The emphasis in the primary school has shifted from naming geometric shapes and memorising geometric facts and vocabulary to that of understanding the properties of geometric shapes, developing spatial sense and using geometry in solving problems. Authors of "A National Statement on Mathematics for Australian Schools" state that "[d]uring the early years of schooling the emphasis should be on exploration, both free and structured, of the children's environment and objects within it"(p.81).

While, in recent times, there has been an increased interest in research on children's spatial thinking, research relating to early geometric knowledge seems to be limited. Wheatley and Cobb (1990) presented a group of first- and second- grade students with shape covering tasks and, based on an analysis of the students' actions and anticipations, identified five levels of spatial construction. Diezmann (1993), in a study involving babies, second grade students and fifth grade students, investigated children's strategies in the interpretation of diagrams of three-dimensional (3D) shapes. She concluded that an effort needs to be made to improve children's diagram literacy. Leeson (1995) studied kindergarten students' spatial construction of three dimensional shapes and found that young children were capable of significant geometrical thinking. Thorpe (1995) found that her investigation of the development of spatial concepts in preschool aged children seemed to support Piaget's finding that young children initially explore topological concepts before exploring Euclidean concepts. Mitchelmore and White (1996) found that most Year 2 students are able to perceive abstract angles in physical situations when both arms of the abstract angle are clearly present.

Owens who has been extensively involved with the assessment of children's spatial concepts (e.g. 1992a; 1992b; 1994) has sought to gain a wider range of children's spatial understanding. She (1992a) devised a group 'pencil and paper' test which assessed children's ability to recognise, reproduce and analyse shapes.

This paper reports on an investigation of young children's spatial knowledge as explored in three tasks involving 3D shapes. It forms part of a project which was undertaken in response to a perceived need to gain a deeper understanding of the spatial and measurement knowledge of students in their first or second years of formal schooling. An important aim of the project is to extend to the assessment of students' early spatial and measurement knowledge, an interview-based approach to the assessment of early arithmetical knowledge. Additional aims of the project include the documentation of students' advancements in spatial and measurement knowledge over the course of a school year and a comparison of students' arithmetical, spatial and measurement knowledge. One of the questions addressed by this study was: What spatial knowledge is possessed by students in Kindergarten or Year 1 (i.e. first-grade) in NSW? The project drew on methodologies typically used in the Maths Recovery project (see Wright, in press). These include individual interview tasks designed to

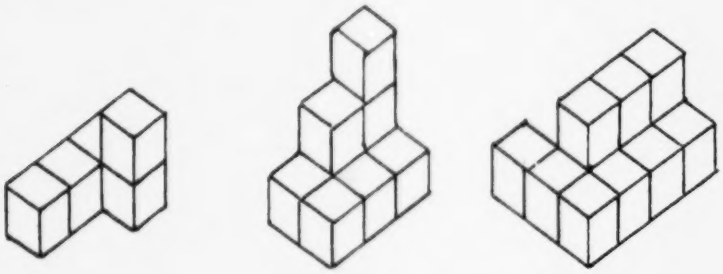
assess a wide range of spatial thinking, and which are problematic for the student. Assessment interviews are routinely videotaped for subsequent analysis.

The project which has been reported by Stewart, Leeson and Wright (1997) involved the development of a space assessment schedule which covered three broad areas of students' spatial knowledge: two dimensional (2D) space, 3D space and position. Within the area of 2D space students' abilities to name, recognise, draw, construct, classify and identify shapes and their understanding of symmetry and tiling were assessed. In the area of 3D space, students were required to sort, classify and name objects and to interpret 2D representations of 3D shapes.

This paper focuses on three spatial tasks (Tasks 15, 16 and 17 of the Space Assessment) in which students' conceptual connections between 2D shapes and 3D shapes are made. For ease of later reference, these will be designated as Tasks X, Y and Z.

In Task X the student was shown a picture of a structure built from five cubes (Figure 1) and asked to build it from a supply of cubes. The student was then shown, in turn, structures made from nine and 12 cubes (Figure 1) and was asked to use cubes to build each.

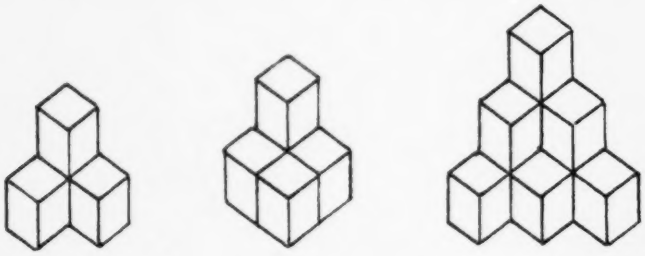
Figure 1. Using cubes to construct shapes.



Task Y assessed whether students could match 3D shapes with their nets. Six solids were displayed in front of the student: a cube, a tetrahedron, a rectangular prism, a triangular prism, a square pyramid and a pentagonal prism. In turn, nets of a cube, triangular prism, square pyramid and tetrahedron were shown to the student who was asked to select the solid which each net could fold to make.

Task Z assessed whether students, by looking at pictures of four, five and ten stacked cubes (Figure 2), were able to determine the number of cubes required to make the structures.

Figure 2. Determining the number of cubes.



Method

Five Kindergarten and five Year 1 students from one school (School A) and five Year 1 students from a second school (School B) participated in the study early in the third school term. As reported by Stewart, Leeson and Wright (1997), it was decided that comparisons could more readily be made if some numerical value could be assigned to each student's performance on the tasks. Analysis of the space

assessments involved writing protocols, i.e. descriptions of each student's solutions as well as student's and interviewer's statements. Subsequent analysis involved assigning a numerical value to the student's solution of each task. The value was determined according to task-specific criteria. This lead to an aggregate score for their performance over the space assessment, out of a maximum of 70.

Results

Students' total scores for the space assessments ranged from 66 to 35. In Table 1 scores for the three selected 3D spatial tasks are shown for the highest scoring first-grader, the highest scoring kindergarten student and the two lowest scoring first-graders, respectively. Their arithmetical stages and their total space scores are also shown.

Table 1
Comparison of Arithmetical Stage and Spatial Scores

Name	Arithmetical Stage	Total Space	Score on Spatial Assessment Task X	Task Y	Task Z
Beth	3	66	3	8	4
Isaac	1	43	2	2	0
Michelle	2/3	37	1	6	1
Mark	2	35	1	2	3

The Case of Beth

Beth, aged 7.0 years at the time of assessment, was a student of Year 1 at School A. She obtained the highest aggregate for the space assessment. Only one other child was assessed as being at a higher arithmetical stage.

For Task X, she successfully built the structure having five cubes, by placing three cubes in a row, then one at right angles to the row and finally one in correct position on top. When attempting the structure containing nine cubes, she initially placed the four visible bottom-layer cubes, then she added a tower of two cubes and a second tower of two cubes, each in the correct position. For the third structure she first placed the six visible bottom-layer cubes, then completed the bottom layer and finally added the three cubes of the top layer.

For Task Y, Beth correctly matched the cube with its net and explained that "if you take the top (of the cube) and fold out the sides it would look like that (the net)". She correctly matched the triangular prism with its net, stating, as she pointed to sides of the solid, "there's a square there and there and a triangle there and there and a square on the bottom". The square pyramid was matched with its net because "there's a square in the middle and this (triangle) out there, this out there, this out there and this out there". She also correctly matched the tetrahedron and its net because "it (the solid) has three triangles .... no .... one on the bottom as well".

For Task Z, Beth correctly stated that the first picture contained four cubes because "that one (the one on top) can't stand there without one underneath". She also correctly stated that five and ten cubes, respectively, were required for the second and third pictures.

The Case of Isaac

Isaac was aged 5.4 years at the time of the assessment. He was the highest scoring kindergarten student for the space assessment. Two other kindergarten students were assessed at a higher arithmetical stage (1/2), though it was only marginally higher than his stage.

When building the structure made of five cubes in Task X, Isaac placed four cubes in a row and then placed a fifth on top of the cube at the end of the row. He then checked the diagram and corrected his structure. For the structure comprising nine cubes, he first placed three in a row, then another three beside them to complete the bottom layer; two cubes were placed on top of these but in the wrong position and with a gap between them; then a cube was placed on top of the latter two. When building the structure made of 12 cubes, he initially placed three cubes in a row parallel to the table edge and three in a row perpendicular to these, to form the 'L' shape that was evident in the bottom row of the picture. He then checked the diagram and turned his construction 90 degrees in an anti-clockwise direction. Then he counted the visible cubes of the bottom layer of the diagram and checked his construction. He then added two cubes to the bottom layer and placed one on top. He paused and, when asked if he had finished, re-checked the diagram and then correctly added two more cubes to the second layer.

For Task Y, when presented with the net of the cube, Isaac remarked "That is a cross now; if you folded that one (a square) away it would make one of these (triangular prism)". When asked if he was certain, he then chose the square pyramid; when asked why it would be that one, he responded "because it would". The net of the triangular prism was matched to the tetrahedron because "they (the net) have got triangles and squares in the middle". When the interviewer questioned "This one (tetrahedron) has triangles and squares in the middle?", he re-examined the shape, reflected and then chose the correct solid. When shown the net of the square pyramid, he stated "If you folded these (the triangles) away it would make a square" and then he chose the cube. He chose the triangular prism to match the net for the tetrahedron because, as he pointed to each triangular face of the prism, "that is a triangle and this is a triangle".

Isaac claimed that the first of the pictures for Task Z depicted a structure comprising three cubes. When the interviewer pointed to the top cube in the picture and asked whether it was on the table or in the air, he insisted that it was laying down flat and he maintained that three cubes only were needed. In the same way he maintained that 4 cubes and 6 cubes were required to make the second and third structures.

The Case of Michelle

At the time of interview, Michelle, who was in Year 1 of School B, was aged 6.8 years. Only two of the first-grade students were assessed with an arithmetical stage lower than hers and these were only marginally lower. Her aggregate space assessment score was better than that of only one other first grade student and it was only marginally better.

Michelle successfully constructed the first structure of Task X by placing three cubes in a row, then one at right angles to the row and finally one in the correct position on top. She correctly formed the rectangular-shaped bottom layer of the second structure but incorrectly placed the remaining cubes by placing them on the two end cubes; when the interviewer turned Michelle's structure through 90 degrees to afford her a different view of it, she maintained that it was correct. For the third structure she placed four cubes in a row and a further four in a row at right angles to the former; she then placed three blocks on top of those in the row formed first .

For Task Y she matched the cube with its net because "when you fold that (the net) up you get that (the cube)". The triangular prism was matched with its net because "these two bits (triangles of the net) are them bits (on the solid) and these two (squares of net) are them (on the solid)". The net of the square pyramid was matched with the tetrahedron.

I: Why did you say that one?

M: `cause that one (a triangle of the net) for this one (a triangle of the solid), that one for this and that one for this.

I: (pointing to square of net) And what about this one?

Michelle looked intently at the tetrahedron while turning it around in her hands.

I: Has it got one of those (a square) in it?

M: No!
I: Maybe there's another shape there that does.
Michelle selected the square pyramid.
M: (enthusiastically) This one!
I: You think that one?
M: That bit (square of net) for this one (square of solid).

The net of the tetrahedron was matched to the triangular prism. When offering her explanation, Michelle realised that the solid "don't have a bottom like that" (the triangles of the net) but she was unable to select the matching solid.

For the first structure of Task Z, she initially stated that there were three cubes.
I: Are you sure?
M: Four
I: Four. Why did you change to four?
M: `cause one square (cube) for that one.
I: Oh, I see and where's this one, up in the air or on the table?
M: On the blocks.
I: Are you saying there's another block under this one?
Michelle does not respond.
I: Is there a block under this one or is it on these two blocks?
M: These two.
She then claimed that there were four cubes in the second structure.
I: No hidden ones?
M: No.
Similarly she claimed that the third structure was composed of six cubes.

The Case of Mark

Mark was also from School B and was aged 6.2 years at the time of interview. Of all the first-graders, he was lowest on arithmetical stage and lowest scoring on the space assessment.

For Task X he was unable to build the structure containing five cubes; he placed four cubes in a row and then placed a fifth on the last one of the four. The structure containing nine cubes was successfully completed quite quickly: he formed two rows of three for the bottom layer, then added the two cubes of the second layer and finally the one on top. For the third structure he first made a "L" shape with a row of four cubes at right angles to a further row of three cubes; a cube was placed on each of the latter three; then cubes were placed on the table, on the outer side of the "L", so that there was one next to the corner cube and each of these towers of two.

For Task Y Mark matched the cube with its net because "that there goes there (as he pointed to the squares * in Figure 3) and you fold the wings (squares # of Figure 3) up and it would come into a big square shape like this". The net of the triangular prism was matched with the square pyramid and explained in terms of folding up two squares and both triangles. When presented with the net of the square pyramid he touched each side and pretended to fold it up; he then selected the triangular prism and claimed that "if you fold them two (two of the triangles of the net) it's still got a point coming out there (pointing to the apex of each of the two triangles of the triangular prism)". The net of the tetrahedron was matched to the square pyramid and he claimed that by folding up a triangle of the net "it comes into one of these (the square pyramid)".

Figure 3. Net of cube.

Mark initially stated that there were three cubes in the first picture of Task Z.

I: Where's this one, up in the air or on the desk?

M: In the air

I: Can it sit in the air?

M: It's got a block under it.

He then correctly stated that there were five and ten cubes respectively in the second and third drawings.

Discussion

A common element of Tasks X, Y and Z was interpreting a 2D representation of a 3D shape. There is a perceptual "distance" between a 3D shape and its abstract 2D representation (Parzysz, 1988). When interpreting diagrams of 3D shapes, it is necessary to offset the loss of information caused by the reduction of a geometric shape from 3D to 2D, by creative imagination.

It seems from the case of Beth that we could expect more advanced first grade students to be able to interpret 2D representations of simple 3D structures of the kind used in Tasks X and Z. Further these students should be able to match basic 3D shapes with their nets.

It is noteworthy that Isaac had reasonable success with Task X but failed to obtain any correct answers for Task Z. Perhaps this indicates that he needs physical materials with which to model 3D shapes from their 2D representations (as in Task X), being unable to simply visualise the shapes (as required in Task Z). For Task Y, which also required visualisation, his score was relatively low. From Isaac's case, then, it appears that some of the more advanced kindergarten students may require hands-on materials to help them in their interpretation of 2D representations of 3D shapes.

In an earlier study by Leeson (1995) most of a group of 33 kindergarten students were able to match a net with its associated solid. In that study, however, both nets and solids were made from Polydron interlocking material, whereas in this study 2D drawings of nets were used while solids were made from Polydron material. It seems that the more abstract presentation of the task, as in this study, makes it more difficult for the kindergarten student to make a correct match.

It might have been expected that less advanced first-grade pupils would also need hands-on materials, yet Mark achieved better scores for Task Z than for the other two tasks. Michelle, on the other hand, had best success with Task Y. It appears that less advanced first-graders are unable to consistently offset the perceptual distance between a 3D shape and its 2D representation.

Isaac's response, "because it would", when explaining his reason for choosing a square pyramid to match the net of a cube, reveals that he is operating at a global perceptual level.

Beth and Isaac made more frequent use of formal geometric language than did the two less advanced students. It appears that spontaneous use of more formal geometric language may be a characteristic of the more advanced kindergarten and Year 1 students.

Conclusion

For this study, kindergarten and first grade students were presented with tasks which were quite challenging. There was no indication that any of the students were unnerved or bothered by the prospect of having to solve these tasks. To the contrary, the students appeared to find the tasks interesting and they seemed to enjoy the challenge afforded by the problem solving tasks. This study confirms that the tasks are appropriate for kindergarten and first-grade students.

The tasks might also be suited to problem solving in small group situations. It would be useful to further explore problem solving in early spatial work, in particular, to document across a range of tasks levels of sophistication in students' strategies.

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Effect of the Different Syntactic Structures of English and Chinese in Simple Algebraic Problems

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This paper compares the performance of first-language Chinese and English secondary students in Hong Kong on a set of items requiring the students to express simple relationships algebraically. It is suggested that the different *types* of response produced arises from the distinct syntactic structure of a given expression when translated from English into Chinese. The results are discussed in terms of the construction of mental models and evidence is also considered from interviews conducted with a small sample of the students.

Background

The analysis reported in this paper arose unexpectedly from the findings of a research study on the causes of the 'reversal' error in algebra (Lopez-Real, 1995). The nature of the reversal error and its possible causes have been well documented (Clement, 1982; Davis, 1984; Mestre, 1988; MacGregor, 1991) and two particular cases can be identified. These may be described as the 'multiplicative' case and the 'additive' case. An example of the former is the problem:

> "In a college there are six times as many students as professors. If the number of students is S and the number of professors is P, write down an equation that describes this relation"

One correct solution is the equation S = 6P, whereas the equation 6S = P is typical of the reversal error. An example of the additive case is the problem:

> "In a classroom there are 6 more girls than boys. If the number of girls is G and the number of boys is B, write down an equation that describes this relation"

Again, a possible correct solution is G = B+6 and the typical reversal error is G+6 = B. The purpose of the study was to investigate whether different syntactic structures affected the facility levels of a range of items. For example, the effect of 'contiguity' was tested with the following pair of items (where the pupils were asked to write an equation for the relation between p and q):

(i) p and q are numbers. p is 6 more than q.
(ii) p and q are numbers. 6 more than q is the same as p.

In item (ii) the phrase '6 more than' has been contiguously linked with the q rather than the p, as in item (i). The study showed that the performance of students on such paired items differed at a statistically significant level. The occurrence of the reversal error was also significantly reduced by using different syntactic structures. Johnson-Laird (1983) suggested that human comprehension and reasoning is based on the construction of 'mental models' that are independent of the syntax of the proposition being processed. MacGregor (1993) supported this theory of semantic models with reference to children's comprehension of algebraic relations. The results of the Hong Kong study suggested that no clear dichotomy of semantics and syntax is evident but rather that the mental models constructed by children are likely to be a combination of semantic and syntactic processing.

The above study was effectively concerned with analysing the *errors* made by children, particularly the reversal error. With respect to *correct* responses, the focus was purely on determining the facility levels of the items and hence the *nature* of these correct responses was not attended to. However, during the recording of the data, some striking patterns in the correct responses to item (i) above, became very evident. Clearly there are a variety of possible correct answers to this item and these can be classified into three types: a) $p = q + 6$ (and variations such as $p = 6 + q$; $q + 6 = p$)

b) $p - q = 6$
c) $p - 6 = q$

Although these equations are mathematically equivalent, the syntax of the question strongly suggests that the first category is far more likely to occur and that the expressions in (b) and (c) would occur rather infrequently. These three categories may be contracted into two classes which could be described as a 'sum-model' and a 'difference-model' of the solution. The impression while recording the data was that the difference-models were appearing almost as frequently as the sum-model. Since this was contrary to intuitive expectation, it was decided to obtain precise quantitative values for the occurrence of the difference-models in similar items of the test.

Difference-Model Responses

Apart from the items previously described, there were two other items on the test that were designed to examine the occurrence of the reversal error in the 'additive' case. For convenience, the three items are listed as Q1, Q2 and Q3 below, although they were numbered differently on the test papers.

Q1: p and q are numbers. p is 6 more than q.
Write down an equation that describes the relation between p and q.

Q2: In a classroom there are 6 more girls than boys. If there are N boys and M girls, write down an equation that describes the relation between N and M.

Q3: The following table shows the weight of a suitcase packed with different amounts of clothing:

Wt of clothing (in kg)	8	10	12	14	
Wt of suitcase *with* clothing (in kg)		13	15	17	19

If the weight of the clothing is Q kg and the weight of the suitcase *with* clothing is P kg, write down an equation that describes the relation between Q and P.

A total of 221 Form 2 secondary pupils answered these three items. The facility rates were Q1 (80%), Q2 (64%) and Q3 (61%). The breakdown of correct responses is shown in Table 1.

***Table 1**: Frequency of Alternative Correct Responses*

Item	Response	Frequency	%
Q1	p = q + 6	94	53
	p – q = 6	21	12
	p – 6 = q	61	35
Q2	M = N + 6	100	70
	M – N = 6	5	4
	M – 6 = N	37	26
Q3	P = Q + 5	112	84
	P – Q = 5	11	8
	P – 5 = Q	11	8

The combined figures (as a percentage of the correct answers) for the difference-model responses on each question are thus Q1 (47%), Q2 (30%) and Q3 (16%). There are large variations here but clearly the figure for Q1 confirmed the impression, during the original recording, that this type of response was almost as frequent as the sum-model response.

Another aspect of interest that was easily obtained from the data was to consider the *total* number of sum-model and difference-model responses and to identify what percentage of these responses were correct. In other words, given that a student tried to use a sum-model description say, what was the likelihood that this produced a correct answer? The results are shown in Table 2 (where S refers to Sum-model responses and D to Difference-model responses).

***Table 2**: Comparison of Sum-model (S) and Difference-model (D) Responses*

Item	Total S	Correct S	%	Total D	Correct D	%
Q1	121	94	78	85	82	96
Q2	151	100	66	48	42	87.5
Q3	137	112	82	26	22	85

For Q1 and Q2, the differences in the proportions of correct responses, given a particular model, are significant at the 1% level. That is, where a student expressed the answer using a difference-model there was a significantly higher probability that the answer would be correct than where a sum-model was used. However, there was no statistically significant difference for Q3.

The results shown in Tables 1 and 2 raise a number of interesting questions. For example, from Table 2, why does the difference-model response produce such a high success rate for Q1 and Q2, compared to the sum-model? And from Table 1, we may first ask why the difference-model response varies so much for the three items and second, are these rates indeed unusually high and, if so, why?

The reason for asking the last question is perhaps better understood when one is reminded of the sample of students being tested. These students are all first-language Chinese speakers being taught in English-medium secondary schools in Hong Kong. Such schools are 'Banded' according to performance levels on entry from primary school and the schools used in this study were Band 1 to 3 (i.e. above-average to average). Hence, although the standard of English of these students is good (and their mathematics textbooks are also in English) nevertheless, English is still their second language. It is therefore quite possible that many students will mentally translate a word problem into Chinese before they try to solve it. To take Q1 as an illustration, the relevant translation is shown below. (A dozen colleagues were independently asked to write down a translation and, with very minor variations, all produced essentially the same form. In particular, the p,q,6 sequence was consistent throughout.)

p is 6 more than q
(Insert Chinese translation here)

Translating back into English on a literal character-by-character basis, this reads: "p compared to q is larger by 6". As can be seen, the syntactic structure of the sentence is now quite different to the original English. The crucial phrase in Q2 (there are 6 more girls than boys) when translated, also gives rise to precisely this same structure. That is, "the number of girls compared to the number of boys is larger by 6". In Q3, since the relationship is 'embedded' within the table, there is no comparable phrase to be translated. Given this form of the sentence for Q1 and Q2, it now appears intuitively more likely that the difference-model may be used when solving the problems. In order to investigate this hypothesis, it was decided to conduct the same test with a sample of first-language English speakers. In the following sections, first-language Chinese or English speakers are designated as C1 and E1 respectively.

Comparison of C1 and E1 Responses

The test containing the three items discussed above was administered to 84 pupils at an international school. All the pupils tested claimed that English was their first language of communication although many were bi-lingual. (In fact, the sample included some ethnic Chinese pupils too). The focus of the analysis below is on the number of correct difference-model responses as a proportion of the total number of correct responses. The comparison between C1 and E1 students for the three items is shown in Table 3. (As before, D refers to Difference-models).

***Table 3**: Comparison of Correct Difference-model Responses for C1 and E1 Students*

Pupils	Q1			Q2			Q3		
	Total Correct	Correct D	%	Total Correct	Correct D	%	Total Correct	Correct D	%
C1	176	82	47	142	42	30	134	22	16
E1	53	6	11	48	5	10	58	5	9

Here also, there is a very significant difference (at the 1% level) between the proportion of C1 students successfully using a difference-model compared to the E1 students for Q1 and Q2. There is no statistically significant difference for Q3. (These significant differences also hold true when the *total* number of difference-models used is compared; not just the successful cases). The fact that no translation effect would logically apply for Q3 lends strong support for the hypothesis that this is indeed the cause of the high occurrence of the difference-models in Q1 and Q2. It appears that the syntactic structure of the Chinese form of the questions emphasises first the *comparison* between the two numbers (p,q or M,N) and then identifies the amount by which they differ. This in turn leads to the difference-model being constructed rather than the sum-model. However, this does not imply that it is simply a *sequential* syntactic process that is being used. If this were so, one would expect answers of the form $p - q = 6$ and $M - N = 6$ to be the more common difference-model expressions. On the contrary, as can be seen from Table 1, the forms $p - 6 = q$ and $M - 6 = N$ were far more prevalent. This indicates that the pupils are constructing a particular mental model (in this case the difference-model) based on a combination of semantic and syntactic information in the problem. This also reinforces the conclusions drawn from the analysis of errors in the original study (Lopez-Real, 1995). In order to test further this suggested reason for the high occurrence of the difference-model, it was decided to interview a small sample of C1 and E1 students.

Interviews

The interviews focussed in particular on Q1 of the test paper. Since there was at least a three month time lapse between the original testing and these interviews, students were first asked to solve the problem again, rather than simply being shown their original solution. Ten C1 and five E1 students were selected for interview, all of whom had used a difference-model expression for Q1 on their test paper. The main purpose of the interviews was i) to try to identify the thought processes that led to the construction of this model, ii) to see whether any visual images played a part in the process, and iii) for the C1 students only, to see whether any translation process had been involved. It is notoriously difficult to analyse one's own mental processes, especially in a situation where there is ostensibly just a single step from problem to solution, and this is illustrated in some of the following examples. Nevertheless, a number of students *were* able to articulate their thinking and such cases can be very illuminating.

For the C1 students, all ten again wrote correct responses although three of them were different to their original answers. In two of these cases the response was now the sum-model expression $q+6=p$, while in the third case the student still gave a difference-model response, albeit an alternative model to the original (i.e. $p-q=6$ instead of $p-6=q$). The first thing this illustrates is that the construction of a mental model is not necessarily stable from one occasion to another. Neither of the two students giving the sum-model were able to explain their thinking. As one of them put it: "That's how it came in my head. I don't know how." Both of them claimed that they did *not* translate the problem into Chinese (e.g. "I can answer in English. I don't need to translate").

Of the 8 students who gave a difference-model response, 6 of them said that they *did* translate while the remaining two were unsure. (Being bi-lingual, the author can sympathise how difficult it is to know whether or not one has translated a given sentence). The following exchange illustrates this. (In the extracts below, I stands for Interviewer and Sn designates one of the students.)

I: Do you translate the problems into Chinese in your head?
S5: Sometimes I do. Um ... yes, I think so.
I: Did you translate *this* problem?
S5: I'm not sure. I don't know. I think I don't have many time.

In their attempts to describe how they arrived at their answer, two of the students (S2 and S8) were unable to give any explanation. Student S7 wrote p-q=6 and explained her thinking as follows: "Because I think .... the equation .. normally the number is here (*pointing to the right-hand side*) and p must be bigger, so it must be p minus q equals six". The striking feature about the remaining five students was that they all began with some reference to the relative size of p and q. Two examples are illustrated below. (The mixture of slightly ungrammatical English followed by a precise mathematical description, as in S4's answer, was very typical of student explanations.)

S4: First of all I know that p is bigger than q. And p is more .... uh ..is more six than q. Therefore p minus q is equal to six.
S9: Well, p and q .... p is bigger by 6. That means I have p ... I must go down six. Then it equals q. (She had written p-6=q)

Naturally, the E1 students were able to express themselves better but the problem of articulating one's *thinking* was still evident and one student was completely unable to do so: "I don't know how I did it. It's just how it came". As with the C1 cases, two of the students emphasised the initial comparison of size. For example:

S11: First it said that p is 6 more than q, so I started to think p is larger than q and it's 6 more so then I wrote p-6=q.

Student S14 gave the most explicit description involving a subtraction construct. (His written answer was p-q=6).

S14: I don't know ..... whenever I .... like .... think of subtraction, I always think of the larger number take away the smaller number.
I: What made you think of subtraction?
S14: Because when I think of left-overs I think of subtraction.
I: And what made you think of left-overs?
S14: I'm not sure .... because there were six more left over.

As mentioned earlier, the students were also asked whether they used any mental images to help them. This question arose naturally with S15 in the course of her attempt to describe her solution process for p-q=6:

S15: Because I think .... I'm not very sure actually. I think it's because I imagined like p, 6 and q and arranged them in my mind.
I: I'm not sure what you mean by 'arranged them in your mind'.
S15: Oh .... OK. Um ... I'm not sure.
I: It sounds as if you have some kind of picture in your mind.
S15: No .... I don't think so.
I: Have you ever used the number line in maths?
S15: Yes ... but because when I was learning algebra I didn't really learn to use it. I didn't use it here.

In fact, not a single student thought they had any visual image in their mind even when their description, as with S15 above, suggested there might be. Another example is the phrase used by S9 (quoted earlier) when she says "I must go down to six". Later in the same interview she was asked to solve the following related problem: "b and c are numbers. b is 4 less than c. Write down an equation that describes the relation between b and c". (As before, the literal Chinese translation is "b compared to c is smaller by 4"). Her solution was given as "b + 4 = c" and she described her thinking process thus: "First I know that b is less than c. It's 4 less. So I start at b and I must go up 4 then I get c". Her up/down metaphors suggest a strong visual image but she too, claimed to have no such picture in the mind. (The ubiquity of Up/Down metaphors in language generally is well illustrated by Lakoff & Johnson, 1980). All ten C1 students were asked this problem. Four of them gave the answer above, two wrote 'c – b = 4' and the other three correct answers were 'b = c – 4'. (One student wrote the typical 'reversal' error b-4=c).

When the students were asked to draw a 'picture' of their "p is 6 more than q" solution, even if they were not conscious of using one, the prevalent image was of two discrete sets of objects with one set having 6 more elements than the other. Perhaps unsurprisingly, a number of these pictures represented the objects as fruit! It was difficult not to avoid the impression that these were retrospective attempts to represent their solutions pictorially and played no part in their mental constructions. A number of the students were explicitly asked if they ever used the number line in mathematics and whether this could be a suitable way to picture the relationship. The responses clearly suggested that they associated the number line only with specific topics in mathematics and did not see it as a useful generic image. For example: "I only think of the number line when we do negative numbers".

Discussion

The very significant differences between the C1 and E1 students in constructing difference-model expressions for their algebraic solutions strongly suggests that a 'translation effect' is one of the causal factors. In particular, this is supported by the fact that *no* significant difference was found for Q3 which was effectively 'translation-independent' (the relation being represented in tabular form). Since the syntactic structure of the English and Chinese sentences have quite distinct patterns, this also suggests that the syntax itself plays a part in determining the mental model constructed. For example, the emphasis on first *comparing* p and q (in the Chinese version) may make it more likely that a difference-model is constructed. The interview evidence also supports this view.

The terms 'sum-model' and 'difference-model' have been used in this paper as a convenient descriptive shorthand. However, it would perhaps be more accurate to use a phrase such as 'sum-expression' since what is being identified here is the symbolic representation of the relationship, rather than the mental model itself. Indeed, it is probably impossible to determine precisely what is the mental model constructed by an individual. All that can be determined with certainty is whether or not the symbolic representation correctly describes the relationship. As mentioned in the previous section, the C1 students were also asked during the interview to describe the situation 'b is 4 less than c'. Given the syntax of the question, the 'expected' solution might be 'b=c-4' and the other responses may again arise from an initial comparison of b and c initiated by a translation of the question. Five out of the six students giving the 'alternative' expressions claimed to be 'translators'.

Whatever the precise nature of constructing a mental model, the argument in this paper is that an interaction of semantic and syntactic processing is involved. (And in the written solution, a further syntactic process must occur in order to express the relationship symbolically). The suggestion from the data is that the different syntax of the Chinese translation has directly influenced the type of description in the symbolic representation. When translation does occur this implies *interpreting* the initial statement and producing an alternative way of expressing that statement. It is this kind of flexibility (i.e. alternative descriptions) that may lie at the heart of constructing a suitable mental model. This is similar to the idea of re-writing a problem in one's own words as a useful problem-solving strategy (Charles & Lester, 1984). Clearly, for most pupils, who are working in only one language, the translation effect is not a relevant factor. However, simply re-expressing a statement in the same language is a comparable process. As Haylock & Cockburn (1989) point out, with reference to early mathematical experiences, comparison problems are more often presented using a 'greater than' expression as opposed to a 'less than' expression, and hence children need the experience of re-writing such phrases using the equivalent alternative form. (Interestingly, they also point out that there are more words in English for describing 'greater' qualities than there are for 'smaller'). Flexible approaches to interpreting a given statement might also involve using some form of visual imagery and this was the reason for examining this in the interviews. Given the ubiquity of the number line image in many mathematics textbooks, it is surprising to find (at least, from the interview evidence) that it is not transferred to

the context of algebraic relationships as tested in this study. It would seem to be a particularly potent image for this situation. This is not to say that it is the *only* useful visual image available but it appears that *no* imagery (at least, consciously) was used by the students.

Finally, as a teaching strategy, it is suggested that whenever pupils need to transform a word statement into a symbolic mathematical statement, they should be encouraged first to try re-stating the word expression in an alternative form. The very act of performing such a re-formulation may in itself help in the construction of the appropriate mental model.

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Assessing Number Sense: Collaborative Initiatives in Australia, United States, Sweden and Taiwan

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Group tests of Number Sense were devised and administered to students aged 8, 10, 12 and 14 in Perth, Western Australia and in Missouri. The development and structure of the tests are described, and some results are presented. Research building on this work and conducted in Sweden and Taiwan is described, and implications for research and teaching are discussed

Purpose of the Research

In 1995 we reported on research into the mental computation of school students in Australia, Japan and the United States (McIntosh, Bana & Farrell, 1995). We were keen to explore these students' understandings of number more widely, and also to investigate the extent to which students' number sense could be meaningfully assessed by group administered paper and pencil tests.

We therefore made an analysis of number sense based on our own understandings and the descriptions and analyses of others and used this to structure tests which were administered to the same cohorts of Australian and American students as had been involved in the mental computation research.

Swedish colleagues at the University of Goteborg became interested in the work and collaborated with us in administering revised versions of the tests to cohorts of students of similar ages in Goteborg. A Taiwanese postgraduate student at the University of Columbia adapted some items and administered them to students in Taiwan, resulting in some valuable insights into the mathematical understandings of Taiwanese students.

The aim was not to produce definitive answers but to probe ways of exploring the very difficult area of student understandings and to acquire a better understanding of what number sense means and how it is exhibited in detectable and describable ways.

What Is Number Sense?

Number sense is not the same as numeracy, though it obviously overlaps with it. In today's discourse numeracy is generally understood to refer to the mathematics that one needs in order to deal with everyday life, and it includes more than number. For example, the definition adopted by the Literacy and Numeracy Benchmarking Taskforce in Australia begins: 'Numeracy is the effective use of mathematics to meet the general demands of life at school and at home, in paid work, and for participation in community and civic life'.

We take number sense to refer to a person's general understanding of number and operations along with the ability and inclination to use this understanding in flexible ways to make mathematical judgments and to develop useful and efficient strategies for managing numerical situations (for more detailed discussions of the components of number sense see McIntosh, Reys & Reys, 1992; Resnick, 1989; Sowder & Schappelle, 1989; Sowder, 1992; Willis, 1990).

Methodology

Sample

Students from two school districts (mid-sized to large school districts near a university) participated in the study, one in Perth, Western Australia, and one in Columbia, Missouri. Schools, then classes within schools, from the districts were randomly selected for participation. The number of students participating at each age level within each country ranged from 110 to 160 students. In both countries students aged 8, 10, 12 and 14 participated in the study. Some of these items together with some new

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items were presented to students aged 10 and 14 in Goteborg, Sweden. A few items were also adapted and used with students aged 12 and 14 in Taiwan.

Instruments

McIntosh, Reys, & Reys (1992) developed a number sense framework based on studying and reflecting on the literature associated with the topic. From the framework six strands were identified (two each from the three major categories of the framework). The six strands represent major components of our definition and understanding of number sense. The strands are:

- Understanding of the meaning and size of numbers.
- Understanding and use of equivalent representations of numbers.
- Understanding the meaning and effect of operations.
- Understanding and use of equivalent expressions.
- Computing and counting strategies.
- Measurement benchmarks.

The framework and identification of these six strands provided a structure for the development of number sense assessment items. After identifying the six number sense strands, a search and review of available assessment instruments was conducted to identify items which might provide a useful 'window' into the thinking of students as related to the various strands. The ideas of Markovitz & Sowder (1994) and Cramer, Post, & Currier (1993) were particularly useful in identifying items which mirrored elements of the framework. In addition, new items were developed to reflect various components of number sense. Items were constructed which were appropriate to the age level in terms of conceptual knowledge and which framed questions in non-routine environments in order to elicit strategy generation based on understanding rather than strategy recall based on familiarity with problem-type.

Early versions of the items were used in interviews with students in the United States. These interviews provided a means to clarify wording and item format and to probe the thinking of individual students as they responded to items. Items which evoked thoughtful introspection (rather than immediate recall of a rule or known fact) and which relied on conceptual understanding, were retained. On the other hand, items which could be answered correctly without indication of understanding were dropped.

Some items were organized into a format which allowed for a group-administered number sense test. Slightly different forms of the test were used in the two countries, but most items were identical. The number of items contained on each number sense test varied by grade level and country (ranging from 30 to 45 items).

The Number Sense Tests were administered during one class period at the beginning of the fourth quarter of the school year (March in the US and Sweden, August in Australia) and at mid-year in Taiwan. Items were read to the 8-year-old students. For all ages, a pacing scheme was used in the administration of the test. Specifically, the instructions were read and students were asked to spend no more than 30 seconds on each item. To encourage students to adhere to the suggested pace, the item number was announced each 30 seconds (for example, the administrator said, "You should be looking at item number three about now."). The pacing scheme was utilised to encourage children to answer each item based on reflection rather than computation and to prompt them to move along in the test rather than spend a lot of time on any one item. We found that this pacing scheme was useful in moving students through the various items and also conveyed the notion that computation proficiency was not the goal. Students wrote responses directly on the test pages. Total administration time was 30 - 40 minutes.

Some Results

To give the flavour of the results each of the six components will be described in more detail and the results of one item associated with each will be discussed. A full account of all items and results is given in McIntosh et al (1997).

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1. Understanding of the meaning and size of numbers (Number Concepts).

Understanding of the base 10 number system (whole numbers, fractions, and decimals) includes patterns and place value which provide clues to the meaning/size of a number (e.g., $\frac{5}{6}$ is a fraction less than one, it is close to one because of the relationship between the numerator and denominator OR 1000 is a large number if you are referring to the population of a school but a small number if you are referring to the population of a town). It could involve relating and/or comparing numbers to standard or personal benchmarks. It includes comparing the relative size of numbers within a single representational form.

Table 1
Sample item (Ages 8 and 10)

If I have $378 in my savings account and withdraw all my money, how many 10-dollar notes would the bank be willing to give me?	_____

51 percent of the Australian 8-year-olds and 70 percent of the 10-year-olds answered this correctly, as did 53 percent of the Swedish 10-year-olds. The item differentiated well between quintiles. Percentages of Australian 8-year-olds answering correctly by quintile were respectively 88, 59, 36, 53, 18.

2. Understanding and use of equivalent forms and representations of numbers (Multiple Representations).

This strand includes recognition that numbers take many different numerical and representational forms (e.g. fraction as a decimal, a whole number in expanded form, or a decimal on a number line) and can be thought about and manipulated in many ways to benefit a particular purpose. It also includes the ability to identify and/or reformulate numbers to produce an equivalent form: the use of decomposition and recomposition to reformulate numbers for ease in processing; relating and/or comparing size of number(s) to a physical referent (e.g. collection of items, shaded region, or position on a number line); and crossing among various representational forms.

Table 2
Sample item (Ages 10 and 12)

You are going to walk once around a square-shaped field. You start at the corner marked S and move in the direction shown by the arrow. Mark with an X where you will be after $\frac{1}{3}$ of your walk.	

One quarter of the American and one half of the Australian 12-year-olds placed a cross within acceptable limits.

3. Understanding the meaning and effect of operations (Effect of Operations).

This strand includes understanding the meaning and effect of an operation either generally or as it relates to a certain set of numbers (e.g., Division means breaking a number into a specified number of equivalent subgroups OR Multiplying by a number less than 1 produces a product less than the other factor). It includes judging the

reasonableness of a result based on understanding the numbers and operations being employed.

Table 3
Sample item (Ages 12 and 14)

Without calculating the exact answer, circle the best estimate for: 87 x 0.09.	A. a lot less than 87 B. a little less than 87 C. a little more than 87 D. a lot more than 87

The percentages of American, Swedish and Australian 14-year-olds answering this correctly were 50, 65 and 82 respectively.

4. Understanding and use of equivalent expressions (Equivalent Expressions).

This strand includes the translation of expressions to equivalent forms, often to re-evaluate and/or more efficiently process a computation. It includes understanding and use of arithmetic properties (commutativity, associativity, distributivity) to simplify expressions and to develop solution strategies (e.g. the use of the distributive property to multiply 7 x 52).

Table 4
Sample item (Age 14)

Write '*is greater than*', '*is equal to*' or '*is less than*' to make this a true statement:	$456 \div 8$ _____ $456 \times \frac{1}{8}$

The percentages of American, Swedish and Australian 14-year-olds answering this correctly were 44, 46 and 48 respectively.

5. Computing and Counting Strategies

This strand involves applying various number sense components previously described in the formulation and implemention of a solution process to a counting or computational (estimation, mental computation, paper/pencil, calculator) situation (e.g. Is 29 x 38 more or less than 400? OR: How many birds do you estimate there are in this picture?)

Table 5
Sample item (Ages 8, 10, 12 and 14)

About how many days have you lived? (Circle the nearest answer.)	A	300
	B	3000
	C	30 000
	D	300 000

The significant digit in this item varied between countries: Australia used 3, the United States 4 and Sweden 5; the most surprising feaure of these results, which are discussed in greater detail in McIntosh, Reys & Reys (1995), is that across all three countries and four age groups the success rate only varied between 19 and 55. Indeed the

Australian percentages correct for the four age groups were respectively 34, 35, 38 and 32. The reasons for this apparent stagnation merit closer scrutiny.

Measurement Benchmarks

This strand was only added at the time of the Swedish testing. It includes applying various number sense components previously described in the formulation and implemention of a solution process to a measuring situation. It requires an understanding and use of standard, non-standard and/or personal benchmark units of measure (e.g. A textbook weighs about a kilogram OR: The angle is slightly less than a right angle so it must be about 85 degrees). It involves measuring attributes such as mass, length, capacity, volume, time and angles.

Table 6
Sample item (Age 10, Swedish version)

How long is a bed for an adult? Circle the best answer	A. 50 cm B. 100 cm C. 200 cm D. 400 cm

Sixty-eight percent of the Swedish children answered this correctly, with a further 20 percent choosing 400 cm.

Developments in Sweden

Researchers at the University of Goteborg carried out similar testing with students aged 10 and 14 and some of the results are given above. However the major innovation was that they published the tests in the journal *Namnaren* together with some discussion of the meaning and importance of number sense and invited teachers to try the tests in their own classroom and to share results and their opinions on the testing and their perception of the relevance of number sense. Thus the exercise initiated a mini-national debate, together with professional development and the involvement of teachers in action research. The project is described in Emanuelsson, Johansson, Reys & Reys (1996).

Developments in Taiwan

Der Ching Yang, a Doctoral student at the University of Missouri-Columbia devised a twenty-item Written Computation Test (WCT) for students aged 12 and 14 and a forty-item Number Sense Test (NST). The first 20 items of the NST were developed to parallel the WCT Items. That is, the same numbers were involved but the format of the items related to these numbers was different. The two tests were administered on different occasions.

The research provided strong evidence that Taiwanese students perform at very different levels on written computation and number sense tests. Basically, the number sense performance level for both 12-year-old and 14-year-old students in Taiwan was poorer than their written computation performance. One example will serve to indicate the difference between their written computation performance and number sense ability. The percentages of the same cohorts of students giving each reponse on two parallel items, one from the WCT and the NST respectively, were as follows:

Table 7
Sample Items (Taiwan, age 12 and 14)

NST Item			WCT Item		
Without calculating an exact answer, circle the best estimate for $\frac{12}{13} + \frac{7}{8}$.			Calculate $\frac{12}{13} + \frac{7}{8}$		
	Gr.6	Gr.8		Gr.6	Gr.8
A. 1	10	20	Correct	61	63
B. 2 *	25	38	Incorrect	39	37
C. 19	36	14			
D. 21	16	12			
E. I don't know	10	16			
F. No response	3	0			

Whereas 61 percent of the 12-year-old students and 63 percent of the 14-year-old students could correctly calculate $\frac{12}{13} + \frac{7}{8}$, only 25 percent and 38 percent respectively of these same students gave 2 as the best estimate for the computation.

Discussion

The research supports the view that there is some identifiable field called number sense and that the number sense of students can, to a certain degree, be investigated and assessed through the use of group-administered written tests. However, often the research produced results whose main value was that they indicated a potentially rich field of inquiry which merited further exploration through structured interviews with individual students.

It was apparent that students had many deficiencies in understandings, particularly when fractions and decimals were involved.

There is scope for much further research, into more sensitive assessment items and procedures, into the performance of students and their thinking, and into ways of translating the implications of the research into effective classroom action.

All of this assumes even greater importance in view of the current moves at state and federal level in regard to the setting of numeracy benchmarks and associated assessment procedures which reveal yet again the tension between the expertise and aspirations of professional educators and those whose standpoint is more remote from the classroom and more simplistic.

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Choosing or avoiding mathematics at the upper-secondary school level: some significant factors

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Over 1995 and 1996, the researchers conducted a study of the factors which influence Australian students' selection or non-selection of mathematics subjects at the upper school level. Some 6 000 students from Western Australia, Queensland and Victoria responded to a survey seeking to determine such factors. This paper describes the outcomes of the survey and suggests that students' cognitive style orientations have a significant role to play in the subject selection process.

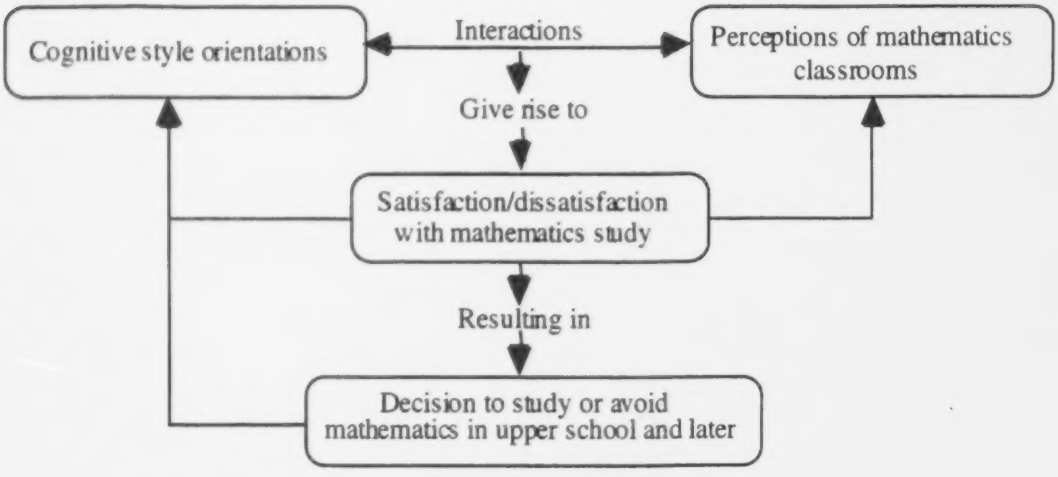
Introduction

The Report entitled *Science and Technology Education: Foundations for the Future* (Brennan, 1994) urged researchers to investigate the decision-making processes that turn students 'onto' or 'off' science and mathematics as a further education or career option. Consequently, this study aimed to investigate the sociological and psychological factors influencing students' choice of mathematics subjects. A literature review indicated that factors which do appear to contribute to students' decisions about their involvement with mathematics include organisational reasons (e.g. school timetable arrangements); personal or individual (e.g. self esteem); confidence with mathematics (e.g. anxiety), perceptions of mathematics as lacking in social relevance and lacking in satisfying career prospects. Another set of contributing factors possibly involves students' cognitive style orientations – that is, those psychological attributes accounting for students' attitudes towards classroom experiences. These latter factors provided a principal focus for the study, for they are potentially manipulable by educators whereas many of the other factors are not (a statement based on the considerable attention given to those other factors in the past, with few discernible results).

The democratic nature of Australian education systems allows students and parents to make decisions about subject choice in secondary education, an arrangement which has provided a mechanism for enabling academically able students to opt out of further study in courses such as advanced science and mathematics. Although there has been a national increase in participation rates for post-compulsory education, there has not been a proportional increase in participation in courses of advanced mathematics and science, and the drop-out in these subjects is now regarded as critical (Malone, 1993; Dekkers, de Laeter & Malone, 1991, 1993; Secondary Education Authority, 1995; de Laeter & Dekkers, 1996; The *Australian*, Feb, 1997).

Apart from the Brennan Report, Australia-wide concern has been steadily growing about this low level of participation in secondary school courses leading to further education and careers in mathematics and science (Brennan, 1994; The *Herald Sun*, Nov, 1996; *The Age*, Jan 1997; The *Australian*, Feb, 1997; *Campus Review*, Feb, 1997). At the heart of these concerns is the desire to ensure that those students who are most capable in mathematics and science continue with these subjects and find themselves well placed for careers in these fields in later life. To this end, this study sought to determine those factors which facilitate this process in mathematics. The study was based upon the assumption that students possess cognitive style orientations which interact with the way they perceive

mathematics classroom environments and hence determine particular levels of satisfaction with the subject. This satisfaction, in turn, is related to students' decisions to continue with (or reject) mathematics as an area of study. The simple non-causative, non-linear model below illustrates relationships between a number of variables exercising reciprocal influences.


Cognitive style orientations
Interactions
Perceptions of mathematics classrooms
Give rise to
Satisfaction/dissatisfaction with mathematics study
Resulting in
Decision to study or avoid mathematics in upper school and later

Students' cognitive style orientations are taken to be psychologically based attributes which underpin their attitudes towards interactions with others, the way they structure knowledge and their approaches to problem solving and self discipline (Denham, 1994, 1995). In this sense, they are associated with particular needs in students which can be either satisfied, to varying degrees, by the way the students perceive their classrooms – in this case their mathematics classrooms. Many of the other factors mentioned earlier which are known to influence students' subject choice continue to be the subject of investigation, but have failed to counter the downward spiral in enrolments and were not considered in great detail in this study. Rather, the study addressed cognitive factors that have not been considered before in the context of student subject and career choice. It was anticipated that the study's outcomes would have implications for teaching and learning approaches and for curriculum development in terms of students' cognitive style orientations.

Theoretical Background

In many Australian secondary schools, the career and subject choice counselling process has been formalised through the introduction of career and vocational education courses. Science subject selection is now part of a structured counselling program based upon career education theories and implemented by specialist school staff. Counselling programs encourage students to identify their personal needs and to reflect upon the extent to which these are satisfied in particular subject areas or may be realised in particular vocations. There are two implications here for mathematics education. Firstly, mathematics curricula and teaching methodologies need to provide students with personally satisfying experiences. If this does not occur, capable students may select other subjects.

Secondly, in order to influence subject choice decisions, mathematics educators need to understand the theoretical grounding and counselling procedures of career education. This knowledge needs to be reflected in mathematics curriculum design and pedagogy. It is strategically important for classroom practitioners and researchers to thoroughly understand how students make subject choice decisions.

Comprehensive theoretical frameworks which explain career and subject choice decision-making have been developed within the discipline of career education (Osipow, 1990; Herr and Cramer, 1992). Theoretical approaches include the matching of human traits and factors, studying the decision making process itself, the application of sociological constructs, the influence of psychological attributes and needs, and the developmental nature

of an individual's ability to make informed career decisions (Herr and Cramer, 1992). Another approach to understanding subject choice is based upon Holland's (1973) theory of occupational choice. This proposes that a choice of occupation should ensure a match between personal characteristics, for example personality, attitudes, ability and interests, and the rewards and experiences coming from engaging in the occupation (Holland, 1973, Baker, 1985; Smithers and Hill, 1989; Ramsden and Ramsden, 1990; Malone and Cavanagh, 1995, Malone and Cavanagh, In press).

The proposed study set out to examine, among other things, the ways in which cognitive style orientations are influential in turning students 'onto' or 'off' science.

Goals of the study

The aims of the study were to:

1. develop a questionnaire to investigate those organisational, socio-cultural, personal and cognitive style orientations which influence students' decisions to study or avoid mathematics at the upper school level,
2. use this information to conduct case studies with students to probe further the reasons for their decision to study or avoid mathematics, and
3. determine the implications of the findings for teachers, school administrators, curriculum developers, students and teacher-training staff.

Methodology

The entire study extended over 1995-1996.

- A questionnaire was developed consisting of items corresponding to the factors of interest mentioned in goal 1 above. As the survey was to be administered to students in Western Australia, Queensland and Victoria (based on the location of the researchers and capitalising on the variety of mathematics curricula and range of mathematics units available in each state, so enhancing the study's 'national' flavour), three versions of the questionnaire were prepared, each taking into account subject titles and other matters idiosyncratic to the different states.
- The questionnaire was used to gather information from a total of 5 817 Year 11 and Year 12 students in public, independent and catholic schools, chosen by means of a proportionate sampling technique, in each of the three states during 1995.
- The second phase of the study (1996) involved developing 21 case studies of individual students chosen from the same schools in the three Australian states (7 per state) in order to probe further their reasons for choosing or avoiding mathematics. This case study data is still being analysed. Emerging insights are discussed later in this paper.

Findings

The Questionnaire (general items):

The data were analysed for themes and issues which significantly contributed to students choosing or avoiding of mathematics in the upper school. Basic statistics such as frequency counts, proportions and cross-tabulations were based upon weighted data, and univariate and bivariate statistics were used to provide a description of the characteristics of the sample and to facilitate an initial exploration of the relationship between the variables. The analysis suggested some strong links between students' decisions to choose or avoid mathematics and the independent variables described below. The dependent variable for the analysis was the mathematics level chosen by students, categories being: High Level Mathematics (the most rigorous); Mid Level Mathematics; Low Level Mathematics (unacceptable for tertiary entrance), and No Mathematics Taken.
Analysis of the questionnaire data suggests strong links between students' decision to choose or avoid mathematics and their

- perceived ability and experiences with mathematics in the primary and early secondary school

- confidence in, and attitudes towards mathematics
- school type
- friends who are studying mathematics, and influential others
- views on the need to possess natural ability in mathematics
- views on the need for mathematics in the career they hope to pursue.

Perceived Ability: A student's perceived ability in mathematics had the strongest effect in him/her choosing or avoiding mathematics. Of major interest is the fact that 35% of students perceived themselves as being excellent when they were in primary mathematics, 19% perceived themselves as being excellent in lower secondary mathematics and only 9% rated themselves as currently being excellent in mathematics. In this sample, 71% of the students had taken mid to high level mathematics. Of those, many indicated that their current ability was only reasonable.
Student Attitudes: Of the variables which make up the students' attitudes, confidence in studying mathematics has the strongest effect on choosing or avoiding the subject and is most strongly correlated with the students' perceived ability (significant at the 0.05 level of confidence). Analysis indicates a weak relationship between a student's enjoyment of mathematics and the mathematics level taken, and that a student's career aspirations are a better indicator as to the mathematics level chosen in upper school. Seventy-nine percent of the students surveyed thought that mathematics would be useful in the future. It would appear from these results that young people with clear perceptions of their future educational and career path are more likely to take subjects in order to meet their career expectations. Thus occupational and career preferences are seen as the determinants of the mathematics level students choose.
School Type: The schools in this study included government and private, coeducational and single-sex, and these types are strongly correlated to the mathematics level taken by students. When adjusted for the covariate *socioeconomic status* which compounds the effect that school type has on mathematics level chosen, the effects are significant at the 0.05 level ($F = 139.22$ and $p = 0.00$).
Effects of Significant Others: Contrary to popular belief, the most influential person in a student's decision to take the most rigorous mathematics units is the student him/herself. Thirty-eight percent of the students nominated themselves as the most important in their decision to take mathematics, and 19% stated that no one was influential in this decision, thus suggesting that parents and others do not have a significant influence on a student's decision to take mathematics in the upper school. This factor is most strongly related to those students who studied a university entry-level mathematics unit.

The Questionnaire (cognitive style items):
Among the most interesting aspects of the results are those related to the responses to the cognitive style orientation items in the questionnaire, despite the relatively small number (16) of such items included. An analysis of variance using three groupings of levels of school mathematics (High level - the most rigorous; Medium level, and Low level - unacceptable for tertiary entrance) indicated that there is a strong relationship between a number of cognitive style orientations among students and the level of mathematics studied. Four orientations were examined, based on the Briggs-Myers (1989, 1992) dimensions of: Extroversion - Introversion (E-I), Sensing - Intuition (I-S), Thinking - Feeling (T-F) and Judgement - Perception (J-P). Simply explained, the Extrovert - Introvert scale describes how people energise themselves; the Sensing - Intuition scale describes those things that a person pays attention to – whether they are realistic or imaginative; the Thinking - Feeling scale measures how a person makes decisions – whether objectively or subjectively, impersonal or personal, and the Judgement - Perception scale describes the lifestyle that a person adopts, be it planned or spontaneous. The data suggests that:

- Students studying high level mathematics tend to have a greater orientation towards introversion on the Extroversion-Introversion scale than students taking low level mathematics;
- The strongest relationship exists between the Sensing-Intuition scale and Mathematics Level Taken ($F = 4.05$ and $p = .007$: significant at the .05 level).

Students studying High Level mathematics have a greater orientation towards intuition on this scale than all other students. These students like to know how things work and hence are more interested in theory and theoretical approaches to teaching and learning than other students;

- Students who tend towards Judgement on the Perceiving-Judgement scale are more likely to take Low Level mathematics or No Mathematics. This indicates that the more capable students had fewer problems with such activity as planning their studies and with working to deadlines ($F = 2.48$ and $p = .05$). Low Level mathematics students have a greater orientation towards feeling on the F-T scale than all other students;
- High Level Mathematics students tend more towards perceiving on the P-J scale than all other students.

The case studies:

The case study data from the 1996 phase of the study has utilised a mixed method design drawing heavily on grounded theory. It has involved 21 case studies of upper school mathematics and non-mathematics students, and has been conducted in the three Australian states. The sample was purposive in nature, with students identified by school personnel. Students were interviewed indepth on several occasions about their feelings on the school, the background to their selection of subjects, their career aspirations and life expectations. The analyses suggest that the selection process for subjects in the upper secondary school is not only multi-dimensional, but more complex than the literature suggests. Relationships exist between students' choice or avoidance of mathematics and how they view their futures, how competent they consider themselves at mathematics, the strength of their sense of identity and the extent to which they have developed a "life-plan".

One way in which these interactions can be explored and investigated is through the concept of "identity". For example, there appears to be interrelationships amongst the students in all three states concerning the relevance they attach to schooling, whether they can imagine where they will be located (socially and physically) in the future, whether they can imagine the first step in how to achieve this image, and how they even think about the future. These factors appear to influence students' views on mathematics as a useful discipline, as well as their decision on whether or not they will study it.

Conclusion

Multi-variate analysis of the cognitive style data suggests that students who exhibit certain cognitive style orientations are likely to be drawn to study mathematics. The case study data suggests that student identity formation also may play a role in this decision. The nature of the relationship between these two factors is not yet clear, and how they interact is less clear. For example, it may be that students with certain cognitive style preferences are disposed towards particular identity formation styles. Also, it would be useful to examine more closely the extent to which cognitive style orientations are a predictor of the nature of the involvement students have with senior secondary mathematics and science. The chief investigators hope to pursue this line of research in a large scale study over the years 1998-2000.

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PEOPLE IN MATHEMATICS EDUCATION

Fred Biddulph
Ken Carr
(Editors)

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EDITORS' NOTE

MERGA 20

The Twentieth Annual Conference of the Mathematics Education Research Group of Australasia (MERGA) is being held in Rotorua, New Zealand, 7-11 July 1997.

Welcome

The Conference Committee welcomes you to Rotorua. This is the first time the MERGA annual conference has been held in New Zealand, and we trust you will enjoy your time. The quality and range of papers is substantial. There is something in this Conference for all mathematics educators.

The conference is organized by mathematics education staff at the University of Waikato, Hamilton. We acknowledge especially the work of Andy Begg as chief organizer.

Paper selection

For the first time these proceedings include abstracts of the short communication papers and posters - there are 28 papers in this category.

The guidelines for presenting papers at MERGA were used to select papers. Two independent reviewers were used for each paper. Where there was disagreement a third reviewer was consulted.

Papers have been published largely as submitted by authors. While care has been taken by the editors, some errors inevitably occur.

We offer sincere thanks to the reviewers of papers who were all mathematics educators from New Zealand. Your speedy and thorough attention to the task was appreciated.

Conference theme

The theme 'He Tangata, he tangata, he tangata' is Maaori for 'Tis people, tis people, tis people'. The expression comes from a proverb, a translation of which appears in the Programme booklet. A parallel is drawn between a flax bush and generations of people, and this symbolises a belief in the importance of human kind as a whole.

Fred Biddulph & Ken Carr
Editors
Hamilton
New Zealand
1997

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Adult Learning in an Industrial Setting

Peter Martin
University of Ballarat

This paper presents an evaluation of a statistical training program which took place at a Ballarat industrial plant throughout 1996. The success of this program, and others throughout America and Europe, has far reaching implications for the way in which we teach statistics in undergraduate courses at tertiary institutions. This paper examines the success of the program in the light of adult learning models and outlines the direction of on-going studies into the effectiveness of such training programs.

Introduction

During 1996 an industrial company quietly embarked upon a specialised statistical training program at their Ballarat plant as part of an overall program initiated by their parent company in America. The training program, which was designed to increase process yields, reduce the cost of poor quality, and to increase capacity through the use of statistical tools, was delivered by a group of consultants from the Six Sigma Academy based in Phoenix, Arizona.. The objectives were to train people to support major process improvement projects and to become proficient in basic and advanced Total Quality Management (TQM) tools.

The training format comprised 5 sessions varying from 3.5 to 5 days in length. Each session used the *plan, train, apply and review* methodology that has been so successful in similar programs conducted in America and Europe. The topics covered included *descriptive and inferential statistics, process improvement planning, failure mode and effects analysis, process capability studies, measurement system analysis, gauge reproducibility and repeatability studies, multi-vari studies, experimental design (DOE), and statistical process control and tolerancing.*

Use of computers formed an integral part of the training. Each participant had personal use of a Pentium laptop, and all were expected to become proficient in Word, Excel, Powerpoint and Minitab. While some aspects relating to these packages were dealt with during the course of instruction, most participants spent many extra hours at home refining their skills.

Participants were also expected to begin training with a work place project clearly identified. In most cases these projects were related to scrap or defect problems resulting from current processes from the factory floor. These projects were the focal point of the training program, and varied in scope and complexity. Having been allocated a project the participants task was to solve the problems associated with the particular process involved. The projects dealt with such problems as *the failure of gaskets to bond adequately, inconsistent densities from a moulding operation, the presence of scratch marks on metal backing plates, curvature of backing plates,* and so on.

Trainees worked on these projects either individually or in pairs. Usually the required result was to reduce scrap or defects by 50%. Successful certification after completing the training was dependent upon the trainee being able to demonstrate command of the appropriate statistical techniques with their application toward success on the selected project.

Financial and Educational Success

For the Ballarat company, as well as others who have embarked upon similar training, successful results have certainly followed. The company have estimated that the training projects alone have resulted in savings of $800 000. Further evidence of the success of the program is seen in the numbers of employees that have volunteered for similar training in 1997. The company is in the fortunate position of having a healthy pool of volunteers from which to select their trainees, thereby creating a premium on positions for this training program. Phase one of company planning will be to ensure that

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every engineer employed at the plant receives this training. The next phase will be to make the training available to shop floor managers, and so on.

There are probably many reasons for the current successes being reported around the world, resulting from training programs such as that undertaken at the Ballarat plant. The implications for how and what we currently teach in our undergraduate courses at University level should not escape our attention. I feel there are four major reasons for the successes reported. The first has to do with the power of the statistical content while the second concerns the learning model embodied in the structure of the training program. Other reasons have to do with company commitment and participant expectations.

i) Statistical Content:

A major focal point of the content was the use of carefully designed experiments involving *Multi-variable testing*. Multi-variable testing techniques allow experimenters to change many variables simultaneously. Traditionally engineers have experimented with the production process by methodically changing one factor at a time and analysing the results. Such a procedure has been shown to be both inefficient and costly in terms of resources and time, as well as being incapable of detecting interactions between variables that often are primary causes of problems in manufacturing processes.

Properly designed experiments where several variables are altered at once are highly efficient in that more use is made of the data collected thereby reducing the number of tests required to isolate the problem. Such experiments are more capable of readily identifying those variables having the greatest impact upon a process as well as the effects of any significant interactions that may exist between these variables.

Statistical analysis provided the tools that enabled the participants to determine those variables considered to be significant factors. Design of Experiment (DOE) techniques provided an opportunity to further refine the list of significant variables and to focus upon an optimal operating window within which process yields and capacity could be maximised, and the costs of poor quality reduced. The material is not new and techniques involving multi-variable testing have existed for many years. However, the power of modern computing software has made much of the analyses more accessible and may well have to do with the growing awareness and acceptance of such data analysis.

Allowing for Process shift: Another challenge to traditional thinking that arose from this particular training program was the concept of allowing for the natural drift of any process mean over time. Research has shown that typically a process mean will shift by a factor of 1.5 times the process standard deviation (Evans 1975). The implications of this with respect to process control directly challenges the traditional concept of process variation in relation to specification limits. Traditionally the objective has been to attempt to keep process variation to about one-sixth of specification width. Allowing for maximum process drift this amounts to 66807 defects per million opportunities. Current best practise throughout the world is to maintain process variation at one-twelfth of specification width, which translates to 3.4 defects per million opportunities, again allowing for maximum process drift.

To what extent in engineering courses offered around Australia is there reference to multi-variate Design of Experiment techniques, or to the statistical tools required for their analysis? To what extent is there reference to procedures for analysing measurement system capability or to the implications of shifts and drifts in processes? In some instances the mathematical and statistical content of our engineering courses has been reduced and yet we have a situation where manufacturing industries around the world are spending hundreds of thousands of dollars putting in place training programs to re-train their engineers in the statistical procedures required so that they can remain competitive. Clearly more time and effort is required on our part to at least pay heed to these trends and to develop accordingly.

ii)The Training Program as a Model of Learning:

My first impressions with regard to the educational worth of the training program were mixed to say the least. Beginning each day at 7:30-8:00am., and continuing until 4:30 or 5:00pm. and maintaining this over a four or five day period was daunting to say

the least. Everybody who participated including the instructors was exhausted at the end of each day with the effort of concentration that had to be maintained.

As mentioned previously, successful training programs in America and Europe have made use of a methodology involving the cycle *plan, train, apply* and *review*. This fits very nicely into what education theorists call the Action Research Spiral (see Fig.1 below) which can be described as following the path of *plan, act, observe, reflect, replan, ...*

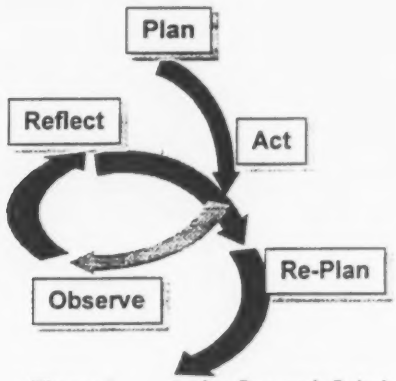

Figure 1: Action Research Spiral

The training program itself was an excellent example of action research being put into practice. It was designed to generate process improvement experts who learn by doing (the heart of the training program involved a project directly related to processes in current operation). Once trained these *experts* are expected to share their training with fellow employees, to continue the cycle anew, and so beginning the development of a culture leaning towards what Oakland would call total quality management.

The learning model adopted by the instructors involved *lectures, course notes, materials and handouts, worked examples, practice examples, hands-on computations, case studies, self review, simulations, application projects and presentations.* The program was structured in such a way that in any 3 or 4 hour period most of the aspects mentioned above would be experienced. For instance, instead of participants completing a set of practice exercises by themselves, one exercise would be allocated to 2 participants who were then required to work together as a team to solve the problem. Solutions and explanations were then presented to the whole group using Powerpoint with appropriate Minitab output inserted as required. Questions from the floor were always encouraged. At first this proved very daunting, to say the least, however, participants soon realised that no matter what they thought they were being heard, it was OK to be wrong and it was OK to ask for help.

This was a model that promoted a deep approach to learning. Throughout the training considerable emphasis was given to

- relating existing knowledge to the project in hand;
- drawing on knowledge from as many sources as possible via project teams;
- placing theoretical statistical ideas into the realm of shop-floor experience;
- relating and distinguishing between evidence and argument - *there is no argument if you have no data* was a common catchphrase.

These points typically characterise the deep approach to learning espoused by Ramsden (1992) and Biggs and Moore (1993).

Characteristics of Adult Learning: An important factor contributing to the success of this training program was that it accounted for the major characteristics that typify adult learning. Roberts (1986) identified five main characteristics of self-directed learning activities for adults. Each of these characteristics were readily identifiable throughout the training program at the Ballarat plant. Table 1 below presents a summary of Roberts' characteristics, with brief explanations of how each was apparent in the training program.

Table 1: Characteristics Typifying Adult Learning (Roberts)

Learning is usually episodic rather than continuous;	*The structure of the training program was such that it took place in concentrated bursts - train for a week, work with it for 3 weeks, train for a week, work with it for 3 weeks.*
Learning is generally aimed at the solution of immediate specific problems of a concrete rather than theoretical nature;	*At all stages of the training examples were brought in directly from the factory floor. Identification of the projects (focal point of the training) was made using specific problems that currently existed in the plant which required an immediate fix.*
Learning is rarely pursued in a systematic way;	*This was more subtle to detect, however it became apparent by observing how participants worked towards solving the problems associated with their particular projects.*
Knowledge from many sources is usually used to solve particular problem (not confined to specific discipline areas);	*The emphasis placed upon establishing an effective team was both strong and consistent. A good team was one that represented all layers of personnel associated with the particular process. Successful projects were often associated with effective teams.*
As material is mastered it is applied immediately	*Again, the structure of the training program ensured this would occur using the one week to learn the material and three weeks to apply the knowledge before the next learning episode.*

iii) Commitment of Company Management:

Undoubtedly an important factor in the success of such training programs is the degree of commitment shown by senior management. The Bulletin (July 16 1996) quotes Mikel Harry (founder of the Six-Sigma Academy) as saying the "key to the process is a company that is prepared to accept change and devote its entire thinking to quality control at every step of production".

Previous successful programs have required the commitment to quality at all levels, and particularly at the senior levels. At the Ballarat plant this was given due emphasis by the senior-level appointment of a *Project Champion*. This responsibility was taken on by the manager of Quality Control, whose role was to devise and oversee each participant's project, and, most importantly to protect participants from negative criticism from above and below.

Total Quality Management: The structure of the program was such that it brought together the essential elements of total quality management, which according to Oakland (1993) and Caulcutt (1995), include *commitment, culture, communication and co-operation.* I feel that this was also a major contributor to the success of the program. A slight modification of Oakland's model of total quality management is shown below.

Figure 2: Model of Total Quality Management (After Oakland)

The processes at the centre of Oakland's model link together complex chains of customers and suppliers. The training program at the Ballarat plant emphasised this and in particular continually stressed the importance of appreciating and understanding the concept of the *internal customer*. Deming himself placed a great emphasis upon cooperation, and Caulcutt (1995) argues that the health of a company culture can be measured by the extent of cooperation that exists. Whether it be between departments or between people at different levels, both are indicative of an open culture with a true customer focus, claims Caulcutt (1995).

iv) Other contributing factors:

Two other important factors contributing to the success of the program were the *anticipation* that this was something very special, and significant, and that it would work, as well as an *aura of credibility* that surrounded the instructors. The anticipation factor was not surprising given the well documented success stories from around the world involving such major companies as Toyota, SONY, Motorola, IBM, ABB, General Electric and other subsidiary companies belonging to the Allied Signal group.

Rick Schroeder of Allied Signal has estimated (Bulletin July 16 1996) that for many companies total product defect costs are likely to be in excess of 20% of sales. Given the enormous potential for improvements and savings in many industrial processes it is therefore no surprise that a well thought out, concentrated, rigorous training program will show very substantial early returns, as in fact has been the case. These early successes are an important factor in the continuing successful application of the initial training received. Any researcher will be buoyed by initial success and will most likely be better equipped to continue with the experimental process.

This was particularly true at the Ballarat plant as early in the program groups of shop floor workers indicated strong negative feelings and attitudes. They argued that why should they bother to cooperate as there had been plenty of previous projects of which none had been completed satisfactorily. After initial successes of the projects directly concerned with this work area the personnel are currently far more tolerant and cooperative regarding future project planning.

The credibility factor was an interesting phenomena to observe. The trainee group consisted of people from a relatively small country city comprising new graduates, with no previous work experience, and some company employees who had been with the company for many years. Fly out some instructors from America who are ready to begin within hours of stepping off the plane, who speak with an accent and who quote from

personal experience stories and anecdotes from companies such as Toyota, SONY, etc. and you have one quite significant credibility factor (the Red Adairs of the manufacturing world). When these people spoke everybody listened.

From a critical perspective they were not particularly good teachers and neither were their explanations always understandable, nor 100% correct. For example, one instructor claimed that interaction effects could not be modelled using regression techniques. He was unaware that multiplying two variables was equivalent to the interaction between the variables. Another instructor used incorrect terminology when referring to ratios of sums of squares from an ANOVA table.

Program Evaluation

The training program involved three groups of trainees, *Black Belts, Green Belts,* and final year engineering students from the University of Ballarat. The *Black/Green Belt* stratification of the training was inherited from the American parent company. The Black Belts were expected to become the expert problem solvers whilst the Green Belts were expected to fulfil a support role. However, the concept of involving final year engineering students in such a specialised way was an innovation not tried before. The objective was to create a pathway between the University and this particular company in the hope of securing a future flow-on of specifically trained engineering graduates, as well as ongoing statistical support from University staff.

The differences between graduates and non-graduates was quite marked, both in the make up of the groups and in the levels of understanding realised from the training. Four Black Belts were selected comprising 3 engineering graduates straight from university, and a young company employee, without formal tertiary qualifications, but who had been identified as a potential rising star. Eight Green Belts were selected comprising people from various areas throughout the company. Collectively the Green Belts had considerable shop-floor experience between them but minimal tertiary engineering qualifications. It was expected that the Green Belts would find some of the statistical procedures a difficult hurdle to overcome.

Probably the most significant difference between the two groups was that the Black Belts were solely dedicated to their allocated projects whereas the Green Belts were not. The Green Belts were required to fulfil their usual work commitments in addition to working on their projects. This proved to be a very frustrating exercise which was eased somewhat by the support provided by the Project Champion. This was usually in the form of ensuring that suitable computing equipment was made available for these people and negotiating with respective department heads for appropriate time release in order to meet commitments.

Despite the endurance test of long days, and theories regarding effective student concentration spans of 45 minutes or less, I believe the overall program was very successful, and that considerable transfer of knowledge occurred. The success of the program is evidenced in the savings returned to the company in the first year of operation (estimated to be in the vicinity of $800 000). The transfer of knowledge may be evidenced in the project reports submitted by the respective participants. An after-the-event survey is currently being analysed to ascertain participants' self assessment of the value of the training. The success and educational worth of future programs will be measured in part by a special questionnaire designed to measure participants' statistical knowledge before and after similar training programs. Of particular interest will be any differences between graduates and non-graduates that emerge, and future research will focus upon identifying the characteristics of those most likely to receive the greatest benefit from future training, and therefore return maximum benefit to the company.

The challenge ahead

The challenge for those of us who teach statistics looms on several fronts. Here we have an enormously successful program involving relatively high levels of statistical analysis made accessible through computer software. Historically as teachers of statistics we have taught students to test whether or not something is significantly different from

something else. Now we need to teach our students to be innovative. George Box (1996), a leading theorist in experimental design believes that statisticians must move out of their "test-oriented mind-sets and take up a critical catalytic role in the process of discovery and development".

The traditional lecture method of instruction is not likely to disappear. It has been seen to be both cost effective and time effective. We must extend the traditional passive role played by the student by actively engaging them in the learning process, and develop programs in which principles and thinking can be taught using applications of immediate interest in a manner allowing the use of familiar learning strategies. As the experimenters of yesterday we need to become the inventors for tomorrow.

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Graphs: Communication lines to students?

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Graph comprehension is considered a basic skill in the curriculum, and essential for statistical literacy in an information society. How do students interpret a graph in an authentic context? Are misleading features apparent? Responses to questions about a graph-based advertisement suggest that students commonly did not appreciate scaling difficulties, relate a graph as relevant in the context of a standard interpretation task, or apply numeracy skills for calculations based on data in graphical representations.

According to a proverb, a picture tells a thousand words, though just which thousand words may depend on how the picture is "read." In our information age, displaying and interpreting graphical information is fundamental to being statistically literate. The increasing presence and variety of graphs associated with claims in the media raises the potential for misleading the public, with the claim of authority, "Statistics show...". Some misrepresentations may be inadvertent errors, such as producing a pie chart accounting for 128% of whole. Others may use purposely misleading representations, such as cutting off the vertical axis of a bar chart to make differences appear relatively greater than they in fact are. As advertisers use more graphs to convince potential customers of the merits of their products, readers need to recognise misleading representations and protest their use.

Recent curriculum documents (Australian Educational Council [AEC], 1991, 1994; Ministry of Education, 1992) have emphasised data representation as a part of the data handling curriculum relevant to mathematics, science and social science. In *A National Statement on Mathematics for Australian Schools* (AEC, 1991), statement B5 suggests students should "represent, interpret and report on data in order to answer questions posed by themselves and others," including examples such as

- Represent data in tables and graphs and compare different representations of the same data, considering how well they communicate the information (e.g. correct, clear, misleading).
- Discuss and interpret information presented in graphs and tables found in newspapers, magazines and text materials. (p. 172)

Similarly, in *Mathematics - A curriculum profile for Australian schools* (AEC, 1994, p. 93), element 5.27 includes the example requirement to

- Interpret bar graphs and histograms for grouped data, including where the scales on the axes must be read between calibrations.

Research involving students' comprehension and interpretation of statistical graphs appears to have been less than prolific. With respect to young children's understanding and use of graphs, Pereira-Mendoza and his co-workers have identified a number of areas of possible misunderstanding: interpreting scale, distinguishing missing values from zero value data, mis-using topic knowledge when predicting, making unfounded assumptions about patterns, finding intermediate values, and manipulating bar graphs (Pereira-Mendoza & Mellor, 1991; Pereira-Mendoza, Watson & Moritz, 1995; Watson & Pereira-Mendoza, 1996).

Research by Curcio (1987) with students in grades 4 and 7 led her to suggest three levels of questions teachers should ask in relation to graph comprehension: (1) reading the data, (2) reading between the data, and (3) reading beyond the data (Curcio, 1989). At level (1), the "reader simply 'lifts' the facts explicitly stated in the graph, or the information found in the graph title and axis labels, directly from the graph" with no interpretation (p. 5). At level (2), the reader "requires the ability to compare quantities ... and the use of other mathematical concepts and skills ... that allow the reader to combine and integrate data" (p. 6). At level (3), the reader must use background knowledge "to predict or infer from the data" (p. 6). Similar distinctions have been used to set levels of questions about a bar chart (Watson & Moritz, 1996a) and a

pictograph (Moritz & Watson, 1997) involving (1) literal reading of data values, (2) calculations using intermediate results to reach a conclusion, and (3) predictions based on the given data as well as topic knowledge. While these levels are reasonably distinct, it was found that some level (2) questions, such as the highest value, were easier for students than level (1) questions, because they did not require reading the scale of the graph for the data value to answer the question.

More broadly in statistics education, Watson (in press) has suggested a hierarchy of statistical literacy which involves (i) having knowledge of basic statistical terms, (ii) recognising, interpreting and using these in applied contexts, and (iii) being able to question unrealistic claims made by the media or others. While applied more widely to concepts relevant to items presented in the social contexts of the media, it has similarities to the requirements of Curcio's levels of graph comprehension. Certainly reading beyond the data is necessary for questioning claims associated with graphs which are made in the media. Examples of the applications of the statistical literacy hierarchy include explaining the nature of a pie chart which sums to 128% (Watson, in press), and drawing a graph to represent a claimed cause-effect relationship between automobile usage and heart deaths (Watson & Moritz, 1996b).

The longer your overseas call, the cheaper the rate.

(a) Explain the meaning of this graph.

(b) Is there anything unusual about it?

Suppose the standard rate is $1.00 for 1 minute.
You have already talked for 30 minutes.

(c) How much would the next 10 minutes cost?

(d) How much did the first 30 minutes of the phone call cost?

Figure 1. Graphing item from media survey about Chance and Data

This report aims to explore the ways students interpret a graph in the context of a potentially misleading newspaper advertisement in the light of the hierarchy for statistical literacy above. The item chosen is shown in Figure 1. The graph presents an interesting challenge to statistical literacy and may cause the observer to question the motives of the company which produced it. Both the vertical and horizontal scales are

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non-linear and the vertical axis is truncated with a somewhat unusual feature of values appearing to decrease with height. Many basic numeracy skills are required to interpret the graph and the implications of the information it contains. In Part (a), students are expected to describe the meaning of the graph to indicate basic understanding of the representation in context. Part (b) asks students to identify any unusual aspects of the graph, which allows students to critique the advertisement at the third level of the statistical hierarchy, by questioning the representation. Parts (c) and (d) require numeracy skills involved in reading values off the graph and then interpreting them in context with calculations. The calculations in Parts (c) and (d) are the sort which might be expected of members of the community who want to know of the benefits offered by the company. Particular attention hence is given to the ways students employ numeracy skills to interpret the data represented and to the ways students evaluate how well the graph and the entire advertisement represent and communicate the information.

Method

The item in Figure 1 is from a media survey which was part of a study of student understanding of concepts in the chance and data curriculum (Watson, 1994b; Watson, Collis, & Moritz, 1994). The survey was administered during 45 minutes of class time to over 1800 students in Grades 6, 8, 9 and 11 at Tasmanian government schools during 1993 and 1995 (sample sizes for groups are given with the results). This report compares performances at different grade levels, and considers change in performance for students who answered the item in both years. Analysis of individual categories of response will use a diagrammatic procedure for mapping responses (Watson, 1994a) to illustrate the structure of responses and analyse common errors. This procedure highlights the data used in problem solving, the concepts and processes involved, and how these are combined to produce intermediate and final responses.

Results

Part (a) Describing the graph

Responses about the meaning of the graph often involved minimal re-wording of the advertisement title, corresponding to Curcio's (1989) literal "reading the data." Students often lifted the title from the advertisement to explain the meaning, or re-worded it slightly, often preferring "cost" to "rate." In this respect, it is not evident whether these students understood this distinction.

> The longer your overseas call, the cheaper the rate. [Grade 9]
>
> The longer you are on the phone, the less you pay. [Grade 9]

Some responses did express in the students' own words a way of interpreting the graph. The first response below appears to initially use the title to make a simplified statement about cheaper calls, but goes on to describe the percentage discount, without clarity as to whether this discount applies to the final time interval or to the entire call. The second response makes clear the discount rate applies for different time intervals.

> The longer you spend talking on the phone to an overseas person the cheaper your call becomes, e.g., if you spend 26 minutes on the phone you get 15% off the cost of your call. [Grade 9]
>
> Standard rate gets cheaper as the minutes pass, while talking on the phone ie. first 3 mins. costs standard rate, next 1-6 mins costs 3% less than standard, next 1-5 mins costs 5% less than standard, next 1-10 mins costs 10% less than standard. [Grade 11]

The following responses illustrate a variety of ways that students misinterpreted the graph, showing they did not appreciate the variables written as the two axis labels.

> It tells you how many calls you make. [Grade 6]
>
> It shows you when to ring, e.g., ring 4-10 October and it's 3% off your normal phone call. [Grade 9]

These disturbing responses may give an indication why many were unable to make sense of Parts (c) and (d) discussed below. Overall less than 10% of responses could be

confidently categorised as going beyond the title to state the relatively complex meaning of the graph.

Part (b) Questioning unusual features of the graph

Most students responded that they considered nothing unusual about the graph. Some responses considered the graph unusual based on what students think a phone or a graph should look like.

> There's no way you can do that to a phone. In the last phone, the ear part of the phone wouldn't reach when you're talking in it. [Grade 9]
>
> They used phone instead of a line graph. [Grade 6]

Some responses had difficulty comprehending that decreasing rate and increasing cost are compatible. The first two responses below questioned the claim, based on the students' misunderstanding that it referred to reduced cost rather than reduced rate. The final two responses appear to understand the distinction, and then proceed to question the clarity of the advertisement and consider how it might be misleading for some.

> Yes, the longer you call it costs extra for lines etc. Why does it get cheaper? [Grade 8]
>
> It should cost more the longer you talk but it doesn't. And if you talk for a great amount of time it might end up being a 100% off. [Grade 9]
>
> Yes, at first glance you'd think you could stay on there all day and not pay anything, but you can't. [Grade 9]
>
> The presentation is a bit silly (the phones). The untrained eye might think their call actually got cheaper! [Grade 9]

Some students were able to comment on the unusual nature of the representation with respect to the reversed scale, which appeared to be recognised as unusual in contrast to standard expectations, almost assuming patterns in graphs.

> The highest rate of discount is at the bottom - the start is higher and it declines when graphs usually incline. [Grade 9]
>
> It's not drawn to scale. It's kind of back to front. You'd think the big phone would represent big saving. [Grade 9]

Other responses commented on the non-linear aspect of the scales, acknowledging that this gives a visual misrepresentation which might mislead people.

> The space from 3% - 5% is the same as the space form 10% - 15%. [Grade 8]
>
> On the graph, the 15% mark is around 1/4 of the original price. Once you reach 26 minutes the charge stays fixed. [Grade 9]
>
> The size of the phone handle is not representative of what percent you get off. [Grade 9]
>
> The prices only go down a fraction, and not alot, like most people would think it was. [Grade 9]

Table 1 shows the distribution of categories of response by grade level. Less than 10% of students commented on the unusual scale or misrepresentation of the graph. The high proportions of students offering no response, or a response that nothing was unusual about the graph, suggest that while between 25% and 49% of students (depending on grade) appreciate that generally this graph may be unusual, few students have reached level (iii) of the statistical hierarchy in relation to this task.

Table 1
Percentage of Responses to Part (b) by Response Category and by Grade

Response Category	Grade 6		Grade 8	Grade 9		Grade 11
	1993	1995	1995	1993	1995	1995
Scale/Reverse	1	2	1	9	6	8
Phone/Cost	24	32	31	40	37	35
No/Yes	58	52	50	37	43	35
NR	17	14	18	15	15	21
N	*286*	*216*	*295*	*329*	*279*	*179*

Parts (c) and (d) Applied use of the graph with calculations

Responses to Parts (c) and (d) varied greatly in structural complexity. The mappings of responses shown in Figures 2 to 9 illustrate paths by which intermediate and final responses may have been reached by students, and assist in explaining errors in calculation or interpretation. For Part (c), the simplest structural form of response involved using the cues "10 minutes" and "rate of $1/min" to yield the response of "$10 cost," as shown in Figure 2. A similar response which also uses the cue "30 minutes" results in a response "$40 cost," is shown in Figure 3. The most significant feature of these responses is that they show no evidence that the graph plays any role in the students' reasoning for the task at hand. The majority of Grade 6 students responded in one of these ways, as is seen in Table 2. The percentage decreased with increasing grade level.

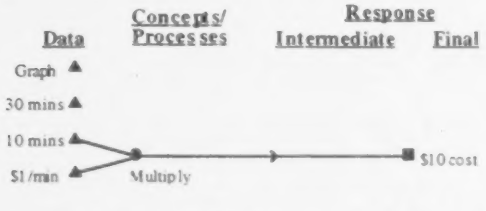

Figure 2. Map of $10 response to Part (c)

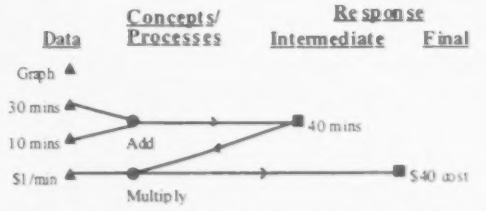

Figure 3. Map of $40 response to Part (c)

Of those who did employ the graph in their response, some students used the graph along with a single other cue to select a value to read off the graph. Those who used the cue "10 minutes" responded with an inappropriate response of "3% off," as shown in Figure 4. Other students responded "15% off" by using the cue "30 minutes" to select the appropriate value to read off the graph, as shown in Figure 5. These students, however, did not go on to use this to calculate a correct response of the cost for another 10 minutes. The category of responses related to "15%" included those who went on to use "15% off" to give responses of 15c, 85c, or $1.50. These inappropriate or incomplete attempts together represent between 3% and 12% of responses for various grades in Table 2.

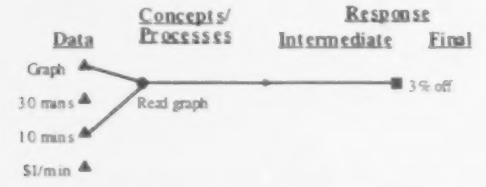

Figure 4. Map of 3% response to Part (c)

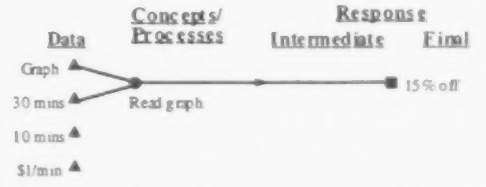

Figure 5. Map of 15% response to Part (c)

The correct answer of $8.50 can be obtained in several ways, two of which are shown in Figures 6 and 7. The need for greater structural complexity to find the correct answer is seen in these figures. The percentage correct increases with grade (Table 2). Other responses which did not fall into one of the categories above were either considered as non-response (NR), or placed in the "Other" category. Included in the latter were responses of $7.50, although it is not clear how students achieved this response, i.e., whether they misinterpreted the graph or made a calculation error.

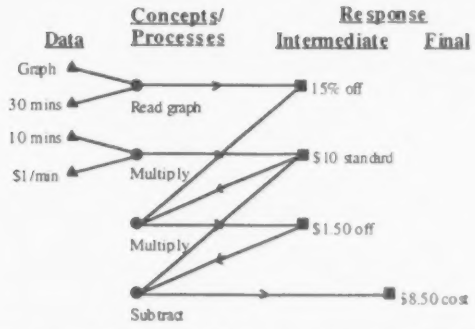

Figure 6. Map of $8.50 response to Part (c)

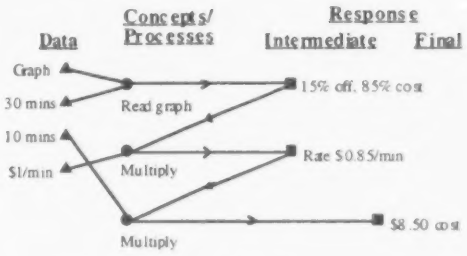

Figure 7. Map of $8.50 response to Part (c)

Table 2

Percentage of Responses to Part (c) by Response Category and by Grade

Response Category	Grade 6		Grade 8	Grade 9		Grade 11
	1993	1995	1995	1993	1995	1995
$8.50	4	3	10	21	14	28
15%/$1.50/85c	2	2	3	6	9	11
3%...	4	1	3	4	1	1
$10/$40	64	64	43	21	25	16
Other	14	14	15	24	20	18
NR	12	15	26	25	30	27
N	*286*	*216*	*295*	*329*	*279*	*179*

For Part (d), similar categories of response were found - those which ignored the graph to respond $30, and those which used the graph to get a percentage value, some of which went on to use to calculate a cost, either based on a 15% discount for the entire call yielding $25.50, or based on the correct interval stages of discounts to get a correct result of $27.79. Figures 8 and 9 show this is a more complex process than Part (c). Figures 8 shows an outline which involves recognising the discount must be calculated in five stages and performing calculations represented by the box for each stage. Figure 9 shows in detail the steps involved in one such calculation of the cost for a time interval, which is very similar to Figure 7, although the time interval must be calculated from the graph, rather than used as given from the "10 minutes" cue for Part (c).

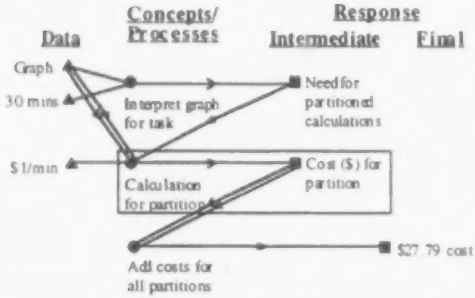

Figure 8. Overview of $27.79 response to Part (d)

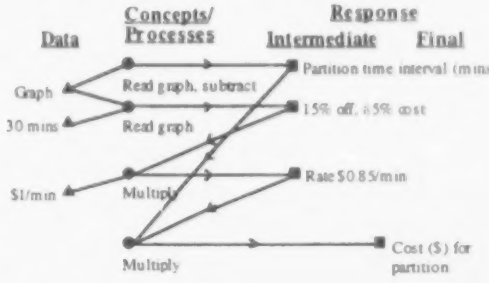

Figure 9. Detail in calculating a partial response to Part (d)

Some of the responses between $20 and $30 were categorised as using a correct approach but involving a calculation error. For example, a number of students responded $23.12. This value is $4.67 less than $27.79 and would result if a systematic error were made in calculating the time interval by subtraction (see Figures 1 and 9), yielding all time intervals one minute less than the correct value. Table 3 shows the percentages of responses in each of the categories above by grade. Again the majority of Grade 6 students responded "$30," apparently seeing the graph as irrelevant to the task. This approach decreased in frequency for older grades but very few were totally successful even at Grade 11.

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Table 3
Percentage of Responses to Part (d) by Response Category and by Grade

Response	Grade 6		Grade 8	Grade 9		Grade 11
Category	1993	1995	1995	1993	1995	1995
$27.79	0	0	2	3	2	2
($27.79)	2	1	1	9	4	12
$25.50	4	1	5	8	9	10
15%/$4.50/...	<1	1	3	4	5	7
$30	62	69	46	25	29	21
Other	15	15	15	26	18	21
NR	16	13	28	26	34	27
N	*286*	*216*	*295*	*329*	*279*	*179*

In analysing responses given by the same students longitudinally 1993 to 1995, many students either did not attempt the questions in both 1993 and 1995, or were of indeterminate level (NR/Other), and so were eliminated from longitudinal analysis. Of 114 students who were assigned a level based on the "highest" level for Parts (c) and (d) for both 1993 and 1995, 35 (31%) of students improved, 8 (7%) regressed, and 71 (62%) stayed at the same level. This latter figure was mainly due to 49 Grade 6 students who gave responses consistently ignoring the graph, that is $10 for Part (c) and $30 for Part (d). This suggests that when minor fluctuations of functional performance are taken into account, very few students actually improved their performance over the two year interval.

Discussion

Several important points arise from the responses of students to this item. One dominant finding is the extent to which responses to calculate cost ignored the graph altogether, suggesting that many students could not interpret the graph in context as it applied to Parts (c) and (d). Asking for "cost" may trigger a calculation mode, which excludes consideration of the graph. These students have not reached the second level of statistical literacy in this context. Experiences with handling raw meaningful data may be useful to these students in establishing links between the data itself and representations of the data in different graphical forms.

Another finding is the lack of use of the numeracy skills which are required for completing the tasks in Parts (c) and (d). Many students who did know how to read the graph could only state individual facts based on extracting one aspect or data value from the graph. To go further, it is necessary to understand rate in context, to understand discount, to calculate unusual interval lengths where subtraction of 'endpoints' is invalid, and to have on overview of a multistep procedure. Putting all of these together was very difficult for these students.

While it may be encouraging that about one third of students noticed aspects of the representation which may be unusual, although not technically incorrect, it is somewhat worrying from a statistical literacy point of view that so few noticed the uneven scaling of both axes. There was also a tendency for some students to look for significant patterns in the graphs (cf. Watson & Pereira-Mendoza, 1996) as a way of finding something "unusual". This is not an appropriate methodology in this context.

The results may disturb educators but please advertisers. If between one third and one half of people see nothing wrong with this advertisement, or believe that it "costs less" the longer you talk, it may be that the advertisers have achieved their objective. Students need to be challenged in the classroom using non-standard graphs, even those with errors, to question why the author has represented a message in certain ways and to be on the lookout for misleading representations.

Acknowledgment

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Using Videotapes in Classroom Research to See Anew

Judith A. Mousley

Videotapes can store a comprehensive data bank that can be revisited many times and examined through different lenses. This paper outlines how videotapes are being used a PhD project, and tells how data were revisited as new theories developed. The taped data supported the development of new levels of understanding about interactions in mathematics classrooms as well as about processes of interpretive educational research.

Videotaping Classrooms

Videotaping is used in classroom research in two main ways. First, videotapes are used to stimulate recall by teachers and students, in order to provoke participants to reflect on and talk about their interpretations of experience. Researchers and their subjects view selected snippets of tape, or perhaps watch whole tapes with either person able to stop the tape at any point (e.g. Marland, 1994). Free association can be used (e.g. Hsi & Hoadley, 1994), but more commonly the researcher asks probing questions (e.g. Argyris & Schön, 1974; Barnett, 1991; Hoyles, Armstrong, Scott-Hodgetts, & Taylor, 1984; Owens, 1996). Interpretations by researchers can be triangulated (see McFee, 1992) or even developed in conjunction with classroom participants. While this "stimulated recall" is a feature of my PhD methodology, it is an aspect of video use that I will discuss in other papers.

Another common use of videotapes is to capture classroom interaction, to enable its close analysis by a researcher in isolation. Here, videotapes have some advantages over field notes in their detail—they hold audio images, complete with pauses, inflections, volume, and relevant background noises. They also hold more complex data that audiotapes because they record teachers' and students' physical gestures, facial expressions and movement around the teaching area. The data can be revisited many times in order to be examined through different lenses or for various purposes. This revisiting, as it has become part of my PhD project, is the focus of this paper.

Methodology

The PhD project is entitled *Teachers' constructions of their roles in building mathematical understanding*. Specifically, the focus of my attention is on what teachers think mathematical understanding is and what they do to develop it. Four case studies of teachers' work are being developed. The teachers are from one school: two have Year 6 classes and two teach Year 2. In this school, two teachers look after each grade level, and they generally plan their work together. Starting with the *Curriculum & Standards Framework* (BOS, 1995) and the school's pre-determined "themes" that are used to integrate subject areas, pairs of teachers outline a work program for the whole year. This involves arranging concepts and skills into what seems to be a

logical order, taking account of what the children have been taught. The teachers then work together in planning specific learning activities. Worksheet and test preparation is shared by taking turns.

Pairs of teachers sharing preparation, aiming to teach the same content through the same activities, and then using the same assessment instruments, provides a useful context for studying how individual teachers in similar positions construct their roles differently. While the intended curriculum and planning are common for many lessons, the implemented and realised curricula are somewhat different for many reasons—so that is one aspect of my analysis.

Videotapes in Use

This paper will focus on data gathered from the two Year 6 teachers, "Russ" and "Jill". I videotaped as many of their mathematics lessons as I could attend over a period of ten weeks. Whenever possible, after taping a lesson I interviewed the teacher about what s/he wanted children to understand by the end of the lesson, how s/he had tried to develop that understanding, which children in particular had understood well or not understood the concept(s), and how the teacher knew this. Periodically, semi-structured interviews were used to explore the teachers' backgrounds, as well as their beliefs about such factors as group work, assessment, parental expectations, and so on. The focus of these longer interviews is on how different elements of the pedagogical environment seem to support or constrain teaching for understanding. All interviews were audiotaped and transcribed.

Initial data analysis involved categorising field notes, snippets of videotape and interview transcripts. The initial categories used were drawn from an initial review of early literature on "Understanding" (e.g. Dilthey, 1927; Heidegger, c.1926; Hume, 1748) and from current literature on understanding in mathematics education in general (e.g. Bastick, 1993; Mevarech, 1995; Skemp, 1992) and of children's understanding of particular mathematical concepts (e.g. Pitkethly, 1994; Truran, 1994)

The videotapes of each lesson were compressed to digital format (see Mousley, Sullivan and Mousley, 1994, for a brief outline of this process). They were then watched several times to identify pertinent pedagogical incidents that seemed to fit the pre-determined categories. When necessary, new "emergent categories" (Glaser & Strauss, 1967) were formed, to accommodate the behaviours observed in the videoed lessons as well as the actions that the teachers claimed they use to develop understanding. Each snippet of video that seemed relevant was entered into a spreadsheet, along with a code summarising its origins, start and stop frame numbers, and a short descriptor to aid future recall (and assist in computer searching for particular incidents, questions, etc.)

Figure 1 shows data from the beginning of one spreadsheet. (As this paper is not being read from a computer monitor, you will need to imagine the video symbol to be interactive, in that when it is clicked the relevant video snippet plays.) Also noted on the spreadsheets were details that I wished to follow up with the teacher during later interviews.

Code	Teaching action	Start	Stop	Details
131.01	Stimulating recall	36	60	"With our work on endangered species …"
131.02	Eliciting children's explanations	60	150	(Various children, until Brad mentions "scale"). * PRE-DECIDED CONCEPT PRIVILEGES RESPONSE. * OTHER PURPOSES OF THIS TYPE OF INTRO?
131.03	Using b/board	100	170	Writes scale of model car, to illustrate Brad's example.
131.04	Emphasising language	90	201	Repeats, stresses terms: *Scale*, *Compare*
131.05	Use of example			"I was just saying … no comparison for whale size."
131.06	Using aid	165	201	"On that chart …"
131.07	Accepting un-invited example	210	260	Ryan (inaudible - do audio - scale used in mapping) * SEEMS TO UNDERSTAND IN CONNECTED WAY -- IS THIS TYPICAL OF RYAN?
131.08	Stimulating recall	210	260	"When we did our map reading …" (Picked up Ryan's context. Look at effect of other uninvited examples, whether used.)
131.09	Eliciting children's explanations	210	260	"How did you work it out in km?" Using scale to calculate length --Essence of the lesson.
131.10	Child's explanation	260	340	Stacey: Clear articulation of her understanding (Idea of ratio)
131.11	Stimulating recall	340	349	Do you remember what the scale we had was? * DID NOT PICK UP STACEY'S RATIO (RELATIONSHIP) IDEA? WHY?

Figure 1. A sample of coded snippets: *Teacher 1 (R), topic 3 (Scale, lesson 1)*

A similar process was used to categorise the audiotapes, typed-up observation and interview notes, and scanned versions of the worksheets used in the observed classes. The spreadsheet software's capability for sorting was then used to group data which had been classified in particular ways, and this sorted data was used to form indices to structure a multimedia framework.

Using the scripting program *Authorware Professional* (Macromedia, 1995), an interactive program was constructed so that categories and several levels of sub-categories of data can now be recalled and viewed, read or listened to with ease. For instance, clicking on the "Explaining" menu (see Figure 2), allows me to access (through menus and sub-menus) all categorised examples of teachers' and children's explanations, the teachers' actions which

promote explanations, and any interview comments or field notes that seemed to be related to explanation.

4. EXPLAINING

Claims about explaining
- Importance of
- Constraints to

Explaining strategies
- Verbal explication
- Using gestures
- Using diagrams or drawings
- Using aids
- Using recall, drawing on previous experience
- Using examples
  - Giving examples
  - Accepting uninvited examples
  - Eliciting examples

Eliciting children's explanations
- Questioning
- Prompting/provoking
- Inviting

Clarifying
- Providing clarification
- Seeking clarification
- Eliciting further information

Explaining activity
- Giving initial directions
- Explaining directions further
- Re-explaining directions

Figure 2. Working category 4: Explanation (at 2/2/1997)

My aim is not merely to describe observable behaviour, but to attempt the more difficult task of understanding action, which (as Wiseman, 1990, notes) includes the *intentions* of behaviour. I am conscious of many levels of subjectivity of this work, such as how the teachers themselves interpret their teaching actions, my re-interpretation of these, and the obscurities associated with terms such as "mathematics" and understanding". I am well aware that no matter how many times I revisit the data, only those actions, comments and events that I have paid conscious attention during the lesson will form part of my final framework. Such are the limitations and complexities of interpretive research.

The video and transcript snippets are often overlapping and multi-dimensional, i.e. applicable to several categories. The categorisation is temporary and transitory—this work is in preliminary stages and there are huge gaps and overlaps in the data as well as the analysis process. Each time I revisit the data in the light of recent reading, reflection, or subsequent data, the framework develops further and requires some reorganisation.

The videotapes have not only helped structure the framework to date by facilitating close examination of the action, but have also supported the development of theories that are "grounded in the data" (Glaser & Strauss, 1967). Being able to revisit the video data (as well as to arrange further interviews with the teachers, see Mousley 1996) has assisted in the development of my understandings, as it has enabled me to "see anew" (Mason, in press). The following is intended to give the reader some sense of how this activity has progressed.

Developing Different Types of Understanding

Water in the Bath is a popular lesson from the Mathematics Curriculum & Teaching Project (Lovitt & Clarke, 1988). I videotaped this lesson, during which groups of children had acted out a scene to give physical expression to

the shape of a particular line graph drawn on a worksheet. I instigated the usual interview after the lesson.

Judy	What understandings did you want the children to develop during this lesson?
Jill	Graphing the level of the water. Reading the graph … the axes. I thought it was a good lesson for the video. We all like hamming it up. I thought it would be a change for you. Different from children doing maths in their seats.
Judy	It was great fun. Entertaining … nice variety for the children. What was it you wanted them to understand about graphing?
Jill	Just the axes. That the line shows change. How the line represents the water level.
Judy	They seemed to understand that well, I think. It is difficult because when the line goes up the water goes down.
Jill	Yes. I had expected some confusion. My demonstration at the start gave them the idea, I think.
Judy	Yes. Do you think they understood the idea of the rate of change in the water level?
Jill	(no response)
Judy	The steepness of the line?
Jill	Yes. Some did.
Judy	How did you know?
Jill	K and P … their stories were spot on… and I was surprised at M and L.
Judy	Can you tell me more about that. What do you mean by "spot on"?

Lesson 261(Int)

The patterns of response here is very typical of that demonstrated in the post-lesson interviews with both teachers. Initially, "understanding" was taken to be understanding of the task, with emphasis on the physical processes involved (acting out graphs, drawing scale models, completing percentage or probability exercises, etc.).

Similarly, the teachers spontaneously suggested the simpler concepts that were involved (that the line shows changing water levels, that one whale is x times as long as another one, that percentage is an expression of a quantity out of a hundred, etc.) More complex mathematical understandings took more probing, and in many cases teachers responded to this questioning by naming children who had understood particular ideas. There was not one lesson where the teachers felt that the more difficult ideas were understood by all children.

This ordering of priorities was also evident in the videos—by far the greatest number of explanations, for instance, was focused on explaining tasks; followed by actions aimed at developing simpler mathematical concepts. The teachers' claims about individual children's levels of understanding were generally supported by the data, with higher-level concepts being developed in individual and small-group situations. Thus initial video analysis enabled me to link the teacher's actions (intentions, behaviours) with their perceptions of the lessons and their claims about what had been understood by their pupils.

Revisiting the Classroom Interaction

After taping about ten lessons, I became aware that the Year 6 teachers had been trying to develop the children's understanding of ratio in almost every

new lesson during the observation period. These lessons included three on scale (the basic idea, scale drawing and models, scaling up and down); two on rates (e.g. $/kilo, km/hr); one on rate of change (*Water in the Bath*); and one on probability. It struck me that these ideas were being introduced as quite separate skills and knowledge strands, but the underlying concept (ratio) was not attended to explicitly during any of these lessons. There were also other uses of rational number in mathematics lessons, such as reducing fractions to lowest terms in the "mental maths" exercises completed at the beginning of each lesson.

Having the ability to review and reanalyse the videotapes became invaluable at this point. Again, I appreciated the multidimensionality of the data collected, with its gestures, drawings, emphases and pauses—as well as the way the movement of teachers at different stages of the lessons, the record of which comments were directed to particular students, etc. This detail enabled my re-examination of the classroom interaction with a very different agenda—it made possible my "seeing anew".

New questions arose along this new avenue of inquiry. Had teachers made links that I had not seen? Was there evidence of children making connections between the various concepts? Was any teachers' or children's language used (or patterns of language use) the same or similar across the development of the concepts?

Such questions led me into a new level of analysis, but also to a broader field of inquiry and theory building—with a new range of questions. Do teachers (more generally) see links between the topics that had been taught? Does the format of national, state an school curriculum documents mitigate against teachers and children developing these connections? How do pre-service teacher education, professional development, and common assessment practices prevent or support the recognition and use of key mathematical ideas which underlie the school mathematics curriculum?

I realised that my questioning about relational understanding (Skemp, 1976) had not been probing enough. It had focused on the lesson content, but not the teachers' own understandings and the reasons for the ways that they think about separate topics and skills. For instance, I had several snippets of interview data such as the following:

Jill	I tried this year to do percentage first and go over the ideas several times before I really hop into probability and things like that, because if the children really have a solid understanding of percentage, hopefully it will be easier to use that understanding when it comes to interpretation of the stats, and probability, and things like that.
Judy	Do you think it is necessary for the children to see how those concepts are linked?
Jill	Yes, but if you do it that way you wouldn't get the learning outcomes done ... you just can't do that and cover the necessary work.
Judy	I noticed that you talked first about percentages being out of a hundred, and then you took fractions like six tenths being sixty out of one hundred, and started talking about equivalent fractions, and simplifying fractions or else making them up to hundredths. Do you think the children understood the links?
Jill	No. I think the hundredths one they do. They understand that link—percentages being hundredths. But it is obvious that we

need to go back and spend a lot more time linking vulgar fractions through to hundredths, through to percentages. So that is something we need to go back to do more work on. They were fine when it came to using calculators to work it out. That wasn't a problem, but that doesn't need much understanding. But we have run out of time. This is just a start, though—they will pick up percentages again in Year Seven, and Eight, and probably Nine.

Lesson 282 (Int)

Judy Do you see these two percentage lessons growing out of any of the work you have done in the last few weeks.

Russ Well, fractions. They have to see a percentage as a part of a hundred. That's why we start with hundreds squares and they color them in.

Judy Yes. That's really useful for modelling percentages. But then it is a big jump from colouring in hundredths to, say, converting three-fifths into a percentage.

Russ Well, they have done equivalent fractions before. It is important to use that knowledge. Some will get it—like B; but others like S won't. We have done them [equivalent fractions] in the mental maths sessions. You just have to hope they will make the links. You could spend a lot of time getting them to see how one works like the other, but you don't have the time to do that all the time. And even if you did, people like S would never get it. It is best just to teach her processes—even then she forgets them.

Lesson 182 (Int)

Conclusion

Luckily, I had arranged for my interviews to become extended "conversations" with these teachers, so I can probe further when I return to the school. I will also be using the categorised videotape snippets, along with discussion points noted during the analysis period, to have the teachers talk more about specific aspects of their work.

It is clear that videotaping can be a valuable tool in mathematics education research. For me, it has given new meaning to Skemp's term "relational understanding", in three ways. First, it raised questions about key concepts in early mathematics learning and how these could be used in the development of teachers' and children's understandings about the links between mathematical topics. Second, it made me wonder about aspects of teachers' lives that support or limit their conceptualisation and use of these ideas. Third, it made me realise that what we come to understand during the research process can move from the instrumental to the relational—and that videotape analysis can support this transition.

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Second grader's representations and conceptual understanding of number: a longitudinal study

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This paper reports the first phase of a 2-year longitudinal study of 120 Grade 2 children, which aims to describe the growth of numerical concepts and processes and children's representations of these. Most children successfully used ten as an iterable unit but were less effective when required to formulate and identify the pertinence of groupings. Performance on ratio and splitting tasks was relatively poor, showing lack of experience in halving and sharing situations. Low achievers were more likely to produce poorly organised pictorial and iconic representations that were lacking in structure. High achievers who used dynamic imagery presented their solutions in well structured but often unconventional ways.

It is not entirely clear how children use their representations in building conceptual understanding of number. Are children's representations of numerical strategies and situations a reflection of their developing concepts, or do their representations actually enhance the development of numerical thinking? Do children make connections between critical aspects of number learning through their representations? How do children's representations change over time?

A 2-year longitudinal study aims to analyse and describe, in the greatest possible detail, children's internal representations of number and the relationships children make between various numerical processes. The study seeks to describe and explain how children impose structure, or lack thereof, on numerical situations, and how imagery influences this process. Clinical interviews focus on aspects of structure in children's strategies and representations when solving tasks involving visualising, estimating, counting, grouping, re-grouping, partitioning and multiplicative reasoning. A 2-year longitudinal study will provide a more coherent picture of how young children construct their numerical representations, and whether these representations change and develop to accommodate more complex numerical structures over time.

Background

Recent studies have focused on the importance of children's construction of number concepts and processes by examining specific areas such as counting, addition and subtraction, or multiplication and division as separate areas of study. Studies focusing on numeration have primarily examined children's counting and grouping strategies (Bednarz & Janvier, 1988; Denvir & Brown, 1986a,b; Fuson, 1990; Hiebert & Wearne, 1992). Other recent studies analysing the growth of number concepts and processes indicate that children's representations of problem-solving situations are closely linked to their conceptual understanding and the way they construct mathematical relationships (Davis, Maher, & Noddings, 1990; Goldin & Herscovics, 1991; Maher, Davis & Alston 1993; Mulligan & Mitchelmore, 1996). Researchers have also highlighted children's representations as evidence of the idiosyncratic and creative ways in which they structure mathematical relationships (Thomas & Mulligan, 1995; Thomas, Mulligan & Goldin, 1996).

In recent years, research on multiplicative reasoning has looked to the early development of multiplication and division and fraction concepts, through essential processes such as counting, partitioning, grouping, unitising and 'splitting' (Confrey, 1994; Lamon, 1996). Studies investigating multiplication and division processes have identified the development of sound problem-solving strategies from an early age and the importance of modelling and representation in this development (Anghileri, 1989; Carpenter, Ansell, Franke, Fennema & Weisbeck, 1993; Clark & Kamii, 1996; Kouba, 1989; Mulligan & Mitchelmore, 1996). Mulligan & Mitchelmore (in press) found that the strategy children employed to solve a multiplicative problem did not necessarily reflect any general problem characteristic but rather the mathematical structure that the student imposed on the problem. The children's internal conceptual structures of multiplication and division, described in terms of intuitive models, were inferred from elements such as

grouping, partitioning, counting and patterning in their external representations and their explanations of these representations.

Imagery and Representations

A number of theoretical approaches are concerned with the way imagery is central to the construction of children's mathematical concepts and processes. The early work of Piaget and Inhelder (1971) referred to the importance of images as internal, holistic representations of actions which could be inspected and transformed. Other researchers have defined the mental processes involved in the child's construction of mathematical concepts as different types of image processing (Brown & Presmeg, 1993; Pirie & Keiren, 1992; Pitta & Gray, 1996; Sfard, 199; Thomas & Mulligan 1995).

Pirie and Kieren (1992) proposed a model of the growth of mathematical understanding described as a dynamic organising process involving 'image making' and 'image having'. The recent work of Pitta & Gray (1996) also draws attention to the way children use imagery to concentrate on different objects, or different aspects of the objects, which are integral components of numerical processes. They claim that the quality of the image formed from enactive approaches, is dependent upon what it is the child chooses to create an image of. This will influence the use to which the image is put. Children's interpretations of mathematical actions are so strongly associated with image formation that it is possible that children's interpretations of mathematical actions may be pre-ordained by their interpretation of the real world. Further research in the acquisition of numerical concepts may then find better explanations for children's difficulties by focusing on the imagery associated with the objects involved in concrete-abstract thought.

However, few studies have provided an in-depth analysis of children's imagery associated with essential and related features of number such as visualising, estimating, counting, grouping, regrouping and 'splitting'. Furthermore, there is growing need to examine children's construction of number concepts and strategies over time through longitudinal studies that monitor growth and change.

This 2-year longitudinal study funded by a large grant from the Australian Research Council, aims to describe the growth of children's conceptual understanding of number by examining qualitative differences in their strategies and representations of number over time. Although the study cannot provide very detailed descriptions of this 'growth' that might well be provided through case studies, it aims to describe patterns of growth and to identify relationships, based on 120 individuals' responses to a set of numerical tasks.

Research Methodology

To date, clinical interviews have been conducted with each child three times– once each in April and November of 1996, and once in April 1997, with the final interview to be conducted in November, 1997. Pilot work was conducted prior to these interviews to trial procedures, develop tasks and coding methods, and for training purposes (Mulligan, Mitchelmore, Outhred & Bobis, 1996).

Sample

The interview sample comprised 120 Grade 2 students, balanced for gender, and ranging from 6.5 to 7.5 years of age at the time of the first interview; of these 117 remained at the third interview. Twelve students were randomly selected from each of ten NSW Department of School Education primary schools randomly selected from three metropolitan regions of Sydney. The sample was representative of students from diverse cultural, linguistic and socio-economic backgrounds.

Interview Tasks

Seven categories of tasks were formulated from those trialled in pilot work that had a common set of characteristics: visualisation/ counting, ratio, grouping, regrouping and partitioning, estimation and 'splitting'. Grouping tasks focused on equivalent grouping using ten as an iterable unit, partitioning and estimating (Bednarz & Janvier, 1988; Cobb, Perlwitz & Underwood, 1996; Lamon, 1996; Mulligan & Mitchelmore, in press; Steffe, 1991). Numeration tasks focused on structural elements of the number system, visualisation and number sense (Hiebert & Wearne, 1992; Thomas & Mulligan, 1995; Thomas, Mulligan & Goldin, 1996). Other tasks focused on ratio and multiplicative reasoning such as arrays and 'splitting', and these were adapted from those used by

Confrey (1994), Mulligan & Mitchelmore (1996) and Outhred (1993). The tasks are shown in Table 1.

Table 1 Task-Based Interview Items

1. Numbers in Context: How old are you? What year is it? How old will you be in 2 years time? How old will you be in 5 years time?
2. Counting sequence; visualisation tasks: **2.1** Ask child to close their eyes and to imagine the numbers from 1 to 100, explain, then draw and write about what they see. Check for other sensory stimulation, e.g., movement, position, colour, or feelings associated with numbers. Repeat the activity to find what the children imagine for: **2.2** numbers past 100; **2.3** before 1; **2.4** before 0 **2.5** between 0 and 1.
3. Ratio: 3.1 You can trade 2 small stickers for a large sticker. How many large stickers are worth the same as 4 small stickers? **3.2** How many large stickers are worth the same as 2 large stickers and 4 small stickers?
4. Grouping/addition: 4.1 Show a collection of pencils in packets of 10 and as loose pencils. Ask students to show 26 pencils. **4.2** Can you show me 9 more than this? How many is that? **4.3** Ask children who are successful with above tasks if they can find another way to show 35. If child does not group; Can you arrange them to make it easier to count?
5. Regrouping, addition-ones, tens, hundreds: 5.1 Show lollies in bags, rolls & loose. Can you show me 125 lollies? **5.2** Show me 10 more than this. How many altogether now? **5.3** How would you make 100 more than this? How many altogether then ?
6. Estimation/grouping: Show a picture of 65 dots randomly drawn. Could you do something so that you will be able to tell me very quickly how many dots there are? What should you do?
7. Multiplicative: 7.1 Splitting (No materials) If you had 24 textas how would you share them fairly between 2 children (of the same age)? **7.2** If two more children came along, how would you share them fairly? **7.3 Multiple Count** Close your eyes and count in 3's as far as you can. What do you see in your mind when you count in 3's? Draw and explain. **7.4 Equal groups** There are 3 children and each child needs 7 textas. How many textas are needed altogether? (Provide textas and allow to draw). **7.5 Array** Show a 3 x 3 array of dots briefly. Cover. How many dots are there? Draw the array from memory. If unsuccessful show array and ask child to copy it. **7.6 Subgrouping** Use playdough to imagine and make a lolly snake. Show how to share 2 lolly snakes fairly between 4 children. How much do they get each?

Interview Procedures

All interviews were conducted by a trained research assistant in a small room separate from the classroom. The tasks were read to the child by the interviewer and were re-read to the child as often as necessary to assist them in remembering details. Paper and pencil were available on the table as well as the necessary materials for each task. The interviewer explained to the child that they could use the materials and pencil and paper to assist in solving the problem if they wished. The child was asked to represent their solutions by modelling, drawing and symbolising their mental images. They were also asked to explain their representation and their solution process. The interviewer asked whether the child could provide any other solutions or images. The child was given ample time to draw representations and were instructed that their drawings needed only be rough sketches.

Task 1 was designed to establish rapport with the child and identify whether the child could use concepts of age and time. The visualisation tasks (Tasks 2.1 to 2.5 in Table 1) were asked prior to other tasks so that the influence of representations used in the other tasks could be minimised. The tasks were administered for all interviews in the order shown in Table 1. The interviews lasted from 15 to 45 minutes with an average time of 30 minutes. When a child gave an explanation that appeared to conflict with the interviewer's observations, they were asked to describe, model or draw what they were thinking at that time.

Data Coding

The interviewer recorded children's responses as they solved each problem, drawing diagrams of children's modelling and noting explanations, gestures and finger movements. A coding system was developed from pilot data for each task category and responses were noted immediately in these categories. This included coding as incorrect,

uncodable and non-attempt. Children's drawings were collected and coded according to the identified categories. Additional categories appeared for Task categories 2 and 7. Each interview was also audiotaped. A second trained research assistant checked the interviewer's coding, notes and audiotapes and clarified uncertainties in consultation with the chief investigator and the interviewer. In all, 19 errors were found and corrected. The coding definitions were used to check for consistency and an 89% agreement rate was found. Two research assistants also checked the data entry and the coding sheets for consistency.

Analysis of Strategies and Representations

Data from the first interview were initially analysed for performance and strategy use to identify some general trends. Additionally, children's drawings and their explanations were analysed for three characteristics:

(a) the type of representation (pictorial, ikonic or notational, including identification of colour and position);

(b) the evidence of structural development of the concepts and processes shown in the strategies or representation; and

(c) evidence of the static or dynamic nature of the image.

For example, pictorial recordings were defined as pictures drawn or oral descriptions of objects given by a child. Iconic recordings included drawings of tally marks, squares, circles or dots that represent numbers. Notational recordings were distinguished by the predominant use of numerals and symbols. The level of structural development was interpreted from structural elements (i.e. grouping, regrouping, partitioning and patterning) given in the responses and in the recordings. The static or dynamic nature of the response was defined according to whether the recordings and children's explanations of their representations describe fixed or moving (changing) entities.

Results

The present paper presents initial analysis of Interview 1 data comparing general performance and strategy use across a range of number tasks. Some exemplars of children's drawn representations will be discussed in the presentation of the paper.

Table 2 shows a general classification of children's representations for the five visualisation tasks 2.1 to 2.5. These data are presented in three main categories: pictorial recordings, notational recordings, and children who explained that they saw 'nothing in their mind'. A separate category of 'dynamic imagery' indicates the percentage of responses using imagery of number that was moving or changing or 'rolling across their mind like a screen'. Table 2 indicates that the majority of children gave notational representations for visualisation task 2.1 (76%), task 2.2 (68%) and task 2.3 (80%) respectively. Notational representations included numerals in a variety of sequences, positions and arrangements. Pictorial responses resembled drawings of animals and objects.

When asked to visualise numbers 'before 1' (task 2.3), it is noted that 70% of children visualised the numeral zero with 10% identifying other numerals. For the visualisation task 2.4 (before 0) there was a clear distinction between those children who could describe and symbolise fractions such as 1/2 and those who explained there was 'nothing', with 64% of children unable to visualise anything. Of the 30% of children who gave notational responses, 23% gave negative numbers and 10% explained that there was an 'infinite number of numbers'. Most children were unable to imagine anything for visualisation task 2.5 (between 0 and 1), but of the 30% who responded, 20% of responses indicated images such as zeros becoming larger and larger, common and decimal fractions, and negative numbers. Table 2 also shows that dynamic imagery was evident in 17% of responses for task 2.1 but diminished markedly for the other tasks. Many of the examples of dynamic imagery were produced by children who performed well and these recordings included drawings, patterns and diagrams that were highly idiosyncratic and well organised.

Table 2 Percentage of Responses by Type of Representation for Visualisation Tasks

Task	Type of Representation			
	Picture	Numeral	Nothing	Dynamic Imagery*
2.1 Visualisation 1-100	14	76	10	17
2.2 Visualisation Past 100	14	68	18	4
2.3 Visualisation Before 1	9	80	11	3
2.4 Visualisation Before 0	6	30	64	0
2.5 Visualisation Between 0 and 1	10	20	70	0

* Percentage of responses using dynamic imagery

Table 3 compares the performance and main strategy type for task categories 3 to 7. The success rate on the grouping and regrouping tasks (tasks 4 and 5) was very high, with the ratio and estimation tasks moderately more difficult. The ratio task 3.2 and 'splitting' tasks 7.1 and 7.2 were the most difficult with 70%, 58%, and 76% of students unsuccessful respectively.

Table 3 Performance by Strategy Type for Ratio, Grouping, Regrouping, Estimation and Splitting Tasks

Task	Performance		Strategies		
	% Correct	% Incorrect	No Stickers	Stickers	
3.1 Ratio 2:1	75	25	38	37	
3.2 Grouping/ Ratio	30	70	13	17	
4.1 Grouping			Tens	Ones	
	95	5	67	28	
5 Regrouping			Ones	Multiples	
100's,10's,1's	91	9	18	73	
6. Estimation/ Grouping	80	20	Tens 42	Others * 27	Random 11
7.1 Splitting/ Share 24/2	42	58			
7. 2 Splitting/ Reformulate	24	76			

n = 118

* Others refers to groups of twos, fives or unitary methods

The ratio task 3.1 was solved by 75% of children by visualising or modeling the task with stickers to show the relationship 'two for one'. Those children who were unsuccessful were unable to model the relationship and either added the numbers or guessed. For the more difficult ratio task 3.2, only 30% of children could successfully use 'two for one' as a unit to determine the total, with many children counting by ones the total number of stickers in an undifferentiated way. Solution strategies for grouping task 4.1 are shown as two categories 'tens' and 'ones' showing 67% of children using ten as an interable unit. However, 28% of children counted the pencils by ones even though they could identify that there were ten pencils in each packet. Similarly, in the regrouping task 5, 73% of children used ten as an iterable unit.

In contrast, the estimation/grouping task 6 distinguished children's grouping strategies in a situation where there were no pre-determined groupings. Only 43% of

children used tens to group dots, with 27% of children using groupings of two or five. The other 30% of children were unable to form groups, systematic or otherwise, even though they could articulate the task requirements. The 'splitting' tasks 7.1 and 7.2 proved to be difficult for children, with less than half able to share 24 textas between 2 children by halving, dealing, counting, or use of known facts. Reformulation strategies in task 7.2 were successfully used by those children who used effective halving and dealing strategies. The results of the splitting tasks are representative of difficulties children experienced in the other multiplicative tasks with the exception of the equal grouping task which was solved correctly by the majority of children.

Discussion of Results

Preliminary analysis of the first interview data has yielded unique evidence of the diversity of children's numerical understandings and strategies and their representations. Although some children were able to form groups of ten and use ten an iterable unit, there was an over-reliance on unitary counting methods. Although the analysis is incomplete, the data showed many cases where individuals were unable to justify the pertinence of grouping in many tasks. These children also lacked a type of flexibility in their thinking; they were only able to replicate models of groups, arrays or situations which had been produced or provided by others. It appears, at this stage, that the poor performance in ratio and multiplicative tasks can be attributed not just to lack of experience in these situations, but to the primitive idea that unitary counting can be used to solve everything.

There was sufficient evidence to show that children's imagery influenced the way they visualised and solved a range of numerical tasks. Low achievers were more likely to produce poorly organised, pictorial and iconic representations that were lacking in structure. High achievers who used dynamic imagery, presented their solutions in well structured but often unconventional ways.

Most striking was the evidence of how children created and imposed structural aspects on their solutions to number tasks. In cases of low achievers, children were unable to visualise any structural characteristics and were so focused on their idiosyncratic interpretation of reality that the mathematical tasks and the concrete objects embedded in them had a completely different meaning to that intended.

On the other hand, there were some cases where the children represented mathematical structures to solve a range of tasks. This included number patterns and relationships involving ratio and proportion and competent use of negative numbers. There were children who had already developed a coherent system for dealing with numbers and these children were also able to visualise numbers in a structured way. They could produce a variety of responses to the same task, and explain that there were similarities between their responses to a number of tasks.

The critical aspect of this investigation is that children's representations reflect their conceptual understanding to some extent. However, it appears from the data so far that we need to focus more closely on why the abstraction of concepts is determined by the concrete or visualised reality of the child. There is evidence that some children are unable to move from concrete to abstract thinking, or visualise mathematical situations at all. Further analysis needs to focus on how children use imagery in the formation of concepts and why some children are unable to create or use an organised 'structure' in developing concepts.

After completion of data collection, the analysis will focus on describing the changes that occur in the development of an individual's number knowledge and how children relate visualising, counting, grouping, estimating and multiplicative processes. In particular, the research will address these questions: What are the critical aspects of imagistic thinking that influence the growth of numerical understandings? Can the difficulties faced by many students in counting, grouping, regrouping, and multiplicative reasoning be attributed to methods of instruction, or intuitive processes in the early formulation of number concepts? How can we use new information about the way imagery influences children's representations of number in the investigation of other mathematical concepts?

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COOPERATIVE PROFESSIONAL DEVELOPMENT FOR MATHEMATICS TEACHERS: A CASE STUDY

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A project of professional development for mathematics teachers was organised in which teachers were employed as facilitators to lead groups of their peers for professional growth in the area of performance-based assessment and reporting. Although a large number of teachers were recruited for the role of facilitator, a small proportion only actually conducted sessions and stayed on in the role. Insufficient training on the topic itself and the skills for the leadership role beforehand and negligible support during the project were the main reasons for the high drop-out rate. Instances of successful professional development occurred where the teachers already had these skills or received adequate training.

Introduction

The literature on professional development of teachers includes many studies of programs in which classroom teachers have taken on roles of responsibility such as facilitator and adviser. These programs fall under Glatthorn's (1987) umbrella term of cooperative professional development, which includes the concepts of cooperative development, colleague consultation and peer coaching. Cooperative professional development is a process by which small teams of teachers work together, using a variety of methods and structures for their own professional growth. The objective of cooperative professional development is to facilitate reflection about practice, helping teachers "become more thoughtful decision makers" (p. 143).

One successful program in which teachers assisted other teachers in their professional development was reported by Kent (1985). In that program, teachers were trained to act as advisers and facilitators to their colleagues, in changing classroom practice and improving the quality of learning in their classrooms. There were three major findings reported: (i) facilitators sometimes encountered jealousies of teachers who were not moving into new roles, because of the additional training and teamwork provided for these new roles, (ii) some schools and districts feared losing control of teachers, because the teachers were empowered to become active problem solvers participating in organisational decisions, and (iii) when teachers took on roles other than classroom teaching, they needed additional skills and knowledge to work effectively with adults, implement change and serve as curriculum consultants.

In another cooperative professional development program, it was found (Goodwin & Lieberman, 1985) that although the teachers taking on professional development roles had extensive skill and experience for their normal class teaching roles, much learning was acquired on the job from their new roles in implementing school improvement programs. In a study of *change agents* in urban school improvement programs, Saxl and Miles (1985) documented the general and specific skills identified in change agents (i.e. facilitators). Among the general skills were interpersonal, group-related, content-related and communication skills. More specific skills gained were initiation, rapport-building, task-oriented, educationally-focussed and autonomy-increasing skills.

Another aspect of facilitator-led professional development was highlighted by Guskey and Sparkes (1991) who showed that the effectiveness of professional development is enhanced if teachers had opportunities to share their best practices, trial new ideas in their own classrooms, assess student understanding and bring feedback to the next in-service session to share with colleagues. A study by McLaughlin (1990) indicated that seven particular strategies were effective when applied in a professional development project. Included in the seven strategies were the notions of professional development occurring over an extended period of time, and regular teacher meetings focussing on practical issues related to the topic.

The case study

The professional development project

In 1994, the Queensland Association of Mathematics Teachers (QAMT) received a grant for professional development of mathematics teachers in the area of assessment and reporting. The purpose of the project was to assist mathematics teachers at primary and lower secondary level to implement Student Performance Standards (SPS), the local version of the National Mathematics Profiles (Australian Education Council, 1994), to broaden their assessment and reporting practices, and subsequently to enrich their teaching methodology. Full details of the project can be found in Cooper, Bleicher, Dole, Nisbet and Warren (in preparation).

It was assumed in the organisation of the project (i) that for teachers to change their teaching practice, there was a need to change their knowledge, beliefs and attitudes to new techniques (Clarke & Hollingsworth, 1994), (ii) that beliefs and attitudes change when student outcomes are seen to improve (Berliner, 1986), and (iii) that opportunity should be provided for teachers to reflect on the techniques. Also, it was assumed that successful professional development occurred in peer cooperation situations (Glatthorn, 1987), in which discussion of best practice and sharing of ideas would take place. Hence, the project was built around employing teachers as facilitators.

A complication for the project came as a result of a major review of the Year 1 to 12 school curriculum in Queensland, commissioned by the state government. Among the many recommendations of the report by the review team (Department of Education, Queensland, 1995) was a proposal to appoint 94 educational advisers (EAs) in each of the areas of literacy and numeracy, to assist classroom teachers with the identification of students having difficulties and the correction of these difficulties. The applicants for these positions included many of the teachers targeted as facilitators for the project. The result was a major depletion of the pool of facilitators available to the project.

The project described in this paper can be classified, in Glatthorn's (1987) terminology, as a cooperative professional development program. Classroom teachers and advisory teachers were recruited to take on the role of facilitator for professional development of teachers of Year 1 to 10 classes, in the area of performance-based assessment and reporting, and SPS in particular.

The role of facilitator was seen as crucial in the network of people involved in this professional development project. Key people were nominated in each region of the state to assist in identifying and recruiting classroom teachers and advisory teachers who could act as facilitators for the project. The role of facilitator was to conduct workshops for participating classroom teachers on various aspects of performance-based assessment and reporting, and SPS specifically. Facilitators were expected to offer to conduct workshops in their areas for teachers of all school systems on relevant topics, and after approval by the project officer, prepare and conduct the workshops. A fee of $50 per hour was paid to facilitators for their seminars and workshops, which took a variety of formats. Some were held during the week, and these usually lasted between 2 and 3 hours. Others were held at weekends, and lasted one or two days. All workshops were conducted out of school hours, due to restrictions imposed by the funding body.

It was envisaged in the initial conceptualisation of the project, that facilitators would not be mere instructors but sharers of their wisdom of practice (Shulman, 1986) in implementing SPS in their own classroom. Ideally, a facilitator would conduct a series of workshops with a group of participating teachers, and over time, the facilitator and the participating teachers would meet to discuss their experiences with the implementation of performance-based assessment and SPS in their own classes and schools.

Recruitment, training and support for facilitators

The recruitment of facilitators began during the official project launch, which took the form of a video conference conducted by satellite across 38 centres in Queensland. Also, invitations were issued to all members of QAMT (the professional association in the state) to volunteer as facilitators; key people in each region of the state were invited to nominate classroom and advisory teachers who were seen as potential facilitators.

Just weeks after the project launch, facilitator training events were organised and conducted in Brisbane (the capital and largest centre of the state) and four other regional centres throughout the state. The program for these weekend training events included (i) Background to SPS; (ii) Materials available; (iii) Planning a professional development activity; and (iv) Teaching approaches for mathematics learning. It was planned also that a package of purpose-designed materials to assist facilitators run seminars and workshops be developed and available at the training weekend. However, this did not eventuate.

Just under 300 potential facilitators attended the training weekend in Brisbane, 40 in Townsville, 35 in Cairns, 20 in Rockhampton and 20 in Mt Isa, which made a total of over 400 potential facilitators expressing an active interest in the project at that stage. However within six months of the training seminars, only 48 of the initial volunteers had conducted a course or remained actively interested in being a facilitator.

A suite of in-service training packages were planned for the project, covering topics such as SPS, assessment and reporting techniques, and the implementation of SPS in all the major strands of the mathematics curriculum (Number, Space, Measurement, Chance & Data, and Working Mathematically) at the three school levels of lower primary, upper primary, and lower secondary. However the suite of in-service packages did not eventuate, due to various factors, such as lack of support in the management committee, changes within the Education Department, and later the resignations of the project officer and liaison officer. Only four of the 18 packages were written, and only three published. A Newsletter was started in 1994, but was not kept up during the course of the project. The planned computer-based communication network was not started until late 1995 and was not developed to any extent to be useful to facilitators or teachers.

Courses (workshops/seminars) conducted by facilitators

A total of 107 professional development events (seminars and workshops) were conducted by facilitators in the first half of 1995. The content of these courses centred mainly on Student Performance Standards (SPS) and its implementation.

A high proportion of the courses conducted contained material of a general and introductory character either about assessment planning and techniques in general or explicitly in terms of the elements of SPS. The remaining courses covered topics that involved problems of making judgments about levels of performance demonstrated in work samples in various strands and SPS implementation issues. The numbers of courses conducted by individual facilitators ranged from 1 to 12. Most facilitators conducted one, two or three courses, and only five facilitators conducted more than four courses.

By the end of Semester 1, 1995 (June 1995), only 10% of the potential facilitators had actually conducted at least one course, and only a handful of others were still interested in the role. The fact that only a small number of facilitators conducted a series of workshops over a period of time with the same participating teachers indicates that this aspect of the project (sharing classroom experiences relating to SPS with a facilitator on a regular basis) was lost in many cases. Approximately half of the events conducted during the project were one-off workshops or seminars, the other half were components of series of two, or more events.

The best examples of series of professional development events occurred in regions outside the metropolitan area. For instance, one key person in a rural area gathered a group of local facilitators (EAs), sought extra training for them and then supervised a program of significant professional development for classroom teachers. In the words of the key person -

> "We were able to provide extra training for the EAs here because we saw that as part of an on-going commitment. I encouraged them to go to conferences and I lobbied with their co-ordinators that they would be released to go to conferences wherever possible. ... Each of them were then able to pick up the professional development that suited where they were at. I think that's a very important part. ... You have to have on-going support."

Another example of significant professional development occurred in a collaborative writing group which was set up to produce assessment tasks (Dole, Cooper, Nisbet & Warren, 1996). A group of 12 primary teachers were brought together to produce a package of exemplary assessment tasks, rich in their potential to provide data on students'

mathematical understanding and knowledge, and thus link with the national profile as a reporting framework. The group had two facilitators, one of which was a university mathematics education lecturer and the other was a member of the project team. The first facilitator acted in the role of participant observer, and the project's research officer attended the meetings as an observer. Data on the project were collected through fieldnotes based on researchers observations combined with ad hoc group interviews, plus an end of in-service survey to ascertain effect of the program on participants.

The writing team met together four times over a four month period during the school year. Meetings were for the whole day and were organised in a three session format: (1) an opening session where participants discussed assessment issues in the first meeting and shared their experiences in trialing the assessment tasks with other teachers in the second, third and fourth meetings; (2) a second session where the teachers brainstormed new assessment ideas and techniques in small groups in the first, second and third meetings and refined tasks in the last meeting; and (3) a final session in which individuals or dyads planned the trials of the assessment ideas from the second session in the first, second and third meetings and organised final writing and publishing in the final session. This format afforded the opportunity for teachers to share their classroom experiences of trialing new ideas and techniques of assessment with critical friends/colleagues. At the end of the final session, participants completed a feedback survey which covered positive and negative aspects of the program, changes in teaching and assessment practices, attitudes to SPS, support networks, suggestions for improvement, and confidence in running similar programs.

Through involvement in the writing team, it was found that teachers actually guided their own professional development. The writing team situation provided teachers with (i) support, (ii) feedback, (iii) opportunities for reflection; as well as assisting them gain (iv) confidence in using, and (v) knowledge and understanding of, the national profile. The results support the following three conclusions: (1) the collegial support of the collaborative writing process resulted in increased confidence for the participants with respect to teaching and assessment; (2) the collegial feedback from the classroom trials and discussion with other teachers improved the teaching and assessment ideas of the participants; and (3) the reflection encouraged by the process on classroom practice improved the teaching and assessment practices of the participants and improved students' learning outcomes.

Feedback about facilitators' training and support

Written feedback from the first facilitator training weekend (held in Brisbane) indicated that the majority of the facilitators believed that they did not receive adequate training. Only 19% reported that the training seminars were good overall, while 59% indicated that they believed that they needed further training if they were to become trainers in SPS and related topics. A large proportion (41%) wanted to learn more about performance-based assessment and SPS and the curriculum strands in particular. It was clear from the feedback that the participants believed that they should have been given greater information, training and guidance in the area of conducting workshops and seminars for teachers on these topics. Many would have appreciated receiving support materials such as a facilitator handbook, presentation packages on recommended workshops topics, handouts and overhead transparencies, in order to conduct seminars and workshops. Some predicted that they would probably need to keep contact with other facilitators and receive support from the project team from time to time, in the form of a newsletter, for example. Feedback indicated that the majority of participants left the training weekend with a lack of confidence in being a facilitator, and a need to know more about SPS and workshop presentation method. They were not sure about how the project was going to be organised in their local areas. There was, nonetheless, a positive attitude in most participants that these things could be remedied before they began any seminar or workshop presentations. There was a general feeling of enthusiasm and a positive attitude that this professional development project had the potential to really make a difference in the classroom if facilitators received more training and support during the project.

Following analysis of the feedback from the Brisbane training weekend, it was recommended that the program for the subsequent training events be modified. It was suggested that (i) opportunity be given for participant discussion of their expectations of the

weekend, the project and their involvement, (ii) greater emphasis be given early in the weekend to the essentials of SPS and student-based assessment and reporting, and (iii) other priority sessions be planned on guidelines for workshop presentation, and setting up support networks between facilitators and key people in regions. Because of the short time between the Brisbane training weekend and three of the other four, these changes were not able to be implemented, except for the weekend at Mt. Isa, which occurred much later. Not surprisingly, feedback from the training seminars in Rockhampton, Townsville and Cairns was very similar to that from the initial Brisbane weekend. In particular, the potential facilitators needed greater knowledge of the essentials of SPS, more suggestions on how to implement SPS in mathematics topics in the school syllabus, more training in how to present in-service sessions, and more resources to use in planning and conducting in-service in SPS.

It is significant to note that of the 400 (approximately) teachers who initially had expressed interest in being a facilitator, only 10% went on to conduct a course in first semester 1995 (February to June). A key question was - What happened to the potential facilitators? It was known that some had become numeracy advisers but there was no information about the others. A telephone survey, therefore, was conducted with a large sample of potential facilitators to determine what had happened to them. It revealed that 40% of those interviewed were unsure about being a facilitator, 30% were not interested any more (other commitments or lack of interest in SPS), 20% had taken a new position or had left teaching, and only 10% were willing to continue as facilitators. Of the 40% who were unsure about being a facilitator, many indicated that they might become interested if they received adequate training in SPS and the role of facilitator. Fortunately, a number of teachers who had been appointed as numeracy advisers retained their interest in being a facilitator and conducted courses for the project. In fact, the greater proportion (almost all) of the facilitators who actually conducted courses were Educational Advisers (EAs) or people in positions of responsibility such as heads of department, regional in-service coordinators or university lecturers, or members of the project team. Those who conducted more than three workshops or seminars were all in this category.

By the time a training weekend was held in Mt Isa, the structure of the sessions had been modified taking on board the comments and criticism of the previous events. Hence, the sessions were perceived by the facilitators attending as being more relevant and useful than those conducted at prior events. The majority thought that they received the information than they expected and needed to conduct professional development.

The importance of appropriate training and the positive effect it had on facilitators was noted by the key person from the rural area example quoted above.

> "I think the EA positions demonstrate that classroom teachers do make excellent mentors to their colleagues. But, you've got to take them out of the classroom and give them some kind of training and on going support.

Reflections and implications

The findings of this study confirm a number of the conclusions from published studies of cooperative professional development. These findings relate essentially to drop-off rate and the need for training and support, and are considered firstly in relation to the project overall and then in relation to two successful cases.

Project overall

The large drop-out rate of facilitators was a significant feature of this professional development project, and should not come as a surprise considering the results of previous studies documented in the literature and the limited extent of preparation of the facilitators in this project. The literature has shown that when teachers take on a role such as facilitator, they need additional skills and knowledge (Kent, 1985), including general skills such as interpersonal, group-related, content-related and communication skills, and more specific skills such as initiation, rapport-building, task-oriented, educationally-focussed and autonomy-increasing skills (Saxl & Miles, 1985). Analysis of the feedback and inspection of the training program agenda, reveals that there was insufficient emphasis on these topics, as well as the target topic, itself, namely SPS. Moreover, it is clear that the training program for

facilitators was too limited in time: clearly, one weekend is too short to prepare teachers for the facilitator role in terms of the recommended skills. Pressure of time on the project team forced a compromise in this regard.

The limited training of facilitators lead to very few of them being able to conduct a sustained program of workshops over a period of time. Consequently, the impact of the project on the implementation of SPS and performance-based assessment in Queensland was quite limited. Very few participants took part in a *series* of seminars and workshops, and hence most participants had no opportunity to share their best practices, trial new ideas in their own classrooms, assess student understanding and bring feedback to the next in-service session to share with colleagues, as recommended by Guskey and Sparkes (1991), and to do this at regular meetings over an extended period of time, as recommended by McLaughlin (1990).

Successful cases

On the positive side, the impact of the project was quite significant in some particular cases. In some areas of the state a number of facilitators, who were also educational advisers (EAs) conducted programs of seminars and workshops, and many teachers participated in a series of events. Feedback from participants showed that these events were worthwhile in assisting teachers learn about SPS and implement performance based assessment and reporting procedures.

In the case from a rural area quoted above, the key person personally recognised the need for facilitator training in SPS itself and in seminar presentation skills, as advocated by Kent (1985). As a result, the facilitators gained much knowledge about performance-based assessment and were in a position to conduct professional development for local teachers. Regular seminars and workshops over an extended period meant that participating teachers benefited from these in-service activities.

In the case of the collaborative writing group, the facilitators were experienced in the role and had appropriate background in assessment in mathematics. The collaborative writing process proved to be successful in providing participants with significant professional development. Factors that were strongly affected throughout the program were the attitude and beliefs of the participants, particularly confidence and self-esteem. Therefore, the program supported the findings of Clarke and Hollingsworth (1994) that changes in teaching practice is preceded by changes in attitudes and beliefs. As well, the program supported the findings of Berliner (1986) that beliefs and attitudes change when student outcomes are seen to improve, and the findings of Guskey and Sparks (1991) that trialing and sharing are crucial for effective in-service.

Final comment

Overall from this project, it can be said that the potential for significant effect did exist initially with the model of cooperative professional development as it was envisaged with this project, and that it did occur for a limited number of facilitators. These facilitators had, or gained, knowledge about performance-based assessment plus skills in the role of facilitator beyond that of regular classroom teachers. However, the potential was not realised with the majority of volunteers due to the lack of knowledge about performance-based assessment, and lack of confidence, skills and experience in the facilitator role. The project did not adequately deliver the required training program or support material. Hence, the major proportion of potential facilitators did not elect to participate in the project. It can be said that the project did not fulfil the purpose of helping the majority of teachers in the state reflect on their assessment and reporting practices and, in Glatthorn's (1987) terms, "become more thoughtful decision makers" in this area of classroom practice. The implications for future programs of professional development are significant in terms of selection, training and support of facilitators.

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Problem Solving in Kindergarten: The Development of Children s Representations of Numerical Situations

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This paper describes the development of nine kindergarten children's representations of numerical problems over five months. The children solved a range of problems by modelling them with concrete materials, then drawing their own representations of the problem situations. The representations became more structured, with quantities delineated in various ways, including the use of letters and labels. By the end of the year, all the the children were writing their own problems and many of them were also representing problems in symbolic form.

Introduction

Representation must be considered to be a key theoretical construct in mathematics education because much of mathematics may be regarded as the representation of one structure by another and an understanding of the extent to which the original structure is preserved in the representation (Kaput, Luke, Poholsky and Sayer, 1987). The term "representation", however, is difficult to define as it can be considered to encompass physical situations, symbolic systems and mental constructs (Goldin, 1992). Until recently the importance of representations has not been emphasised in mathematics teaching despite its central role in mathematics. Lopez-Real and Vello (1993) found that the Year 5 and 6 children in their study drew diagrams for only 5% of 693 "diagram-suitable" problems. When the children were asked to draw diagrams and use these to solve problems, children gave correct solutions to approximately one third of the problems that they had previously answered incorrectly.

The relation between internal and external representations in the traditional view of mind as opposed to a constructivist approach has been explored by Cobb, Yackel and Wood (1992) who believe that mathematical meanings given to representations result from interpretation and construction by students. The notion that students develop individual interpretations of concepts, has changed the emphasis in mathematics teaching from students' answers to their solution methods. While individual perceptions may differ and representational systems may be personal, Thomas and Mulligan (1995) suggest that the further the representational system has developed structurally, the more coherent and well organised will be students' external representations.

At present little is known about the ways in which student's internal conceptions and representations generate external representations, such as pictures, during problem solving (Mariotti & Pesci, 1992). A crucial question would seem to be "What factors influence the development of student's representations and what is the teacher's role in assisting this process? The use of concrete materials has been often suggested as a way young children can model solution processes. The value of concrete representations is that they can mirror conceptual structures so a child can use the structure of the representation to construct a mental model of the concept. However, students may not see the correspondence between the structure of the material and the structure of the concept. This inability to recognise structural similarities has been suggested as a reason why concrete representations do not always assist students to learn about particular concepts (Hart, 1987; Janvier, 1987; Lesh, Landau, & Hamilton, 1983).

Perceiving and representing structural similarities requires an abstraction of the essential features of the problem and such abstractions may be difficult for students. Beveridge and Parkins (1987) found that provision of a diagram indicating a solution method was effective in helping students solve a problem when the representation was such that students recognised the structural correspondence between diagram and problem. Dubinsky (1989) suggests a major concern with visual representations,

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especially those that are created by students, is that "It may be true that a picture is worth a thousand words, but what if it is the wrong picture?" Key information may have been omitted or be shown in a way that students find difficult. Thus, if a representation is provided students may not recognise structural similarities between a situation and its representation, however, for students to create their own representation requires knowledge of the conceptual structure and articulation of its essential features.

One important factor in recognising and representing structural features would seem to be experience of translations among the different types of representations, as well as translations within each. This factor has been emphasised by Lesh, Landau, and Hamilton (1983). These authors consider that the act of representation may facilitate the emergence of concepts and representations during problem-solving sessions as students use different representation systems to solve problems. These authors found that in realistic problem-solving situations good problem-solvers were usually able to switch to the most convenient representation at any point in the solution process.

It is not easy to teach students to create and use representations. Two factors have been suggested as important: incorporation of numerical information from the problem into the representation and clear depiction of the relationships among problem quantities (Lopez-Real and Vello, 1993). The nature of the problem itself may also influence the representation. If students cannot draw a problem representation they may not understand its conceptual structure. Lester (1996) believes that the following results clearly emerge from the literature on problem solving:

- "Students must solve many problems to improve their problem solving ability.
- Problem solving ability develops slowly over a prolonged period of time.
- In order for students to benefit from instruction, they must believe that their teacher thinks problem solving is important.
- Most students benefit greatly from systematically planned problem solving instruction.
- Teaching children about problem solving strategies and heuristics and phases of problem solving does little to improve students' ability to solve mathematics problems in general." (p. 666)

The first two points would suggest that problem solving should start as early as possible in a child's schooling. Indeed Carpenter, T., Ansell, E., Franke, M., Fennema, E., & Weisbeck, L. (1993) showed that kindergarten children,when encouraged to model or represent actions or relationships can solve multiplication and division problems. These authors felt that consolidating and extending children's intuitive modelling skills might provide a framework for developing problem solving in the primary school. Another crucial factor may be the role of the teacher in assisting students to develop representations of mathematical structures. Lester (1996) suggests that teachers' attitudes to problem solving as well as their knowledge of planning problem solving instruction are key findings from research.

The present study builds on the findings of Carpenter et al. (1993) as it provides information about the development of kindergarten student's representations of problems over a period of approximately five months.

Methodology

The children in this study all came from one kindergarten in a medium socio-economic area of Sydney. They were separated into four ability groups at the beginning of Term 2 in Kindergarten, although these groupings were not fixed and some children later changed groups. In Terms 2 and 3 each group worked on problem solving once a week during the mathematics activity time. The groups rotated through different activities over the week; the teacher worked with the problem solving group while volunteer mothers worked with the other three groups.

The collection of the data on which this paper is based was serendipitous. The teacher of the class kept all drawings the students made in problems solving sessions and just before these were given to parents at the end of the year, the first author made copies of the annotated drawings of nine children as examples for student teachers. These

included four children from the most advanced group, and two from the least advanced group. The children whose drawings were copied were selected because they used dark pencils that enabled the drawings to be photocopied. Only later were the drawings examined with the aim of examining the changes in the representations over time.

The teacher who collected the data for this study was in her first year of teaching. She has a strong commitment to developing children's independence and to involving them in thinking about their own learning. She endeavours to put these principles into practice in her teaching and exemplifies the finding reported in Lester (1996) that for students to benefit from instruction in problem solving, they must believe that their teacher sees it as important.

The contexts of the problems (see Table 1) were based on either literature read in class or familiar situations such as the children's bedrooms or the school playground. The first problems the children solved sharing situations because the initial impetus for beginning problem solving began with the nursery rhyme "Baa, Baa, Black Sheep". The teacher modelled three people and three bags of wool with different coloured cubes and said "We have three bags of wool, how many would each person get? The children replied "one", so she repeated the question for six, then nine, bags. As the children had little difficulty with this concept she asked "What if there were ten bags?" One response was to give a bag back to the sheep, the other was to share out the wool.

In the problem solving sessions the teacher began by reading the problem to the children and showing how it could be modelled with cubes. Next, children modelled the situation individually. This step was crucial because the children could not read the problem so the coloured cubes acted as a memory aide. The children were then asked to draw their own picture to show how they solved the problem. Considerable emphasis was given to showing their thinking and aspects of children's pictorial representations were discussed in their group session. Groups generally attempted the same problems although some children attempted additional problems while others did not attempt them all and required more support.

As well as problems involving repetition of a number pattern and representing three dimensional situations in two dimensions, the problems (see Table 1) included the following types (Carpenter & Moser, 1984; Carpenter et al, 1993): addition (combine); subtraction (combine and separate); multiplication (equal groups); division (partitive); and fractions (one half). After one month the teacher introduced the children to the idea of writing their own problems by asking them to make their own shopping list using a specific number pattern and imagining situations (a journey, the bedroom of their dreams). Later, as well as solving given problems, they also wrote their own problems.

Results

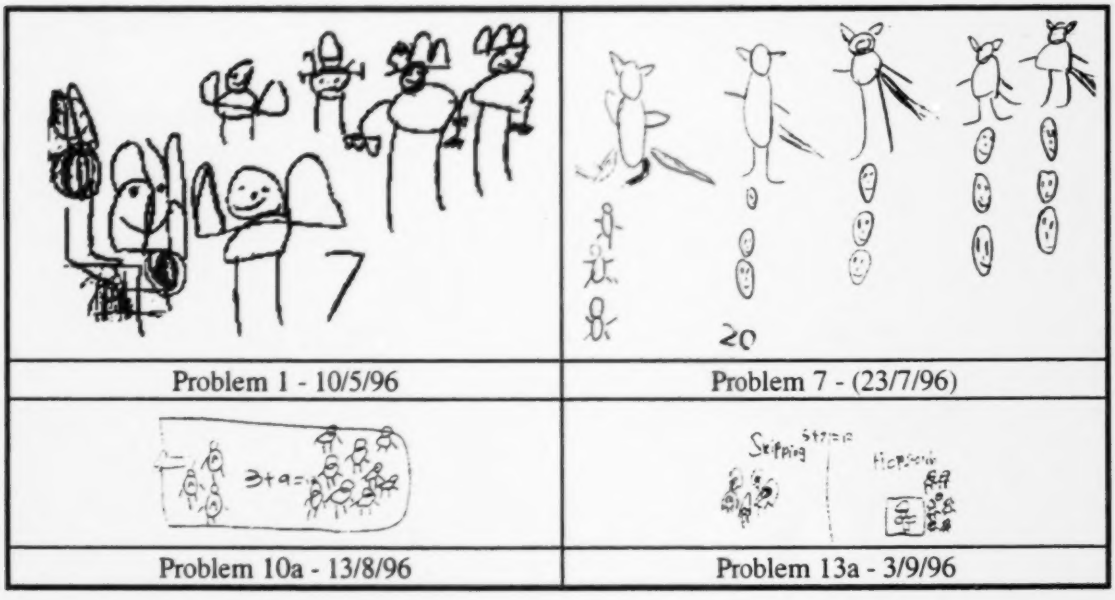
Since the teacher gave the children differing levels of support and there are results for only some children in the class, the number of children correctly solving each problem is not given. While the results will focus on the children's drawn representations, it should be noted that by the end of the year all the children in the class modelled the problems using concrete materials and understood the concept that one cube represented one unit.

The children's drawings developed markedly over the two terms. None of the children's initial drawings were structured but the later drawings of all children showed evidence of their solution strategies and most children's representations were increasingly organised in the way groupings were depicted.

Table 1 The problems used in the problem-solving sessions

Date	First problem	Second problem
10/5	1a Two monsters were playing, two monsters were flying, three monsters were dancing. How many monsters were at the monsters' party?	1b Four monsters are at a party. There are eight little cakes with cherries on top. How many cakes will each monster get?
22/5	2a Grandma has baked 6 cookies. Tom, Lizzie and David love her cookies. How many will they get each?	2b There were 18 cookies on the plate. Tom was very hungry and ate half of them. How many cookies were left on the plate?
29/6	3a Three dinosaurs each had two mittens. How many mittens altogether?	3b Derek the Dinosaur knitted 4 pairs of socks. How many dinosaur feet were kept nice and warm?
4/6	4a In the story "The Shopping Basket", Jeffrey had to buy lots of things. How many things did he have to buy?	4b Make up your own shopping list using the same number pattern.
11/6	5a In the story "A Lion in the Night", people chase the lion out of the castle, over the fields, into the forest, past the church, into the boat, across the sea, over the mountains and into the fields. Imagine that you are going on a journey to a magic castle. Draw a map of your journey.	
18/6	6a In "Alex's Bed", Alex rearranges his bed to make more space. Think about your bedroom and the things that are in it. Draw a plan of your bedroom.	6b Imagine that you could have the bedroom of your dreams. Draw a plan of how it would look.
23/7	7a In "Amy's Place", Amy discovers some possums investigating her new treehouse. If there were 5 mother possums, and each had three babies, how many would there be altogether?	
30/7	8a In the book "Bear and Bunny Grow Tomatoes", Bear grew lots of large, juicy tomatoes and shared them with his friend Bunny. Bear gave Bunny two boxes of tomatoes. Each box had 8 tomatoes inside. How many tomatoes did Bear give Bunny?	
6/8	9a Creepy Crawly Caterpillar had 16 legs to walk with. After he came out of the stony prison, he only had 6 legs. How many pairs of legs did Creepy Crawly Caterpillar lose?	
13/8	10a There are twelve people at Sammy's swimming party. Three are adults and the rest are children. How many people at the party are children?	10b Eight children at the party are trying not to get sunburned noses, 5 are wearing suncream and the rest hats. How many are wearing hats?
20/8	11a In her jewellery box, Janet has 6 gold rings. She also has 7 silver rings with diamonds. How many rings does Janet have in her jewellery box?	11b Make up your own problem about Janet's jewellery box.
26/8	12a One afternoon in the rainforest, there were 9 flying foxes hanging from a gum tree. Some of the flying foxes flew away to look for food, leaving four flying foxes hanging from the tree. How many flying foxes flew away?	12b Write your own problem about the KS rainforest
3/9	13a There were twelve children in the playground at lunchtime. Seven children were playing hopscotch and the rest were skipping. How many children were skipping?	13b Write your own problem about the school playground.
17/9	14a Geoffrey went to the Zoo to see all the animals. He brought along ten slices of bread for the elephants. When he had finished feeding them he had two slices of bread left. How many pieces of bread did the elephants eat?	14b Write your own problem about the zoo.
5/11	15a In the story "Alexander, Who Used to be Rich Last Sunday", Alexander spent all the money that his grandparents gave him. If Alexander had twenty dollars and he bought three lollipops that cost two dollars each, how much money would he have left?	

Figure 1 shows the change in one child's representations (Jeffrey) over four months. His first drawing has no structure, but in the second problem session he drew lines to indicate the links between cookies and people. Later drawings show increasing use of structure, for example, the array showing the five mother possums, each connected to a group of three babies for Problem 7. By session 10 he had begun to label group elements ("A" for adult) and writing equations, first with assistance (10a), then independently (10b). In an independently completed assessment task (13a) Jeffrey's drawing shows clearly separated, labelled groups as well as the corresponding equation.



Problem 1 - 10/5/96 | Problem 7 - (23/7/96)

Problem 10a - 13/8/96 | Problem 13a - 3/9/96

Figure 1 Jeffrey s representations of four problems

The concrete materials (cubes) that the children used to model the problems were coloured, so it would be expected that they would use colour as a means of depicting problem quantities and relationships. Indeed, use of colour was one of the first strategies children used. However, they developed a variety of other strategies in their drawings including :

- size and pictorial details (large; small; skipping ropes, etc.);
- separation for subtraction and addition;
- crossing out and partitioning of sets for subtraction;
- drawing lines to indicate sharing relationships;
- array structure to show equal groups in a multiplicative situation;
- letters and words to label elements of sets or sets.

While the use of size, pictorial details, and separation to show groupings would seem to be a natural part of drawing a picture for young children, the use of letters to label group elements or words to label the groups themselves, might not be predicted. The impetus for labelling came from one child who suggested using letters to show adults and children. This idea was adopted by a number of the other children, including Jeffrey (see Figure 1). The stimulus for this strategy may have come from the emphasis on initial sounds in early reading. It was noticeable in the drawings that children adopted representational strategies used by other children in the same group.

Use of equations

In Session 10 one child asked how equations were written using "plus" signs so the teacher worked through an example with the group and several children began writing equations, at first with assistance. Soon six of the nine children did not seem to have difficulty writing both addition and subtraction equations when these could be clearly

related to the children's representations of problem structure. The children who wrote equations tended to draw groups that were delineated in some way; that is, separated, labelled, or shown with different pictorial details (e.g., figures wearing hats). Because the children were translating from a written problem to a concrete, then a pictorial and finally to a symbolic form, production of an equation was a mapping of the concrete or pictorial representation rather than of the written problem. Thus, it is not clear whether the children could have solved the equations if there was no accompanying representation. However, one child (Craig) realised that the same problem could be represented in two different ways, writing 4+5 =9 and 9-5=4 for Problem 12a. Therefore, it would seem likely that the children would be able to dispense with the intermediate representation as they gained counting skills. At this stage the children were using a count-all strategy to determine the answers to the problems.

Children had difficulty when they attempted to write equations for multistep problems such as Problem 15. The teacher did not model this problem for the group in which the children were most confident about writing equations. The children who successfully wrote equations (Craig and Briony) worked out the first step (how much the lollipops would cost) with concrete materials, marked off six dollars in their drawing, then wrote 20-6. However, children who tried to show the relation between dollars and lollipops in their drawing became confused when they tried to write the equation. They focussed on the number of lollipops and wrote the following expressions: 20+3=23; 20-3=14; and, 20-3=16.

Self-generated problems

The week after the children began writing equations the teacher asked them to write their own word problem about Janet's jewellery box (Problem 10b). For this problem all the children replicated the operation of the problem that they had just solved (Problem 10a) and wrote addition word problems. In general, the operations used in the self-generated problems were modelled on those in the immediately preceding problems. Problems 12a, 13a and 14a were not simple structures to model as the initial and final states were given and children had to determine the change. The seven children who attempted Problem 12b generated take away situations (birds flying away, koalas climbing down trees, etc.) whereas four of the five children who completed 13b wrote addition problems. Problem 14b, however, produced a variety of responses. Two children from the group that needed most support dictated addition problems:

> There was a giraffe, a tiger and a monkey. How many animals were there? (Naomi)
>
> There was a person, a parrot and a monkey walking at the zoo. How many were there? (Adam, with prompting)

Four other children modelled the same operation as the original problem (subtraction) but the structure of their problem was simpler; instead of a change situation (10-x = 2) they made the result the unknown.

> There was one lion and 10 fairy penguins. The lion ate two of the penguins. How many were left? (Craig)
>
> There were 5 kookaburras and one snake. The snake ate two kookaburras. How many were left? (Anthony)
>
> There was one monkey and 4 bananas. The zoo keeper took away one banana. How many bananas can the monkey eat? (Isabella)
>
> There were 4 birds and one lion. The lion ate one bird. How many birds were left? (Kay)

All these children included three parameters in their problems. This did not occur in earlier separate situations except for Craig who dictated the following problem about a rainforest (Problem 12b): "There was one Tasmanian tiger and five echidnas. The Tasmanian tiger ate two echidnas. How many echidnas were left?" Initially Craig wrote the equation as 6-2=4 but self-corrected this to 5-2=3. For Problem 14a he wrote 10-2= , then crossed this out and wrote 10-8=2. For his own problem about the zoo (see above) Craig confidently wrote 10-2=8. Albert and Kay also had some difficulty writing equations for their problems; Anthony first wrote 2-3=3, then crossed this out and wrote

5-2=3, while Briony wrote 4+3=3, then 4-1=3. Three children wrote their own problems and these were more complex:

> There wor 5 lin and there wor 8 tigrs tace awae too of them ho mach wor there left (David)
>
> There was 10 stars and 15 children ho memey mor stars do we need? (Alison)
>
> Threr worer five benarnas and 2 muncies. bothe of the 2 muncies aete 1 eche. How meny benarnas were left? (Jeffrey)

None the above structures had been discussed with the children: David and Jeffrey developed multistep problems: addition (David); and multiplication (Jeffrey) followed in both cases by subtraction; Alison wrote a compare subtraction structure. These children did not draw representations of their written problems, nevertheless, Alison and David attempted to write equations. David wrote 5-8=12 while Alison first wrote 10+15, then crossed it out and wrote 10-15= but she did not know how to solve her equation.

Only one child generated a combine subtraction problem in response to Problem 12a. Jeffrey wrote, "There were 2 children in the school. 6 of the 12 children were bad children and they were leifin there rubish on the playgrowid. How mene children were good?" and confidently wrote the equation as 6+6=12. The other children who attempted this problem wrote addition problems, e.g., "There was 3 peepl plaine with the hoops and 4 peepl plaine with the scipping ropes hoomen peepl wr there? 4+3=7" (Alison).

Conclusions

Although the sample is not representative, the results for these children support those of Carpenter et al. (1993) who showed that after eight months in kindergarten children who were taught problem solving could solve a variety of quite difficult word problems. The children in this study had less time spent on problem solving than the children in Carpenter et al.'s study, yet they were remarkably successful in representing and solving complex word problems, both using concrete materials and in drawings.

The children's drawings of the problem situations show that they used a variety of strategies to represent aspects of the contexts including showing properties of the problem elements (colour, size, pictorial details); separating groups or crossing out individual elements; partitioning sets and drawing lines to indicate sharing relationships; drawing array structures to show equal groups in a multiplicative situation; and using letters and words to label elements of sets or sets.

All nine children had written equations for single step problems with a direct relation between the quantities by the end of the year, and some had a good grasp of representing problems symbolically. The teacher also reported that the children became very motivated to write their own problems; most children in the class were enthusiastic about writing and painting their own problems in free time sessions.

These results suggest that both mathematics educators and teachers should question their assumptions about kindergarten children's understandings of mathematics and their skills in representing their solution processes. The results presented in this paper suggest that children can achieve a great deal when the following conditions are met:

- problem solving is seen as an enjoyable activity that is valued by the teacher;
- the teacher has an expectation that all children will benefit from problem-solving, not just those children who are seen as "bright";
- problems are linked to a familiar context;
- children model situations with concrete materials and they are encouraged to show their thinking;
- scaffolding is provided to support children at different levels; and
- problem solving skills are consolidated and extended over a period of time.

If these conditions are part of the teaching environment, then some children achieve far more than would be expected at kindergarten, both in terms of representing problem situations symbolically, and in generating problems that they have not previously encountered. The concern for young children who have been encouraged to enjoy problem solving is to ensure that their initial enthusiasm for problem solving is fostered.

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Attitude and Attribution in Mathematics:

Gender Factors with Low Achievers

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This research examined the attitude towards mathematics, motivation to do mathematics and the attribution of difficulties in mathematics of a group of low achieving students at a Brisbane metropolitan high school. At this school, boys comprise the majority of students in remedial and special needs classes. The group selected consisted of low achieving students in years eight and nine who were receiving support in the form of one-on-one help in a withdrawal situation, in-class support and a modified curriculum.

Gender Issues in Mathematics Education

Gender differences in mathematics generally relate to differing performance or achievement, participation rates in different courses, and differences in attitude, motivation and attribution of success or failure. The present study focuses on the last of these three categories of differences.

Differences in Mathematical Achievement and Participation

Since the early 1970's there has been a global concern about gender differences in mathematics performance and participation in advanced mathematics courses in high school and college as well as the under-representation of women in maths-related careers. Subsequently there have been many reports in the literature of gender-related differences in achievement in mathematical topics in which the achievements of girls, has in general been below that of boys. (See for example Burton, 1986; Carroll, 1995; Clements, 1989; Fennema, 1985, 1990; and Leder, 1984, 1989). Willis (1990) found differences in performance on mathematical tasks, gender differences dependent upon task types, under-representation of girls in mathematics in the senior secondary school, and differences favouring males in high levels of mathematics achievement. Leder (1992) noted also that although achievements of males and females tend now not to show strong noticeable differences, more males continue to enrol in the "higher level" upper secondary courses and that performance difference in favour of males continue to be found on mathematical tasks that require high cognitive skills.

Differences in Attitude, Motivation and Attribution

The issues of attitude, motivation and attribution are complex and the aims and objectives of the present study were confined to the examination of the students' beliefs and the determination of whether observable differences were present. Girls' under achievement and low levels of participation in mathematics have been related to negative attitude. Bradberry (1989, p. 313) claimed that girls are "less likely to see mathematics as relevant and practical." Moss (1982) has suggested that intervention programs to promote greater female participation will need to address girls' attitudes towards mathematics and problem solving tasks.

According to McLeod (1992), gender differences in attributions also reflect differences in participation. Furthermore, he claimed that mathematical gender differences are influenced by confidence which in turn relates to enrolments. Leder (1984) reported that females tended to attribute failure to low ability, yet did not relate success to high

ability. Willis (1989) explained that girls more often than boys attribute their failure in mathematics to lack of ability by saying that their successes are due to external features such as luck, an easy test or hard work. Consequently, they doubt their capacity to avoid failure or achieve success. In a later study Willis (1990) denied that it is appropriate to say that girls
have lower self-esteem in mathematics than boys. Fennema (1989) identified differences in beliefs held by males and females. She reported that in general males tend to believe that their success in mathematics is due to their innate ability, whereas, by contrast, females tend to attribute failure to lack of ability.

Recent Developments in Gender Issues

There is now evidence to suggest that at the secondary level, not only have girls "caught up" with the boys, but are now achieving better. At the school chosen for the present study, for example, the principal (O'Dempsey, 1994) noted "At our Senior Awards Night, 71% of all awards (all types) went to girls ... This trend also replicated itself at the Junior Awards Night." Leder (1992) now believes the performance gap for males and females is narrowing, with little differences being found in many areas. At the lower end of the achievement scale we have evidence that males outnumber females in remedial and special needs mathematics classes and there is growing concern about "underachieving" males (Koopman & Langan, 1995).

Aims of the Research

From the above considerations, research questions were formulated: What are the attitudes to mathematics and the motivation to learn mathematics of a group of low achieving male and female secondary school students who are in a special needs support program? Are there any differences between the male and female members of the group of low achieving students in their attitudes to mathematics, their motivation to learn mathematics, and the factors to which they attribute their achievement?

The answers to these research questions are expected to generate further questions for discussion and possible research. For example, do these attitudes held by students involved in a special needs support program differ markedly from other low achieving students who do not receive any such support?

Qualitative Methodology

Since the questions called for research into attitudes, motivation and attribution, it was recognised from the outset that the data needed for the study should be qualitative in nature, and therefore it was decided that qualitative research methods would be employed. Generally speaking, qualitative research data are not suited to statistical analysis and it was not the intent of the present investigation to make inferences regarding the populations from which the samples were selected. Nevertheless care in the identification of the population and the description of the sample was made since the study is expected to generate hypotheses which, in later studies might be tested for their validity and degree of possible generalisation to the population from which the samples were selected. Any observed differences in attitudes, motivation and attributes between the two groups may reasonably be expected to be representative of other similar groups throughout the state.

The Parent Population: The school involved in the present study is an outer Brisbane region metropolitan high school drawing a population of mixed socio-economic background but predominantly lower socio-economic background. It has been described in the Australian Bureau of Statistics Data for School Profiles (1995) as being of low family income and high unemployment.

The Sample: The low achieving group were a group of students from mixed grades (8 and 9) who were receiving support from the special needs support group at the school. These students all had been identified as low achievers in mathematics requiring high support needs. The students' needs were being met through partial withdrawal from class, in class support and the provision of a modified curriculum. As noted previously,

the proportion of males in special needs and support classes at the school is greater than females and in the group for the present study there were 10 males and 7 females.

Data Gathering

Each of the 17 subjects identified for inclusion in the sample was interviewed using a structured clinical interview schedule in which a predetermined set of questions relating to attitude, motivation and attribution were asked. The verbal responses to these questions were recorded and the verbal protocols so obtained were categorised for analysis. Some of the questions were drawn from the literature, others were composed by the present writers, based on experience in mathematics education and in special needs support teaching.

Results and Discussion (Selected Questions)

The results of those questions yielding relevant data for discussion are presented. These are listed by category:

Questions Relating to Attitude

Do you like / dislike maths?

Five students, three males and two females, stated that they liked the subject. It has been well documented in the literature that students who have a history of failure such as these students, rarely express a liking for the subject. In view of this it may be surprising that as many as five of the 17 expressed a liking for the subject. There were no noticeable gender differences in the responses to this question.

Do you like learning new mathematical skills?

Fourteen, five girls and nine boys, of the seventeen answered "Yes" to this question. It would appear that despite a general dislike of the subject, the majority of students nevertheless enjoy it when they experience some measure of success.

How do you feel when you do maths?

The responses to this question were categorised and presented in Table 1

Table 1: *Feelings when doing mathematics*

	Male	Female	Total
Negative			
Bored	4	3	7
Frustrated	4	0	4
Other	2	1	3
Total negative	10	4	14
Neutral			
Alright	0	3	3
Total	10	7	17

Not surprisingly, 14 of the 17 students expressed negative feelings when doing mathematics. However, of the ten males, all expressed negative attitudes whereas the responses of the females were evenly divided between negative and neutral. Not unexpectedly, no positive feelings such as "good" were observed. Two of the three females who responded positively to this question also answered that they enjoyed the subject. One female made a somewhat contradictory response that she "disliked" the subject, but felt "alright" when doing it. Somewhat more contradictory were the response of three of the males who reported liking the subject but felt "bored". Digressing interview questions

revealed that they were "used to being bored" with most of their school work and saw no reason for mathematics to be different.

Do you think maths is important? Do you think maths is a useful subject? Why or why not?

Sixteen of the seventeen students answered "Yes" to both of these questions. One student, a male answered "No" to both. Clearly the vast majority of students viewed mathematics as important and useful. Most responses to the first part gave a reason relating to mathematics and job prospects. This result is not totally unexpected since many of these students would be contemplating leaving school at the end of year 10 to seek employment. However, the perception of importance and usefulness by such a high proportion of the students, especially all of the girls is encouraging in view of the overall claim that girls are "less likely to see mathematics as relevant and practical" (Bradberry, 1989, p. 313) .

Do you think that it is more important for males to study mathematics than females? Why or why not?

Only two students, one male and one female, answered "Yes" to this question. The remaining students all commented to the effect that mathematics is equally important to both males and females. Clearly, the vast majority of these low achieving students recognised the importance of mathematics for both males and females.

Do you feel confident doing mathematical tasks in class?

Only six students, two girls and four boys responded "Yes" to this question. Five students, two girls and three boys answered "Sometimes", while the rest answered "No". Only two of the seven girls responded "Yes" to this question. Fabricant (1990) found that confidence when doing mathematics was a crucial elements in girls' choices to continue with mathematics. Nevertheless, the proportion of boys who responded "Yes" to this question (four of the ten) was no higher.

Questions Relating to Motivation

Do you care whether you do well at maths? Do you want to improve in maths?

Only two students, one male and one female answered "No" to both these questions. Eight, three females and five males answered "Yes" to both, and seven, four males and three females answered "No" to the first and "Yes" to the second. There were no noticeable gender differences in the responses to these questions.

Other data relating to an aspect of motivation, that of willingness to seek help and spend time studying, were obtained from the responses to several other questions. Responses were classified as being "well motivated", "motivated" or "unmotivated". Only two students, one male and one female, were considered as unmotivated and three were well motivated. Surprisingly, the majority of students indicated some positive motivation in their willingness to either seek help, spend extra time, and attempt homework. Again this is encouraging. Furthermore, no gender related differences were observed.

Questions Relating to Attribution

Have you always had poor results in mathematics?
Do you think that maths is too difficult a subject for you?
If or when you did not do well in a test, would you think that it was because
(a) the content was too difficult, or
(b) you didn't study or put little effort in

Fifteen of the 17 students, including all of the females, answered "Yes" to both of the first two questions. The responses to the third question in this group are presented in Table 2

Table 2: *Attribution of difficulty*

Attribute	Male	Female	Total
Too difficult	3	4	7
Didn't try hard	7	3	10
Total	10	7	17

All of the females and most of the males reported finding the subject difficult. However most of the males attributed this difficulty to lack of effort rather than inability, while the females were more evenly divided in their attribution between lack of effort and the subject being too hard for them. These results are consistent with, though not as marked as, those reported in the literature by Willis (1989), Fennema (1989), and Leder (1984).

Do you think you could you do better at maths? Why or why not?

The results for this question are presented in Table 3.

Table 3: *Question 19 responses*

	Male	Female	Total
Yes	8	4	12
No	2	3	5
Total	10	7	17

Three of the girls who answered "No" to this question stated that they were "trying as hard as I can". None of the boys gave this response. Four girls who answered "yes" said that they could try harder, listen more, or make more effort. Eight boys answered "yes, by working harder etc." The two boys who answered "No" attributed their lack of success to their own inability. These results are consistent with though not as pronounced as, those reported in the literature.

If or when you did well on a test, would you think that it was because

(a) the test was easy
(b) you "fluked it"
(c) you had studied & put more effort in

The responses to this question are presented in Table 4.

Table 4: *Attribution of success*

Attribute	Male	Female	Total
Easy Content	2	2	4
Chance/luck	4	0	4
Self	4	5	9
Total	10	7	17

The group was evenly divided in their attribution of success to either content or self (i.e. hard work). However, four boys attributed their success to luck, while no girls made this attribution. This is contrary to earlier findings reported in the literature. However, it should be noted that these previous studies were of females who largely had experienced some measures of success in mathematics in order to determine to what they attributed such success. The girls in the present study had rarely experienced success in mathematics and for them the question was somewhat hypothetical. Nevertheless, it is noted that if and when success is experienced by the girls, the majority attribute their performance to their own ability.

Summary and Conclusions

Clearly most of the students disliked the subject and experienced negative feelings about it. However, they nevertheless cared about their performance, expressed a desire to do better, and viewed the subject as important and useful. Nearly all lacked confidence when doing mathematics, and though not overall highly motivated, expressed a willingness to spend extra time and seek help with the subject.

There were no pronounced differences between male and female attitudes, liking of the subject, or desire to improve. Importance and usefulness were seen by all. In particular the vast majority of the low achieving females recognised the importance of mathematics for both males and females. Overall, the females' attitudes were no worse than those of the males. From the point of view of the special needs teacher, these are encouraging results. However, further research to investigate the generalisability of this conclusion is recommended before we become too excited about it. It may be that this better than expected perception of the females is a direct result of the students' involvement in the special needs support program, or it may be that overall female perceptions of mathematics as a male domain are, in fact, finally changing.

Recommendations

Further research is recommended to determine whether the decline in the view of mathematics as a male domain shown by this group of students is now more widespread and whether the conclusions of the present research can be generalised to larger sections of the student population. Further research could test how teachers involved with special needs activities such as the withdrawal of students from regular classes and in-class assistance can best help students improve their attitude towards mathematics and motivate them to succeed in the study of the subject. It is also recommended that research continue to investigate the link between the over representation of boys in low achieving groups and their attribution of difficulties with mathematics.

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Broadening the Descriptors of van Hieles' Levels 2 and 3

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Despite controversy, the van Hiele levels continue to be used as an important framework for developing geometry teaching programs and for interpreting students' understanding of geometrical ideas. However, as they stand, the level descriptors offer a restrictive base. These cause problems when questions are posed outside of the direct notions of properties of figures, class inclusions and deduction about which the Theory is explicit This paper is an initial attempt to broaden the level descriptors in a way that is consistent with the original ideas of the theory but which allows for more inclusive criteria. To assist in this process ideas drawn from the SOLO Taxonomy are employed.

Introduction

The van Hiele Theory has been the focus of continued research attention since the late 70s. In particular, many theses, reports and articles have been published concerning issues related to verification and exploration of the five hierarchical levels, and their associated characteristics, that are an important basis of the Theory.

The levels have proved a useful tool in (i) identifying problems in students' understanding of certain geometrical concepts, (ii) evaluating the structure or development of geometric content in secondary text books, and (iii) guiding the development of syllabuses. However, the theory has not been without controversy. Some (Gutiérrez *et al.* 1991) have challenged the discontinuous nature of the levels. Others have queried the apparently simplistic one-dimensional nature of the levels and believe the theory fails to describe the diversity evident in students' behaviour. Despite these forms of criticism, there is considerable empirical support for the levels (Clements and Battista, 1992).

The van Hiele Theory hypothesises five levels of thinking identified as Levels 1 to 5. Of interest to this paper are the first four levels. These are described below with the modification to Level 2 suggested by Pegg (1995).

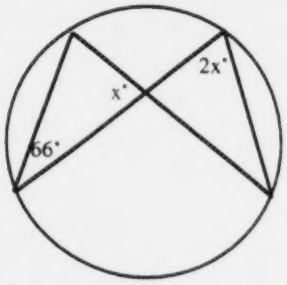
Level 1:	Figures are identified according to their overall appearance.
Level 2A:	Figures are identified in terms of a single property (usually sides).
Level 2B:	Figures are identified in terms of properties which are seen as independent of one another.
Level 3:	Relationships between previously identified properties of a figure are established as well as relationships between the figures themselves.
Level 4:	Deduction is understood and students can develop logical proofs. Multiple definitions in terms of minimum properties can be provided.

There are at least two important reasons for splitting van Hiele's Level 2. Many students, especially up to the first or second year of secondary education, respond at this level when asked to describe a figure. Second, attaining Level 2A represents a culmination in the thinking process of the development of a single concept or property. As such, it represents an important interface between the visual/intuitive thinking at Level 1 and the identification of several isolated concepts/properties at Level 2B.

Background

One issue that has confronted the writer is that the current level descriptors are narrow and not easily generalisable to a range of question types common in school geometry. When one considers some of the typical questions posed of students in the junior secondary school there does not appear to be much guidance from the van Hiele Theory in allocating levels. Reference to van Hiele writings or the two most commonly used tests, i.e., those devised by Usiskin (1982) and Mayberry (1981) do not help. For example, the following six questions, taken from the 1985 and 1986 State School Certificate Reference Test of New South Wales, typify the concern.

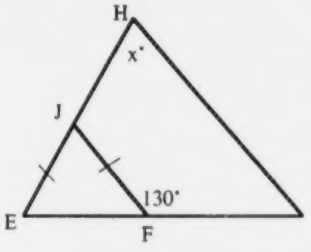
Question 1



x = ?

A. 33 B. 38 C. 48 D. 57

Question 2

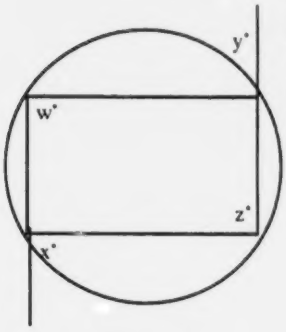


In the diagram
EJ = JF
JF ∥ HG
<JFG = 130°

x = ?

A. 50 B. 80 C. 100 D.130

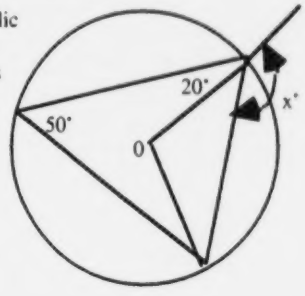
Question 3



The diagram shows a cyclic quadrilateral.
Which statement below is always true?

A. z = w
B. x = w
C. x + y = 180
D. x + w = 180

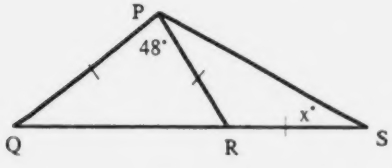
Question 4



The diagram shows a circle centre 0.
x = ?

A. 110
B. 120
C. 130
D. 140

Question 5



x =

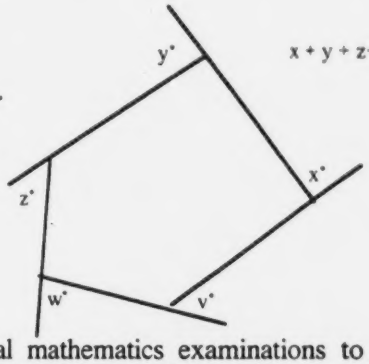
A. 33 B. 42 C. 57 D. 66

Question 6

PQ = PR = RS

Angle QPR = 48°

Angle RSP = x°



x + y + z + w + v = ?

A. 180
B. 360
C. 540
D. 720

These items were given, as part of general mathematics examinations to some 35,000 students (who represent 40% of the Year 10 (16 year old) student population) doing the advanced mathematics course in the state of New South Wales, Australia. Significantly, a consistent pattern emerged in the nature of responses. This showed (i) students who were correct on Questions 2 and 5 (approx 75% of the candidature) were also able to handle correctly and consistently typical Level 2 items, and (ii) students who were correct on Questions 1, 3 and 6 (approx 45% of the candidature) performed

correctly and consistently on more typical Level 3 items. Approximately 60% of the candidates were correct with Question 4.

While there was clearly a close relationship between these items and the two van Hiele levels, van Hiele's descriptors give little guidance to support coding the nature of the thinking associated with these five questions. Hence, the focus of this paper is on considering how the descriptors of the van Hiele levels, as previously provided, can be expanded so that while they remain true to the original descriptions they can also be inclusive of, and able to categorise, the type of questions posed above.

The SOLO Taxonomy

One approach to addressing the issue of more inclusive level descriptions is to refer to a theory that is seen to have some sympathy with the van Hiele Theory. The SOLO Taxonomy (Biggs & Collis, 1982, 1991) has been identified by several writers (e.g., Jurdak, 1989; Olive, 1991; Pegg & Davey, 1989) as having strong similarities with the van Hiele Theory, despite some apparent philosophical differences.

The SOLO Taxonomy, as with the van Hiele Theory, has its roots in the Piagetian tradition and both theories are relevant to and designed to facilitate school-learning activities, albeit in different ways. The van Hiele levels are a series of signposts of cognitive growth reached through a teaching/learning process as opposed to some biological maturation. SOLO, however, is particularly applicable to judging the quality of instructional dependent tasks. It is concerned with evaluating the quality of students' responses to various stimulus items. While it is possible to set questions which encourage a response at a particular level, it is students' attempts at an item that are of paramount interest as well as the many natural groupings of answer types. This focus represents an important departure from, say, Bloom's Taxonomy where levels have an *a priori* quality and students are deemed either successful or not. With SOLO, the data (students' responses) are treated polychotomously (as opposed to dichotomously, i.e., true or false) and the categorisation of answers into multiple groupings with similar characteristics reflects various stages of cognitive growth.

This represents a philosophic shift from van Hiele's (and Piaget's) ideas, as the levels describe responses, not people. With SOLO, a response provides a measure of a student's attainment at a particular time and in a particular circumstance, it does not, necessarily, determine some stable stage of cognitive functioning. This approach overcomes the décalage problem identified by Piaget in which a person may respond at a different level to the same (or similar) tasks from one testing episode to the next.

Fortunately, this difference in focus does not represent a fatal flaw to any comparison of level descriptions across the two frameworks. The main difference between them is manifested in the conclusions that are drawn about the overall nature of a students' level of thinking. This contrasting view is not the focus of this paper and useful comparisons about levels can be made without confronting this issue.

A SOLO classification combines two aspects. The first of these is the mode of functioning and, the second, a level of quality of response within the targeted mode. Of relevance here is the mode referred to as concrete symbolic and the three levels within the mode referred to as unistructural, multistructural and relational.

In the concrete symbolic mode a student is capable of using, or learning to use, a symbol system, such as a written language and number notation. The important feature of this mode is that there is an empirical referent available. This is the most common mode addressed in learning in the upper primary and secondary school. This mode becomes available after an individual has progressed to a certain level of attainment within the ikonic mode. (The ikonic mode is where a person internalises outcomes in the form of images. It is in this mode that a child develops words and images that can stand for objects and events.) Hence, early responses in the concrete symbolic mode carry with them the need to base judgments on observable, physical experiences that make sense to the real-world understanding of the person.

The three levels within the concrete symbolic (C.S.) mode (Biggs & Collis, 1991), represent a growth from the more 'concrete' to the more 'abstract'. Brief descriptions of these levels are:

The *unistructural level* of response is one that contains, or draws upon, one relevant concept or datum from among all that was available.
The *multistructural level* of response is one that contains several relevant but isolated concepts or data from among all that was available.
The *relational level* of response is one that contains an over-riding linking concept. Alternatively, each relevant concept is woven together to form a coherent structure.

The development encompassed by the three SOLO levels moves from a focus on a simple (one aspect) and more tangible aspect, which is closely aligned with an individual's real-world experience, to the less tangible aspects, namely, relationships between concepts. This growth is, in part, determined by the availability of working memory for the completion of the task. At the unistructural level the student has only to understand the question, relate the question and the answer, and use one concept. The multistructural level response requires a similar ability except the student needs to be able to access a number of concepts. The relational response requires, in addition, an overview of relevant concepts while being able to monitor the process or task from beginning to end, thus allowing for a logically complete conclusion.

Finally, notions of consistency and closure are also strongly related to each level. The former refers to the need felt by individuals to make a response that is not contradictory, either from the perspective of the answer or the data provided. The latter refers to the desire to provide an answer and hence finish the task. These two needs represent opposing forces that impact on the nature or quality of the response.

The link between the level descriptors of SOLO and the van Hiele Theory can be summarised as: unistructural responses (C.S. mode) are consistent with Level 2A thinking; multistructural responses (C.S. mode) with Level 2B thinking; and, relational responses (C.S. mode) with Level 3 thinking.

Results

Level 2A

In terms of the five questions posed at the beginning of the paper, none can be seen as requiring Level 2A thinking. None require an application of a single concept related to a (visually) verifiable situation. Examples of questions in which a unistructural response or Level 2A thinking is required are given below in Figure 1. Here, for those students who have access to the relevant notion, the diagram acts as a cue to prompt a single appropriate concept involving 'real' angle measurements. Further, the result involves a unique number, which can be obtained quickly, and hence the demands on working memory are low.

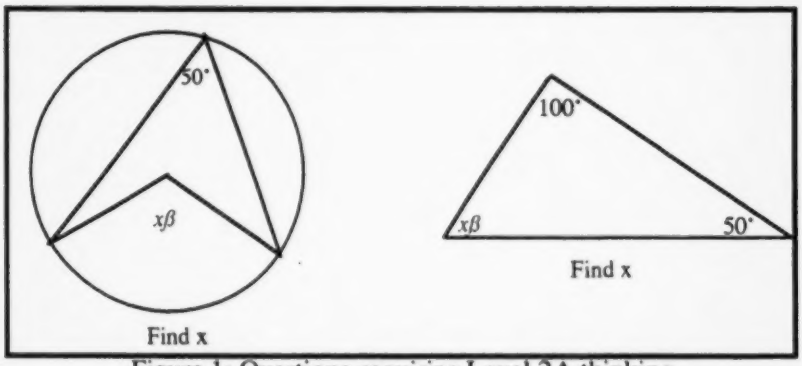


Figure 1: Questions requiring Level 2A thinking

The purpose of these questions is to ascertain whether the respondent has knowledge of the appropriate concept and can understand the question. In both cases a single concept is needed, and real angle measurements are used. A student can 'close' quickly if the demands on the working memory are small. Consistency can be assisted by the visual support given to the answer that would confirm it being unique and 'real' to the student.

Level 2B

The three questions, Questions 2, 4 and 5, each belong to this category. To complete successfully these tasks, the student must employ more than one concept. There is a clear, unique end-point, the use of actual angle measurements ensures a series of 'real', meaningful closures for the students. The questions can be solved without the need of an overview of the questions and the available data. Students are able to move logically along a path from the given angle to the answer without needing to know in advance how they might proceed. Indeed the path to the answer would be unexpected until the completion of the activity.

The reason for the lower correct response rate for Question 4 (60% as opposed to 75%) is related to the distraction caused by the 20° angle. This sets up some interference to the solution process with students attempting to start with this angle or, alternatively, use it part way. Absence of an overview means such information can set a student on an unsuccessful solution path from which they can not recover.

Level 3

The three questions, Questions 1, 3 and 6, each belong to this level. The major advance in thinking appears to be the ability to refrain from seeking individual closures (answers) before proceeding to the next step. This represents an ability to form a generalisation based on previous 'concrete' experiences with specific cases although final uniqueness of the result still needs to be guaranteed. This level represents an ability to work with pronumerals. In each case this usage can be supported by replacement of some unique numbers, which make sense to the student within the questions' context, if the student feels pressured. In both cases, students who are consistently successful at this type of problem have an overview of the elements of the question. They are aware of the solution strategy which relates the use of various concepts and, unlike those who perform at Level 2B, the arrival at the final answer would not come as a surprise.

Conclusion

The above analysis has identified a way forward to expand upon the descriptions offered by van Hiele for his Levels 2 and 3. In essence, this development means, in the case of Level 2, that the characteristic of thinking in terms of independent properties can be interpreted within SOLO as an aspect of a broader thinking category in which concepts are addressed in isolation. These concepts need to have an obvious visual basis and individual closures (answers) must have a strong real-world referent for students. Aspects of processing which occur towards the end of a task appear independent of any initial processing. The individual steps (or closures) leading to a final solution can be performed in sequence without concern given to some general overview. A distractor in the process can cause concerns.

At Level 3, the characteristic of thinking concerns the acceptance and use of relationships between properties and figures. The broadening of this level using SOLO is associated with the ability to have an overview of relevant elements and to form, on this basis, appropriate generalisations. Consequently, for a given task, relevant data are identified and the student can monitor these data. This allows for the reasoning at a later stage in a question to be adjusted in the light of earlier thinking. Also at this level, students have a notion of a generalised number by using algebraic symbols which can stand for 'real' numbers. Hence, they do not, necessarily, replace pronumerals with numbers but students feel secure as they have the option to do this in cases in which they perceive to be more difficult. This ability to work with pronumerals allows students to refrain from the need to calculate particular answers for each step of a problem (a characteristic of Level 2 thinking), and opens the way for relationships between different concepts to be utilised.

Interestingly, support for these latter ideas can be found in van Hiele's own writings. He stated "The differences between the objects of the second and third levels can also be demonstrated by different ways of writing. At the second level, calculation deals with relations between concrete numbers: $4 \times 3 = 12$, $6 + 8 = 14$. At the third level

of thinking it deals with generalisation of results: $a \times (b + c) = (a \times b) + (a \times c)$. In these generalisations you do not return to the original objects of the second level, namely the concrete numbers"(1986, p.54).

Here, we can see van Hiele broadening his own level descriptions. Surprisingly, van Hiele chose not to pursue this line of thought. However, with the support of SOLO and the discussion in this paper, this quote can assume a much deeper meaning. For example, at the second level there is a focus on actual numbers, single concepts are involved, and working memory demand is relatively light. At the third level, the working memory demands are heavier, the example given is more abstract in nature and actual numbers are not used. The rule cannot be known by its separate parts only by an overview of all the elements and the structure of the relationships can the pattern be understood. The letters in the equation each represent an entity in their own right, there is no direct concrete referent for the rule although substitution of numbers remains an option which would guarantee uniqueness of outcome.

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The relationship between the place value understanding of seven-year-old children and the strategies that they use to solve written addition problems

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Individual interviews were used to explore 128 seven-year-old children's understanding of place value, to see how this related to the strategies that they used to solve written addition problems. Children who understood the place value of two digit numerals were more successful at solving written addition problems, but a surprising number of children with face value understanding could also solve the problems. The variety of strategies (successful and unsuccessful) that the children used are described, and the implications for teaching algorithms are discussed.

In New Zealand, as children progress from the Junior school to the Standards (at around age seven) they are working on some key concepts such as decade structure and place value (Faire, 1992). The nature of these concepts may give rise to particular problems, especially where the written numerals and the spoken number words are two related but different systems (Carpenter, Fennema, & Franke, 1991; Kamii, 1985; Miura et al, 1993). This has implications for children learning mathematics in English (as opposed to those learning mathematics in languages like Maori, where the structure of the number system is explicit in the number words). As Carpenter et al (1991) explain, in English, while the number words larger than one hundred explicitly designate the units and the number of each unit, with numbers less than one hundred the designation of units is less explicit. The number names do not clearly emphasize the groupings of tens, and the problem is even more acute in the teens, where numbers are designated with a single word, and the first syllable of the word denotes the *units* rather than the tens (e.g. *four*teen). This is reflected in the way that many young children write the English "teen" numerals (e.g. fourteen written as 41) (see Peters, 1994), and may have implications for place value understanding. A study by Miura et al (1993) indicated that the place value understanding of children in countries like Japan and Korea, where the numerals correspond exactly to the spoken number words, was better than in countries like the United States, France and Sweden, where the number names do not clearly show the groupings of numbers.

In addition, many children seem to have problems with the formal language of mathematics and the use of symbols (Hughes, 1983, 1986; Skinner, 1990). Work with five to seven-year-old children showed that the children were reluctant to use the operator signs to represent addition and subtraction, even though these symbols were being used in their maths books (Hughes, 1983, 1986). It has been suggested that a major difference between high maths aptitude children and those who are poor mathematics learners, is the extent to which they are able to make sense of the rules and symbols they are taught (Resnick, 1986).

If children are involved in using symbols that they do not understand, they may conclude that mathematics is not supposed to make sense. If this happens, they are likely to stop monitoring their work thoughtfully and will not be troubled by answers that are clearly wrong (Baroody & Ginsburg, 1990). Instead of focussing on the mathematics they may adopt a passive kind of involvement in the classroom and learn the advantages of self preserving behaviours (King, 1993). These may include adopting systematic routines, even if they yield wrong answers (Brown & vanLehn, 1982; Resnick & Omanson, 1987). For example, strategies which children employ to cope with story problems that they do not understand may include; guessing, trying all the operations and looking for the most reasonable answer, and looking for isolated key words and phrases that signal operations (Sowder and Sowder, 1989).

If the children resort to these 'coping strategies' rather than coming to understand the mathematical concepts that are involved, it may help to explain why the gap between children at the top, and those at the bottom, widens as children progress through school (see Fogelman & Goldstein, 1976; Young-Loveridge, 1991).

The purposes of this study were, firstly, to explore children's understanding of the place value nature of written numerals and secondly, to note how this related to the strategies that they use to solve written addition problems.

Method

This study involved 128 seven-year-old children from three New Zealand primary schools. All of the children were in their third year of school (J3). The children were from a variety of ethnic backgrounds but all had English as their first language and were being taught in English.

Individual task based interviews were used to explore the children's understanding of place value and the strategies that they used to solve written addition problems.

The place value task used in this study was adapted from a task used by Faire (1992) and Kamii (1985), and was designed to assess the children's understanding of the place value nature of a two-digit numeral. (For full details of the task see Peters, 1994.)

Figure 1 shows the four written addition problems that were used. The researcher recorded the strategy that the children used to solve these problems, and the direction in which they worked (e.g. from right-to-left, adding the units first, or from left-to-right, adding the largest numbers first).

37	75	84	325
+21	+15	+39	+285

Figure 1. Written addition problems

Results and Discussion

Place value understanding

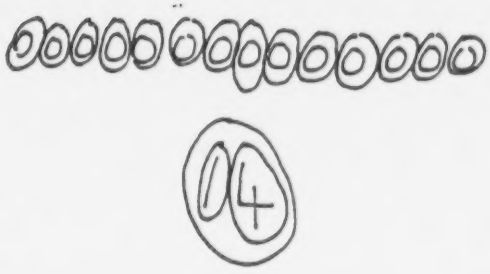
Data from the place value task were analyzed, using a series of levels of place value understanding (described below), which were adapted from those described by Faire (1992, pp45-46), Kamii (1985, pp56-57) and Ross (1989, pp49-50). Copies of the children's drawings illustrating the different levels of place value understanding are shown in Figure 2.

Level 1a, Cardinal Representation: Here objects are counted out to represent numbers. That is, the set of counted objects is the meaning of the number. No meaning is assigned to the individual digits.

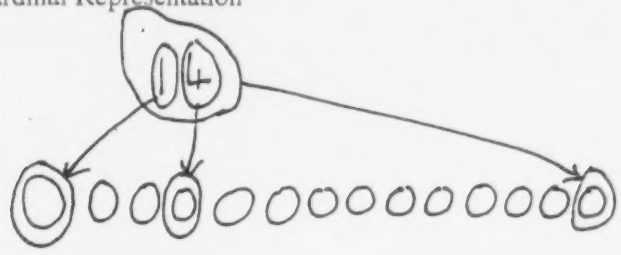
Level 1b, Ordinal Representation: The individual numerals represent ordinal positions. For example, the one in 14 represents the first counter in a line of 14 counters, the numeral 4 represents the fourth counter, and the whole numeral represents the fourteenth counter.

Level 2, Face Value Representation: Children at this level recognize the face value of each of the digits and that both digits taken together represent the whole set of objects, but do not recognize that the number represented by the tens digit is a multiple of ten. For example, the one in 14 represents one counter, the four in 14 represents four counters, although the whole numeral represents 14 counters. The fact that nine counters remain unaccounted for is of no concern to these children

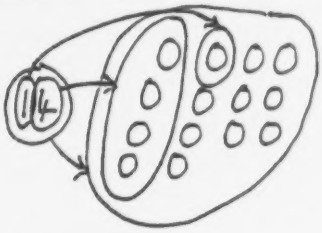
Level 3, Positional Understanding: The individual digits making up a two digit numeral stand for the amounts that are determined by the place or position in which the digits occur. The tens digit is interpreted as a multiple of ten and the right hand digit as ones. The total of tens and ones is understood as the cardinal number of the set of objects.



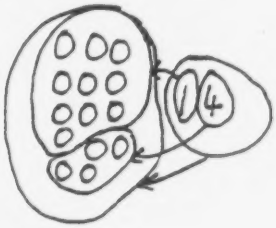
Level 1a. Cardinal Representation



Level 1b. Ordinal Representation



Level 2. Face Value Representation



Level 3. Positional Understanding

Figure 2. Examples of children's drawings showing different levels of place value understanding

Table 1 shows the number and percentage of children at each level. The majority of the children were at Level 2, they recognized the face value of the digits and that both digits taken together represented the whole set of objects, but did not recognize that the number represented by the tens digit was a multiple of ten. A small number of children had not yet developed any understanding of place value and were not able to show that the cardinal value of a set of objects can be represented by a multi-digit number. These results were consistent with previous findings, which indicate that some children are still constructing meaning for place value after several years of schooling (Faire, 1992; Kamii, 1985; Ross, 1989; Seirink & Watson, 1991).

Table 1
Number (and percentage) of children at each of the levels of place value understanding

	N	(%)
No understanding	9	(7)
Level 1a	9	(7)
Level 1b	8	(6)
Level 2	75	(59)
Level 3	27	(21)

Some of the topics that were covered during the mathematics lessons observed in this study, such as two-place addition and subtraction with renaming, require an understanding of place value if they are to be fully understood. As these results show, very few children had reached a full understanding of place value when they started to work on these topics. Although it was not discussed, some of the teachers involved in this study may have over-estimated the children's ability to understand the place-value nature of two-digit numerals. Ross (1989) suggests that it is easy for teachers to over-estimate children's understanding in this area because children may appear to understand more than they do. When faced with a collection that is already grouped in standard place value partitioning of tens and ones, a student who is asked to make correspondence for the digits in 52 for example, need only look for "five of something and two of something else" (p50).

The fact that such a small proportion of the children in this study interpreted the tens digit as a multiple of ten had implications for their ability to do written addition which required regrouping and renaming. The following section looks at the results of the written addition tasks, and the relationship between place value understanding and the strategy used to solve written addition problems.

Strategies used to solve written addition problems

More than half of the children (61%) were able to complete 37+21, a simple addition problem which does not require regrouping. Thirty-eight percent were able to solve 75+15. This problem does not involve renaming the tens, and the addend was small enough for some children to solve this using a counting-on strategy. Only 23% were able to solve 84+39, which required regrouping and renaming. Although very few children had been taught 3-place addition, a similar number (20%) were able to solve 325+285. It appeared that once the principles of regrouping were understood, this knowledge transferred to problems involving larger numbers.

Table 2
Number (and percentage) of children at each of the levels of place value understanding who were successful on each of the written addition problems.

	37+21		75+15		84+39		325+285	
	N	(%)	N	(%)	N	(%)	N	(%)
No understanding	2	(3)	0	(0)	0	(0)	0	(0)
Level 1a	6	(8)	2	(4)	1	(3)	1	(4)
Level 1b	3	(4)	3	(6)	3	(10)	1	(4)
Level 2	44	(56)	23	(47)	13	(43)	10	(40)
Level 3	23	(30)	21	(43)	13	(43)	13	(52)
Total	78		49		30		25	

Table 2 shows that most of the children who were successful on the written addition problems were either at Level 2 or Level 3 of place value understanding. A surprising number of children who only recognized the face value of digits (Level 2) were able to solve these problems. Figure 3 shows a strategy which was commonly used by Level 2 children, where the face values of the digits were added and the total written below the column. This strategy resulted in a correct answer for 37+21, but the children needed to carry out some form of regrouping to be successful on the other three problems. Figure 4 shows an example of a Level 2 child using a traditional carrying procedure to successfully solve all of the problems. The fact that many Level 2 children used the procedure shown in Figure 4 indicates that children were learning to apply this 'carrying' procedure before the place value nature of larger numerals was fully understood. Other authors (e.g. Resnick & Omanson, 1987) have also found a lack of association between ability to perform the 'carry' procedure, and an understanding of the number system.

3 7	7 5	8 4	3 2 5
+ 2 1	+ 1 5	+ 3 9	+ 2 8 5
5 8	8 10	11 13	5 10 10

Figure 3. Strategy commonly used by Level 2 children to solve written addition problems (adding the face value of the digits).

3 7	1 7 5	1 8 4	1 1 3 2 5
+ 2 1	+ 1 5	+ 3 9	+ 2 8 5
5 8	9 0	12 3	6 1 0

Figure 4. Level 2 child using a traditional 'carrying' procedure.

Although many children were able to correctly use the 'carrying' procedure for solving written addition problems, many others showed that they were trying (unsuccessfully) to apply a procedure that was not fully understood. Figures 5, 6. 7, and 8 show some examples of common errors. Aaron (see Figure 5) was unsure where to put the carried amount, and, when he did carry an amount to the next column, he did not include it in his total. Terri (see Figure 6) attempted to carry the 0 (ones) when adding 75+15, instead of the 1 (ten).

3 7	1 7 5	1 8 4	1 1 3 2 5
+ 2 1	+ 1 5	+ 3 9	+ 2 8 5
5 8	8 0	1 2	5 0 0

Figure 5. Aaron: Example of an incorrect carrying procedure.

37	75	84	325
+21	+15	+39	+285
58	81	1113	

Figure 6. Terri: Example where 5 and 5 were added (getting the answer 10), the child carried the 0 (ones) instead of 1 (ten)

When children worked from left-to-right, instead of from right-to-left, regrouping or 'carrying' became even more problematic. When solving 75+15, Jo (see Figure 7) added the tens column first, writing 8 below the column. He then added the ones, getting the answer 10. He carried 1 across to the top of the tens column, and wrote 9 (ten minus the carried 'one') in the answer line. The carried amount was not incorporated into his answer.

37	1 75	84	1 325
+21	+15	+39	+285
58	89		510

Figure 7. Jo: Example where 5 and 5 were added (getting the answer 10), the child carried 1 and wrote 9 (10-1) in the answer line.

Some children who worked left-to-right carried an amount to the previous column, but because their answer was already in place, they did not incorporate it into their answer. Rene's answers provide an example of this (see Figure 8). When solving 75+15 she added 7 and 1 to get 8, and wrote 8 beneath the tens column, she then added 5+5, getting the answer 10. She wrote 0 in the ones column, and put the 1 above the tens column, but did not incorporate this into her answer. She used a similar strategy on the other two tasks (84+39 and 325+285).

37	1 75	1 84	1 325
+21	+15	+39	+285
58	80	113	5100

Figure 8. Rene: Example showing addition left-to-right working where carried numbers are not incorporated into the answer.

The results showed that many children were using systematic, but wrong strategies. Previous research has identified similar errors when children attempt to use the standard algorithms (Brown & vanLehn, 1982; Ginsburg, 1981; Kamii, 1985; Resnick & Omanson, 1987). This is clearly an example of when the formal mathematics of school fails to mesh with the children's informal understanding of mathematics. Perhaps because of a lack of place value understanding of the numerals, children were failing to check if their answers were realistic (for example, Aaron's 84+39=12 shown in Figure 4).

Directional strategies

Although all of the teachers taught children to work from right-to-left when solving this type of problem, only 39% of the children worked in this way. Thirty-eight percent of the children did not have a clear directional strategy and the remaining 23% tackled the addition problems working from left to right. Although working in the traditional right-to-left (<-) direction was the most successful strategy, many children who did not work in this way were also able to complete these tasks. Table 3 shows the number and percentage of the children using each directional strategy who were able to solve each of the addition problems.

Most of the children who did not have a clear directional strategy were unable to solve the problems, although five children were able to solve the first two problems by counting-on in ones. It was interesting that many children were able to successfully work from left-to-right (i.e. adding the largest numbers first). Some of the most able children worked in this way, holding the numbers in their heads, and then writing the final answer. Other children wrote the totals beneath each column, adjusting the previous figure if necessary. An example of this was Simon. On the second problem (75+15) he added the tens column (7+1=8) first and wrote 8 beneath the tens column. He then added the ones (5+5=10), wrote 0 in the ones column, and added one to the tens column, changing his 8 to a nine. He used the same strategy on the hundreds column of the last problem (325+285). These children appeared to have developed their own, meaningful strategies for doing this type of problem.

Table 3

The number (and percentage) of children using each of the directional strategies who were successfully able to complete each of the written addition problems.

	No directional strat.		Left to right (->)		Right to left (<-)	
	N	(%)	N	(%)	N	(%)
Total	48		30		50	
37+21	5	(10)	25	(83)	48	(96)
75+15	5	(10)	10	(33)	34	(68)
84+39	0	(0)	4	(13)	26	(52)
325+285	0	(0)	8	(27)	17	(34)

Sowder and Schappelle (1994) also found many children successfully using a left to right strategy, and suggest that working in this way reinforces place value understanding and is less likely to lead to errors than standardized procedures which have been memorized but poorly understood.

It is important to remember that choice of algorithms is arbitrary. There is a tendency to feel that the standard procedure selected by a particular culture is the only "right" way, and yet people from different cultures, and from different periods in history, often use very different algorithms (see Philipp, 1996). Awareness of the range of alternative algorithms which are in use around the world may make teachers feel more comfortable in accepting the alternative strategies they see children using.

Conclusion

The results of the present study are consistent with the literature that stresses the importance of place value understanding (Faire, 1990; 1992; Miura et al, 1993; Ross, 1989). Less than a quarter of the 128 J3 children interviewed in this study could show that the tens digit was a multiple of ten. This is consistent with previous findings, which indicate that some children are still constructing meaning for place value after several years of schooling (Kamii, 1985, Ross, 1989; Seirink & Watson, 1991). The fact that most of the children were being taught the conventional algorithms for addition, including the 'carrying' procedure suggests that the teachers may have overestimated the extent to which the children in their classes fully understood the place value nature of numbers.

Ross (1989) suggests that the nature of standard place value tasks makes it easy for children to appear to know more than they actually do. Teaching children conventional algorithms before they fully understand the place value nature of the number system may have consequences for their long-term progress in mathematics. Several authors have found that some children develop systematic errors when attempting to use the standard algorithms (see Brown & vanLehn, 1982; Ginsburg, 1981; Kamii, 1985; Resnick & Omanson, 1987), and Baroody & Ginsburg (1990) suggest that there is a danger that such children will conclude that mathematics is not supposed to make sense. When that happens they are likely to stop monitoring their work carefully, and will not be troubled by answers that are clearly wrong.

Many authors have described activities which could be useful in enhancing children's place value understanding (e.g. Faire, 1992; Sowder & Schappelle, 1994; Ross, 1989). It may also be important for teachers to allow children to have lots of experience working with large numbers and solving meaningful problems, using their own informal strategies, before the conventional algorithms are introduced. Where children develop their own successful algorithms teachers may need to consider what advantages conventional algorithms offer these children. It may better to allow children to use their own algorithms, which focus on number meaning, rather than rote learning conventional routines which may be poorly understood.

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Statistical Thinking: One Statistician's Perspective

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This paper discusses some characteristic ways of reasoning within the discipline of statistics from the perspective of someone who is both a practising statistician and teaching statistician. It is conjectured that recognition of variation and critically evaluating and distinguishing the types of variation are essential components in the statistical reasoning process. Statistical thinking appears to be the interaction between the real situation and the statistical model. The role of variation in statistical thinking and the implications for teaching are also discussed.

Background

The development of students' statistical thinking is considered by statisticians (Bailar, 1988; Barabba, 1991; Snee, 1993; Wild, 1994)) to be crucial for the enhancement of student learning in statistics courses. This challenge to the current statistics teaching practice of teaching procedures raises questions as to what are the characteristics of statistical thinking and what approaches should be used in teaching for the development of statistical thinking?

After interviewing some undergraduate students in two separate case studies (Pfannkuch & Brown, 1996; Pfannkuch, 1996) on their solutions and reactions to statistically based information and problems some hypotheses were formed on the development of their statistical thinking. In order to broaden and understand the findings from these studies it was decided to investigate the nature of statistical thinking from a practitioner's or 'expert's' perspective. These statisticians would be in the position of commentating either from reflection on their own statistical thinking and/or on their clients' or students' statistical thinking. Through this interaction with 'experts' it was hoped that further insights would be gained into the characteristics of statistical thinking and that this would be informative for conjecturing the type of statistical thinking that should be developed in students. It would also confirm or refute previous findings which had been solely derived from student data.

Method

Several statisticians were individually interviewed in depth for approximately ninety minutes. Broad questions were asked; the interview following a semi-structured protocol based on the statistical enquiry empirical cycle. The interviews were audio-taped and transcribed. Each statistician had his own perspective on statistics, dependent upon the type of statistical work he was involved in and his own beliefs and experiences. Each one focussed on different aspects in a statistical investigation to explain his perceptions.

For the purposes of this paper one statistician's view is presented and analysed. Ray (not his real name) is a private consultant, working mainly in quality management and is also a supervisor and lecturer for Stage 3 (final year) undergraduate project students. In the Stage 3 course he has specifically developed a module on statistical thinking. Therefore Ray was considered to be in a position to comment on statistical thinking from a practitioner's and teacher's perspective. An analysis and discussion of his responses, including implications for teaching, is presented in this paper. The analysis and interpretation of Ray's comments were presented to him for corroboration.

Analysis

Two interlinked components emerged from an analysis of Ray's interview on statistical thinking. The first component is that statistical thinking involves 'noticing', understanding, critically evaluating and distinguishing the types of variation. This is a very broad component but it is all pervasive throughout the statistical enquiry process from posing the question through to the conclusions and appears to be the core of statistical thinking.

> *Ray: "Statistics is the science of variation basically. Statistical thinking is about evaluating variation, I guess. Variation contains all the information about what's going on."*

The second component is the realisation that a sound judgement of a situation can only be made by gathering and analysing data.

> *Ray: "Statistical thinking I think is about knowing the only way we get any information about the world is by taking samples of data in one form or another . . . It's about saying how would I find out about that . . . I can gather some data and turn it into information that can help me understand this."*

The analysis of Ray's interview has been categorised from a modelling perspective of statistics. The three categories are: understanding the dynamics of the real world problem; moving towards a statistical model; and using statistical tools. Each of these is considered below in detail.

First Category: Understanding the dynamics of the real world problem

In order to understand the context of the problem, an enquiry process is set in motion, that depends upon an ability to ask questions and to 'notice'. The 'noticing' of variation, the wondering why and the thinking about how to collect data to answer the questions is perhaps a scientific way of viewing the world.

> *Ray: ". . . [questions].should be generated by you walking around and bumping into processes . . . you've got a dehumidifier you could start thinking gosh there's a lot of water in it . . . you're walking along the pavement and you're seeing bits of moss growing on the pavement . . . you're in a coffee bar and you're served coffee spilt into the saucer . . . most of the data isn't in numerical form. It's impressions and fleeting thoughts.*
> *Int: . . . in noticing the moss, you are noticing variation.*
> *Ray: Yes.*
> *Int: When you're served spilt coffee, you are noticing variation. This coffee has spilt, this one hasn't.*
> *Ray: Yes, that's right."*

When Ray's students started their projects they realised that defining and understanding the problem that was to be investigated required an investment of time, thought and 'noticing'.

> *Ray: "[my students] have a company who's got this vague interest in improving the quality of their operation somewhere along the line. They don't know how, they don't know what's important to their customers, they don't know who their customers are. The first six months they've got to spend learning about the business, understanding it."*

The development of a broad background knowledge about the business or situation from which data is to be collected is considered essential by Ray. Therefore context knowledge as well as subject knowledge becomes important in defining and understanding the problem. To determine, for example, what 'improve' means in a specific situation students need to adopt a view of the enterprise that 'notices' variation. With this 'noticing' of variation is the understanding and recognition that there are different types of variation, such as, common cause and special cause variation. Common cause or stable system variation is hard to link to any particular source and knowledge on how to improve such a system can only come from an in depth investigation. Special causes can be assigned to an identifiable source.

> *Ray: "They go in and talk to people who are operating the machines or whatever they are looking at. Finding out what the little nagging problems are because they are all things which are special cause variation. . . . there are special causes where we get something from a supplier which doesn't match the specifications . . . when they go through their projects they recognise all these different types of variation, that impacts on what's going on."*

Recognition of the type of variation is important as it determines what strategy to pursue in addressing it and the action to take.

Second Category: Moving towards a statistical model

Once the problem has started to crystallise, measurement issues arise. Students are faced with the conundrum of what can be measured that will capture the essence of the problem, that will reflect a partial truth or model of the actual situation.

> *Ray: "Probably one of the biggest problems they [students doing projects] have is taking what is a reasonably complex situation whether it's a management situation or a production situation or a customer service situation and finding appropriate measurements. Measurements which reflect what is important about the process . . . the things we are interested in are always fuzzy things. . . . the customer wants prompt service; what's prompt service?"*

The narrowing down of a problem was a difficult process for both the students and company executives. Invariably both tried to tackle a problem that was too big. The refining of the problem, to explanatory and response variables, to measures and to stratifications that inherently reflect the process under investigation required an ability to recognise variation and understand the real problem

In their interactions with clients, students were faced with the fact that their clients' perceptions didn't match with the reality.

> *Ray: " It causes a lot of problems for them. The really good ones manage to work their way around that and become a team with the client . . . the others say things like these people are too difficult to work with ...they have a different agenda . . . at some point we try to teach them that there's always more than one perspective for looking at something. Don't for goodness sake think you've got the main eye on the truth here. . . . Study their [everyone in the organisation] perspective as it is going to be informative. . . . so different to yours, maybe they know something you don't. You can always learn."*

The importance of being aware that one's own beliefs and interpretation of situations and statistical information are dependent upon a personal viewing lens and that one should actively seek alternative explanations is stressed to Ray's students. Multiple interpretations of information required an ability in students to distil and to analyse the differing points of view and in their final analysis to capture the essence of the problem through proposing what measurements to take.

> *Ray: "[the good students] are good at talking to people and finding out what really matters . . . and saying well it seems that **these are the key measures** we should be concentrating on we should train people on how to do them . . . so they reflect what's important to the customer, to the organisation. "*

Many of Ray's project students are presented with data that has already been collected. The first aspect some noticed was the variation in the data. Because the recorders of the information had no stake in improving the system or did not get feedback as to why the data was important, glaring errors were found. This facet, that measurement errors overwhelmed any signals to such an extent that the data was useless is an eye-opener to students. Many organisations collected data with little attention being paid to the data collection process or appeared to collect and store data for no reason at all. These organisations had not defined what questions they wanted answered and therefore what information they wanted to know. The 'poor' students tended to take the data at face value

and use it whereas the 'good' students delved more deeply into finding out about the data and what was relevant and what was irrelevant. The distinction that Ray drew between the two types of students, those that treated statistics as an algorithm for data and those that did not, was critical thinking.

> *Ray: "I guess they're just not critical enough. They're not picking up on that thing John Tukey said, the more you know what's wrong with the figure the more useful it becomes. Mostly figures are wrong. Deming said, if you're going to use numbers boy you've got to understand them. You ought to know what's going on."*

Those students who understood what the data meant in context were noticing, understanding and critically evaluating the variation in the collected data and thus were able to propose measures that could model part of the real problem.

Third Category: Using statistical tools

In Ray's experience many students have an inadequate understanding about sampling when planning their investigation. Theoretically the students knew how to take a random sample from a population but when the reality was presented they wanted to measure and analyse everything, often in a specific time interval. Nor did they appreciate that a small sample, in proportional terms, was often adequate. *"It's only based on a very small, in proportional terms, sample of the population so they don't want to make any great statement about the reliability. So there is a lack of statistical faith" (Ray).* This 'lack of statistical faith' could suggest an inadequate understanding of variation and knowledge that statistical tools took into account the variation from sample to sample. That samples will say something about the population from which they are drawn and will help gauge whether the system is stable (common cause variation) or not.

Data representation also presented problems. Ray stated that students could respond with ease to a given graph and point out all its failings yet when they were in the position of having to produce and create their own graphs they resorted to bar graphs, pie charts and chart junk. *"They'll only do it in that sort of critical fashion so that it's an analytical tool rather than a synthetic tool. So that if they are creating something they'll go back to their old habits" (Ray).* Thus a critical attitude was not adopted for their own work. He commented that the students had difficulty in interpreting basic graphical tools such as histograms and time series plots. They did not realise that a histogram said something about what was going on inside the process, that there were clues to be read. He believed that this was because the key element of distinguishing between stable variation (common cause variation) and special cause variation was missing from their statistical education. Graphs should be inspected and fundamental questions asked, such as, what is going on here, is this common cause variation or is this special cause variation? The reading and interpreting of graphs required context knowledge so that reasons could be put forward as to why the data looked like this, so that data could be split in another way and so that data could be explored graphically.

The recognition of the difference between common cause variation and special cause variation was not easy. Ray reported that many people including statisticians got it wrong because they looked at individual data and said it was exhibiting special cause when it in fact it was common cause. There was a tendency to examine the extremes and call them special cause. For example, Ray said, they will say something went wrong there, but it wasn't that something went wrong it was the way the system was managed. As an illustration Ray gave the example of a bank account that was causing the owner problems as it had become overdrawn three times, which could, at first glance, be attributed to the special cause of a large dentist bill. However on thinking about it, the owner of the account was a certain age and knew the state of his teeth. Hence he should have been able to reduce the impact of these dental bills through financial planning. Thus what had seemed to be a special cause was now drawn into the system of common causes. However there were some things, such as accidents, that could not be planned for. *"This distinction between common cause and special cause is not taught well" (Ray).*

Another point that Ray raised was that it did not seem to occur to some students to use a significance test. Instead they considered a visual test of the graph sufficient. Underlying this aspect seemed to be a lack of understanding of variation in relationship to significance testing. The recognition that special causes can be revealed in what is thought to be common cause variation in the summary statistics he considered to be important. Context knowledge and subject knowledge were very powerful in making those interpretations, making the thrust beyond information to understanding.

> *Ray: "What always struck me when I became involved in quality management was that I had got this degree in statistics. I'd done a lot of consulting over the years of all sorts, mostly biomedical areas. And I hadn't really made the connection from the summary statistics in the information through to the understanding of what was going on. What must be going on to generate those patterns. There was more diagnostic information in that stuff than I'd ever realised."*

An interpretation of the summary statistics requires context knowledge and the desire to make the connections to achieve a deeper level of understanding of the problem situation. Subject knowledge and context knowledge seem to be inextricably linked if the findings are to be communicated in a meaningful way.

Discussion

Variation in its many forms appears to be a key component in statistical thinking. In Ray's view it starts with an ability to 'notice' variation, that is be aware of variation in everyday activities, in specific activities, in planning procedures for the gathering of data, in data collection, in data sets, in graphs, in statistical summaries and in interpretations and conclusions. Many statisticians concur with this perspective that variation is central to statistical thinking.

> *"Pupils in the future will bring away from their schooling a structure of thought that whispers 'variation' matters." (Moore, 1992, p. 426).*
>
> *"Variation is not a new concept. What is new is the awareness of variation and how it affects everyday activities is infiltrating the workplace. . . . Knowledge of the theory of variation alters people's view of the world forever. It influences practically every aspect of how companies are managed". (Joiner & Gaudard, 1990, p. 35).*
>
> *"I define statistical thinking as thought processes, which recognise that variation is all around us and present in everything we do . . ." (Snee, 1990, p. 117)*

Moore (1990) gave five core elements of statistical thinking: the presence of variation in all processes; the need for data; the design of data production with variation in mind; and that statistical analysis seeks to quantify and explain variation. The two interlinked components that emerged from the above analysis would agree with these tenets. However, according to Ray and these statisticians, to understand variation in its many forms requires a new awareness, and possibly a new conceptualisation of variation. Ray emphasised that an understanding of variation required the ability to evaluate the difference between common cause and special cause variation. Thus it would appear that an understanding of variation in data may be based on these factors: (1) the recognition of variation in all processes; (2) the distinguishing between the types of variation; (3) some variation may be explained on current knowledge; (4) other variation cannot be explained on current knowledge.

The question then arises on the place of random variation in this construct. Random variation is often defined as 'unexplained' variation (Konold et al, 1991), yet at an intuitive level, particularly with data not obtained from random devices, individual data can be explained. In quality control the view taken is: *"All variation is caused. Unexplained variation in a process is a measure of the level of ignorance about the process".* (Pyzdek, 1990, p. 102). To overcome this conceptual barrier random variation may need to be defined as a multiplicity of causes that are chosen to be ignored in the data

(Falk & Konold, 1991), or that when a pattern cannot be discerned the phenomenon is modelled as random. From these various definitions it would appear that random variation is a model which can be superimposed as a means of coping with unexplained variation and that randomness is a convenient human construct which is used to deal with variation in which reliable patterns cannot be detected.

Statistical tools appear to be seen by some students as descriptive tools rather than analytical tools. For example, some of Ray's students preferred to make inferences from visual tests of the graphs. This may be because there is a confusion about the type of statistics they are undertaking. Much effort must be spent in dialogue with the data, looking for and extracting persistent patterns or signals and thinking of multiple explanations. One explanation that does not seem to naturally occur to some students is the possibility that the difference is due to chance and that some statistical tools will evaluate that possibility or if there is a difference, that statistical tools will take into account the size of the sample (Konold, 1993; Pfannkuch, 1995). Behind these analytical tools lies the concept of variation.

Measurement issues, such as how to capture data that are relevant to the problem and what is realistically measurable, yet still reflect a way of modelling the situation was one of the biggest problems facing the students. This is an area that does not seem to be addressed in some statistics courses (Pfannkuch, 1996).

Throughout the statistical enquiry process, interpretation is a key factor in statistical reasoning, from the defining of the problem through to the conclusion. Interpretation of information in all its forms will be largely based on the students' experiences, perspectives and context knowledge. Therefore Ray thought that it was essential that his students search out alternative explanations. Other education researchers (Hancock et al, 1992; Pfannkuch, 1996) and statisticians (Bartholomew, 1995; Barabba, 1991) concur with this idea. The development of students' metacognition as well as their cognition and disposition in the specific area of interpretation would seem to be advantageous.

Some Teaching Implications

Teaching statistics where variation is the central tenet implies that when students are using statistical tools to explore data, subject knowledge and context knowledge are essential for interpreting the variation. Steinbring(1991, p. 506) believes that *"[stochastic] knowledge is created as a relational form or linkage mechanism between formal calculatory aspects on the one hand, and interpretive contexts on the other."* This may mean then that to conceptualise variation a strong linkage has to be created between the statistical thinking tools such as graphs and statistical summaries and the context of the situation. The assumption behind this epistemological triangle (Fig. 1) is that the concept of variation would be subject to development over a long period of time with different tools and different contexts.

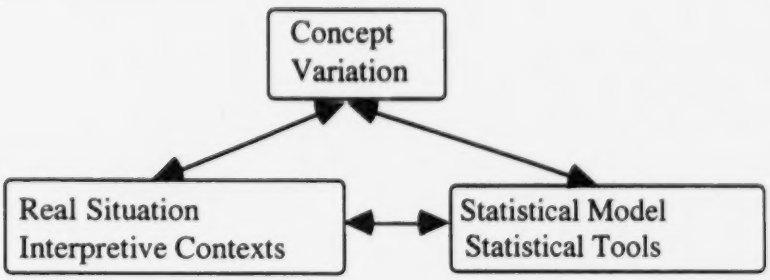

Figure 1. Epistemological Triangle

The development of statistical knowledge and concepts in this way would seem to suggest that it would also develop statistical thinking which could be regarded as the interactions between the real situation and its statistical model and between these and the resulting conceptual development. This interdependence between similar elements has been noted

by statisticians such as Bartholomew (1995) who stated that statistical reasoning was based on the interplay of data and theory and educationists such as Pfannkuch (1996) who stated that context knowledge and subject knowledge were required to operationalise statistical reasoning.

The implication stemming from this is that teaching variation through games of chance alone will not give students sufficient understanding of variation. Understanding variation in data is quite different from understanding variation in random devices (Pfannkuch, 1996). The link between the two is very subtle. Historically probability has roots in two different lines of thought: the solution of gambling problems and the handling of data (Lightner, 1991). Today the gambling root of probability dominates teaching and textbooks. The other root in data needs to flourish alongside with the emphasis being on exploring variation rather than exploring probability as it is now in some curricula (Ministry of Education, 1992). Variation in all its contexts needs to be central to statistics teaching and thinking. And since context knowledge may be needed to operationalise statistical thinking then this implies that students should be taught from contexts that they have knowledge about.

Conclusion

Looking at the world from a statistical perspective means that there is a way of knowing, a way of thinking about the world that is uniquely statistical. Moore (1990) and Ullman (1995) state that statistical thinking, that is reasoning from uncertain empirical data, is a fundamental independent intellectual method. It would appear that this reasoning is based around the interplay between the real situation and the statistical model. Underpinning this, is the assumption that data is needed to understand the real situation. Statistical reasoning may not occur unless there is recognition of the underlying variation and an understanding of how to critically evaluate that variation from a real situation and a statistical model perspective. Distinguishing the types of variation and conceptualising variation may require the development of a new way of communicating these ideas in ordinary language and in intuitive models that are intellectually accessible (Fischbein, 1987). From this one statistician's perspective it would appear that an understanding of variation in the context of the original problem is a key element in statistical thinking.

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The Potential of Computer Manipulatives for Overcoming Place Value Misconceptions

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The difficulty in teaching and learning place value concepts is well documented. One suggestion to overcome this is to use computer manipulatives to help students "bridge the gap" between symbols and concrete materials. A teaching experiment investigated Year 3 students' place value learning using either conventional or computer-generated place value blocks. Observations of one lower-achievement student using computer software indicate that well-designed mathematics software offers advantages to teachers for overcoming students' misconceptions not possible with conventional materials.

Background

The importance of place value in the primary mathematics curriculum and the difficulty teachers experience in developing place value concepts in their students are well documented. As Resnick (1983) stated,

> The initial introduction of the decimal system and the positional notation system based on it is, by common agreement of educators, the most difficult and important instructional task in mathematics in the early school years. (p. 126)

Ross (1990) likewise noted that "children find place value difficult to learn and teachers find it difficult to teach" (p. 13). One reason for this mutual difficulty is that, as Skemp (1982) pointed out, mathematics' "[conceptual structures] are purely mental objects: invisible, inaudible, and not easily accessible even to their possessor" (p. 281). Thus teachers are often not aware of what their students are thinking of with regard to mathematical objects. To answer questions about students' mental representations, researchers investigating place value understanding generally use observations of participants' behaviour to posit "various cognitive structures and processes believed to produce the behavior" (Putnam, Lampert & Peterson, 1990, p. 65). When observed behaviour reveals misconceptions about place value concepts, models of faulty internal structures can be proposed.

Misconceptions of Multidigit Numbers

The place value literature "is replete with studies identifying children's difficulties learning place-value concepts" (Jones & Thornton, 1993, p. 12). Lack of space precludes a detailed description here of the misconceptions shown by students; as Sinclair and Scheuer (1993) noted, misconceptions of place value concepts held by children are "very diverse" (p. 200). One misconception in particular, called the "face value" construct (Ross, 1989) has been reported by many researchers, and is considered in some detail here.

The "face value" construct: Though it is efficient to compute answers to multidigit questions *as if* each digit was a single unit, many children apparently believe that each digit *actually represents* only its face value. One set of tasks used by researchers to identify students using this construct is "digit correspondence" tasks (Miura & Okamoto, 1989; Ross, 1990). Participants in a number of studies were asked to show what was represented by each digit in a two-digit number showing the number

of objects in a collection. Students operating from the face value construct would indicate unit objects as representing each digit, rather than tens and ones (i.e., a student might represent 43 as a set of 4 ones and a set of 3 ones).

This faulty conceptual understanding of multidigit numbers has been observed by researchers who were investigating a variety of mathematical abilities. These included counting (Cobb & Wheatley, 1988), representations of two-digit numbers (Miura & Okamoto, 1989; Ross, 1989, 1990), comparison of pairs of two- and three-digit numbers (Sinclair & Scheuer, 1993), handling two- and three-digit numbers in novel problem-solving exercises (Bednarz & Janvier, 1982), and completing written algorithms (Fuson & Briars, 1990; Kamii & Lewis, 1991). The presence of this construct points to a critical misconception of multidigit numbers, but one that may be difficult to detect (Ross, 1990). It was found to be present in the thinking of one participant in this study, reported below.

Teaching Place Value Concepts

There is widespread support for the idea that the key to the development of higher level conceptual structures for numbers is teaching students to make connections between numbers and their referents (Fuson & Briars, 1990; Hiebert & Carpenter, 1992). Traditionally these connections have been made via concrete materials: MABs have been used for over thirty years now (Dienes, 1960) as the predominant place value materials in many countries, including the USA, the UK, New Zealand and Australia. Yet despite the popularity of MABs among teachers and academics, research shows learning effects from their use are equivocal (Hunting & Lamon, 1995). Hart (1989) summed up an idea that a number of authors seem to share:

> Many of us have believed that in order to teach formal mathematics one should build up to the formalization by using materials, and that the child will then better understand the process. I now believe that the gap between the two types of experience is too large, and that we should investigate ways of bridging that gap by providing a third transitional form. (p. 142)

Several authors have recommended the use of computer manipulatives for bridging this gap between concrete materials and symbols (Champagne & Rogalska-Saz, 1984; Clements & McMillen, 1996; Thompson, 1992). However, there is little research that investigated precisely what the advantages or pitfalls of using computer manipulatives for place value teaching may be, and how they compare with standard MABs. The present study is planned to address this gap in the literature. A software application (Price, 1997) designed to help students develop accurate models for multidigit numbers was used in this study with one cohort of students, and MABs with another.

This study investigates the use of two different forms of representations for multidigit numbers, computer-generated and standard place value blocks. The aims of the research are: (a) to identify the conceptual structures for numbers held by the participants; and (b) to look closely at the interactions between individual students' conceptual structures and their use of either conventional MABs or an innovative software application. Specifically, we are interested in whether either material will *support* or *confirm the validity* of accurate models of multidigit numbers, and on the other hand, whether they will *challenge* or *contradict* students' misconceptions.

Research Design

Research Questions

Two broad questions are being addressed in this research:

a) Are there differences in children's development of conceptual structures for multidigit numbers when using two different representational formats (place value blocks and computer software)?

b) What differences emerge as children of different ability groups learn place value concepts using two different representational forms and associated processes?

Assumptions

A number of assumptions underlie this research. First, whilst deliberately avoiding the troublesome question of what numbers actually are, and where they exist, it is assumed that most people will by their actions treat numbers as entities, and so for them numbers do exist in some form (Sfard, 1991; Sfard & Thompson, 1994). Second, the conceptions that a person holds for numbers form a system that has a structural form; the system incorporates rules by which the conceptions may be manipulated (Ohlsson, Ernst & Rees, 1992; Resnick, 1983). Third, there is some relationship between internal and external representations of numbers (Hiebert & Carpenter, 1992; Putnam et al., 1990). Fourth, the nature of internal representations of numbers may be deduced from a participant's responses to particular mathematical tasks (Resnick, 1983, 1987).

Method

The basic methodology used for this study was a teaching experiment (Cobb & Steffe, 1983). Ten daily teaching sessions were used, with the author taking the role of teacher for all participants. A pretest and posttest developed for this study were used to analyse specific features of place value knowledge held by each student. Participants worked either in pairs or singly, and used either conventional MABs or a computer model of MABs. Both types of material were used as a teaching tool with an experienced teacher; it is not assumed that the materials themselves will be effective in improving children's ideas of numbers. As Hunting and Lamon (1995) pointed out, the mathematical structure of multidigit numbers is not contained in any material. Learning can only take place when the "[learning] experience is meaningful to pupils and . . . they are actively engaged in thinking about it" (Baroody, 1989, p. 5).

Software design: The software application has been designed to provide up to five different representations of any whole number up to 999 that may be switched on or off by the user. The software will represent a number as: (a) a standard arrangement of hundreds, tens and ones blocks; (b) a written symbol; (c) a numeral expander that may be opened or closed at each place; (d) an audio recording of the spoken number name; or (e) a variety of non-canonical arrangements of blocks (i.e., with more than nine in a place). Though there are some features of MABs that are *not* present in the software, educational software offers a number of advantages over conventional concrete materials that cannot be disregarded (Clements & McMillen, 1996). In particular, Clements and McMillen noted:

> Actual base-ten blocks can be so clumsy and the manipulations so disconnected one from the other that students may see only the

> trees—manipulations of many pieces—and miss the forest—place value ideas. The computer blocks can be more manageable and "clean". (p. 272)

It is hypothesised that by making multiple representations of a number available to a student, the software will aid children in the development of useful, accurate conceptual structures for multidigit numbers.

Participants

Participants in the study reported here were selected from the population of Year 3 students (aged 7-8 years) at a primary school in a small rural town north of Brisbane. Participants were selected at random from two pools: students of either high or low mathematical achievement, based on the previous year's Year Two Diagnostic Net, a state-wide test used in Queensland to identify students at risk in the areas of literacy and numeracy. Students in pairs were matched for gender and ability, on the assumptions that at this age children will prefer to work with same-gender partners, and that maximum learning would be possible if both children were of similar mathematical ability. Results for only one child are reported here, to illustrate how the specific features of the software enabled her to develop a more accurate understanding of place value.

Case Study: Nina

One participant in the study reported here demonstrated a number of key points already alluded to in this paper. Nina (not her real name) was identified by the Year Two Net as being in need of remedial help in mathematics and language.

The pretest demonstrated that Nina had severe misunderstandings of numbers, and showed great confusion about the meanings applied to symbols and concrete representations of numbers, and about the magnitude of numbers. It was evident from several of her responses that Nina was operating according to a face value concept of multidigit numbers. For example, when asked to show a two-digit number (43) with MABs, she showed 4 tens and 3 ones without hesitation. However, when asked to "show 43 another way", she exchanged the numbers of blocks, and showed 4 ones and 3 tens. When shown 7 hundred blocks and 3 one blocks, she simply counted the blocks as individual entities, and stated that the represented number was 10.

Throughout the ten teaching sessions Nina continued to demonstrate evidence of the face value construct. When considering two-digit numbers, she frequently used the method of adding the face values of the number to decide its value. For example, she stated that 39 was "twelve altogether", and that 72 was "nine". The following excerpt from a teaching session illustrates some of the confusion she had in understanding symbols for two-digit numbers:

I - Nina, can you tell me what these two numbers [36 & 63 written on paper] are?

S - 66 *[puts hand on mouth]* Oh, 36, 63. They're both the same, because the 3, the 3 is at the back, and the 3 is at the front, and [points] the 6 and the 6, and the 3 and the 3.

I - Well, Nina, I was going to ask you which one is bigger?

S - Oh, which one is bigger? 36 is bigger than 63.

I - How do you know 36 is bigger?

S - [indistinct; holds up both hands with fingers outstretched] because . . . [waves hands in the air] I don't know.

Though Nina frequently changed her mind in answering questions, her fundamental (faulty) conceptions were very robust, and would stand up to sustained challenges from

the interviewer. One of her beliefs that seemed to reinforce her misunderstandings about tens and ones was her claim that numbers are different in New Zealand! She evidently started pre-school in New Zealand, and had deduced from her experiences on either side of the Tasman that numbers were different in the two countries.

The interviewer used the idea that misconceptions may be overcome through challenges presented by a teacher (Ginsburg, 1981). Ginsburg used the technique of testing a participant's "strength of belief" by challenging the child when he or she seems uncertain or hesitant in the response to a question. The two techniques recommended by Ginsburg to test strength of belief were "counter-suggestion" and "repetition of the problem or introduction of highly similar problems" (p. 10). Both these techniques were used to challenge Nina's beliefs about two-digit numbers. The computer software was used to augment these techniques, by presenting a number of representations of numbers that countered the participant's assertions. For example, when asked to compare written symbols for 48 and 73, Nina was certain that 48 was larger, saying:

S - That's got 7 and 8 is more than 7, and 4's more than 3 so that *[48]* is the highest number and that*[73]*'s the smallest.

Though the interviewer repeatedly attempted to show her that the computer-generated block representation showed that 48 was smaller, she did not accept it. She persisted in her belief that 48 was larger even when an on-screen numeral expander was used to show that 73 was 7 tens and 3 ones, or 73 ones (Figure 1).

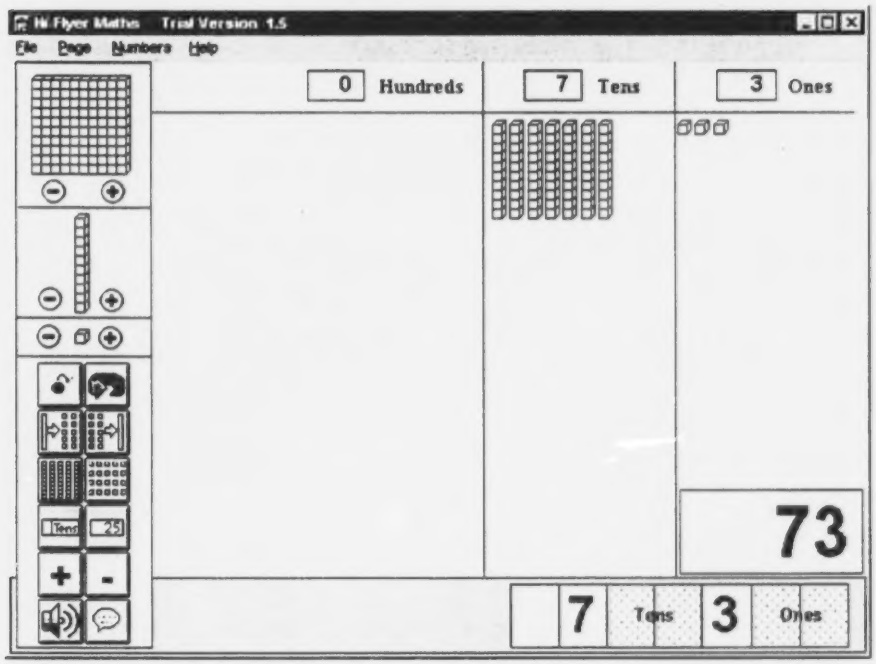

Figure 1 Screen shot of *Hi-flyer Maths* showing numeral expander and conventional block representation of number

Some success was achieved when another aspect of the software was accessed. The software includes a "show as ones" feature, which presents a number solely as ones

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blocks. When this feature was used, Nina was able to count the ones in 73 and see that it included more than 48 ones (Figure 2). The following conversation took place:

I - Stop. That one there is number 48 isn't it?

S - Yep. I said 48.

I - And all these over here are still to come. [points to remaining ones in 73]

S - Ahh...

I - Now remember when we looked at 48 you said 48 was smaller than 73.

S - Yes.

I - Have a look up here. If we count the first 4 rows and the next 8 here, that's 48. And we've still! got all these here. [points again]

S - Oh, I think I'm wrong. I think that's higher. *[points to written 73]*

I - I think that's higher too. [smiles]

S - I was wrong.

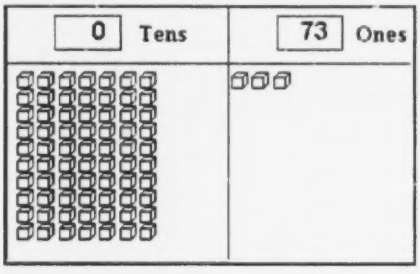

Figure 2 Section of screen showing "tens as ones" representation

It would be pleasing to report that this incident led to an immediate change in Nina's conceptual structures for two-digit numbers. However, the posttest revealed that many of her confusions remained. For example, after counting 36 pop sticks and asked to show which sticks were indicated by the two digits in the number 36, she put out a group of 3 and a group of 6 sticks, indicating that she was still operating according to the face value construct. On the other hand, there was some indication that she had a different understanding of two-digit numbers as a result of the teaching program. When asked in the posttest to compare 61 and 39 she stated that 61 was larger, because of the number of tens.

It seems reasonable to make the conjecture that for a child such as Nina there would need to be many more sessions where her misconceptions were challenged, and correct conceptions reinforced.

Discussion

In the situation described above the software enabled the interviewer to persist in presenting several counter-suggestions to Nina's assertions about the numbers 48 and 73, until she found that her belief could not be sustained in the face of contrary evidence on the computer screen. While that sort of process is possible with physical blocks, it is doubtful whether it would be as effective, for the reasons given by Clements and McMillen (1996) above.

The software offers a teacher great versatility in presenting several representations of a number simultaneously, as needed. In this instance the "show as ones" feature was accessed in order to challenge Nina's belief that tens were single entities, rather than collections of ten ones. In different session with a different child, another feature may

be useful for illustrating an instructional point. The principal rationale behind the software design is that connections among numbers, symbols and other referents are of critical importance in learning place value concepts (Hiebert & Carpenter, 1992). By offering the teacher (and student) a range of available options in representing a number, the software provides a level of versatility in demonstrating connections that is not possible with conventional physical place value blocks.

Salomon (1979) noted that each form of technology offers a different set of "affordances" that set it apart from other technologies; we would extend this idea to individual software applications. This piece of software offered the particular advantage of presenting multiple representations for a number under consideration, which enabled the interviewer to lead the student to confront her own misconception. Without the specific features built into the software, or ones like them, it is likely that Nina's misconceptions may have been unaltered during the teaching session.

The study reported here is still in an early stage, and much data collection and analysis are still to be done. Nevertheless we believe that the evidence from these early observations indicates specific strengths that computer software can have for the teaching of the abstract concepts of mathematics, particularly in the area of place value.

Well-designed mathematical software enables a teacher to draw on resources that otherwise might not be available at all. We assert that software for the teaching of mathematics should be designed with these affordances in mind. The other factor of crucial importance in this endeavour is a clear model of how place value concepts are developed in children's minds; this research is hoped to offer new insights in this area.

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Teacher Presence as a Variable in Research into Students Mathematical Decision-making

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There have been many policy recommendations for students to become more active in their learning of mathematics, and to make sensible choices of computation method. Year 5-7 students were asked to choose among calculator, written and mental computation methods to answer a series of multiplication questions, with a teacher either absent or present. Findings indicate that the students made choices based on what they believed the teacher "really" wanted, rather than on valid mathematical factors.

Rationale

During the 1980s and early 1990s a number of major policy documents for mathematics education were published, including the NCTM's (1989) *Curriculum and Evaluation Standards for School Mathematics*, the National Research Council's [NRC] (1989) report *Everybody Counts*, the UK Cockcroft Committee's (1982) *Mathematics Counts* and in Australia the Australian Education Council [AEC]'s (1990) *A National Statement on Mathematics for Australian Schools*. Given the ongoing technological revolution, it is not surprising that each of these documents should include advice regarding the use of calculators and computers in school mathematics; nor is it surprising to find much similarity in that advice. Overwhelmingly the authors of these reports urged mathematics teachers to include calculator and computer technology in their lessons. Specifically, teachers were encouraged to (a) include calculators as a means of computation; (b) encourage students to engage in learning for themselves, and to take some responsibility for knowledge and understanding gains (NRC, 1989); and (c) teach students to make sensible choices among available methods of computation to use in a variety of situations (e.g., AEC, 1990; NCTM, 1989).

Despite the abundance of advice in the literature to include technological devices in the teaching of computation, anecdotal evidence suggests that many teachers are not following that advice in the manner in which it was intended. It appears that in many classrooms written computation is still the major focus of most mathematics lessons, and calculators are given little place. The study reported on here arose from a perceived need to find more about students' ideas of when the various computation methods should be used. In particular a number of potential influences on student choices were chosen as independent variables: question format (symbols or word problems); student year level; type of number involved; and teacher presence. Though precedence was found for investigating the influence of each of the first three factors listed (question format, year level, and number type), no studies were found which looked at the interaction of a teacher's presence or absence on a student's decisions.

The question of the effect of a teacher's presence on children's choices relates to complex relationships among student, teacher, and subject matter. Four observations seem pertinent at this point. First, there is much evidence that children do not see what happens in school as necessarily making sense, and apparently often do not even expect it to do so (Silver, 1994). Willis's (1990) comment is quite appropriate here:

> Unless students see the relationship between mathematics and its uses as they proceed they may never develop a view of mathematics as making sense in their world and as having relevance to the solution of their problems. (p. 10)

Second, students have traditionally been taught as passive receivers of knowledge "from above," and have not been expected to act or think independently from the way they have been taught (NRC, 1989). Though this is changing, there is much evidence to suggest that echoes of this past model of teaching and learning persist today. In other

words, even though teachers have been urged for some years to foster independence in their students (NCTM, 1991; NRC, 1989; Rutherford and Ahlgren, 1990), in many classrooms students are still taught to be unthinking followers of others, and particularly their teachers.

The third point, following from the second, is that children tend to behave in school as they believe they are expected to. McIntosh, B. J. Reys, and R. E. Reys (1992), for example, told the anecdote of a boy who was reluctant to admit to using mental computation to calculate 37 + 25, telling an interviewer it was "Because I wouldn't get a mark then. I can't understand the way she tells us to do it on paper, so I do it this way in my head and then write down the answer and I get a mark" (p. 2). In other words, the child felt the (very real) need to make a pretence of following his teacher's directions. The child did so even though the teacher's directions made no sense to him, and though he was capable of performing better using another method which to him evidently made perfect sense. Willis (1990) likewise described a child who demonstrated a commendable level of skill at mental computation, but got the same questions wrong when they were presented in written form. Willis commented: "All the power was in the teacher's hands . . . it was his teacher's task and his teacher's rule, not his" (p. 3). The fourth point is that some teachers, parents, and even children see written algorithms as the "proper" way to carry out computation, and believe that using a calculator is somehow "cheating" (Hembree & Dessart, 1992; Price, 1995).

The combination of the above propositions leads to the following conjecture: If children believe (a) that mathematics does not make sense, (b) that they should follow their teachers' directions even if they do not make sense, and (c) that they are expected to carry out computations "properly" without using a calculator, then the presence or absence of a teacher *will* affect children's choices, especially concerning the use of a calculator. The research reported here set out to investigate this hypothesis.

Aims

The aim of the study was to discover factors having an influence on upper primary students' decisions about which method of computation to use to answer multiplication questions. It was hypothesised that students' choices of computation method would be influenced by the form of presentation of the questions, the students' age or maturity and the presence or absence of a teacher.

Independent variables chosen were: students' year level; number type (extended basic facts, aliquot parts, or other two digit numbers); question format (word problem or symbols); and teacher presence. Only the last variable, *teacher presence* is reported on here. As mentioned above, this variable had not previously been included in research of this sort.

Method

A random sample was chosen from students in years 5 to 7 (ages 10 to 12 years) at an independent school north of Brisbane, Queensland. The school in question tends to promote a style of teaching and learning that is teacher-directed, with an expectation that students will respect teachers' directions and follow them closely. This point will be seen to be relevant to the study's findings below. The sample was stratified to include equal numbers of both genders and equal numbers of students of high, medium, and low mathematical ability, based on assessments made by class teachers. The sample comprised 18 students from each of years 5 and 6, and 16 students from year 7. There were fewer students from Year 7 owing to there being too few students available for selection.

Each student was observed individually as he or she answered a series of 12 multiplication questions printed on cards. The questions were written either in symbols or as word problems. Numbers included were in three categories: extended basic facts (e.g., 20 x 50); aliquot parts (e.g., 25 x 16); or other two-digit numbers (e.g., 31 x 29). A calculator, pencil and paper were provided, and each student was instructed to use whichever method of computation he or she desired for each question. Each session was videotaped.

Rather than merely asking students to nominate or describe the computational method they would use to answer each question (as used by B. J. Reys, R. E. Reys & Hope, 1993), the student actually carried out the computation. This method introduced an obvious difficulty, namely that of determining when mental computation was used. This was handled by assuming that if a student paused in the answering of a question without writing anything, and then wrote the answer, that student had used mental computation. Conversely, the benefit from using this method was that the setting was more authentic, being closer to that of actual practice in a classroom. In particular, if a student planned to use a certain method (mental, for example), and found the calculation too difficult, then the student was free to change to another method. By only asking the students how they *intended* to answer questions, B. J. Reys et al. (1993) found it difficult to interpret some students' claimed computational choices. For example, some students stated they would use mental computation to answer a question such as "29 x 31"; the authors offered the opinion that some students may have viewed as the same as "30 x 30", but admitted this could only be found out by using an interview.

To include the variable of teacher presence, for half of each interview the interviewer (a teacher at the school, and known to all the participants) left the room, asking the student to complete the remaining six questions alone. Each interview was videotaped for later analysis.

Results

Table 1 records the participants' choices in the presence of the teacher-interviewer, and in his absence. There was very little difference in the use of mental computation between the two situations; the main difference was in written and calculator computation methods. There was a switch of approximately 10% of all choices from use of paper and pencil to calculator when the teacher was absent which, based on the chi square test, was statistically significant at the .05 level.

Table 1 *Choice of Computation Method When Teacher is Present or Absent* (N°=°52)

	Computation Method (%)		
Teacher Presence	Written	Calculator	Mental
Teacher Present	55.8	25.6	18.6
Teacher Absent	46.2	36.4	17.5
Chi square: 8.74 ($df = 2$)*			

*p <.05

Table 2 shows the breakdown of choices by year level. It has to be pointed out that the video camera recording the participants' actions was not concealed at all. Thus, though results are statistically not very significant, it may reasonably be assumed that trends evident would have been more pronounced had the camera been concealed. The reasoning behind this is that the absence of the teacher did make a difference in students' choices, even though the video camera was obviously recording the students' actions when the teacher was absent. Thus it was just the *presence of the teacher* that made the difference, despite the fact that the students' actions would be plain to the teacher from the video tapes. It is reasonable to assume therefore that the evidence here demonstrates a factor in the students' thinking that may have been more pronounced had the recording camera been concealed.

Year 5 and Year 6 students chose to use the calculator an average of 13% more often when the teacher was absent, than when he was present. Mental computation was also reduced by students of these two year levels in the absence of the teacher, though not by very much; most of the decrease was in the use of written methods.

Year 7 students' choices did not show a statistically significant difference, though the change that did occur was in the same direction as for the younger students. The reasons for this difference can only be guessed without further investigation, such as through

interviews. However it is hypothesised that the older students were more confident than the younger students to choose computational methods for themselves in the presence of the teacher.

Table 2 *Choice of Computation Method When Teacher is Present or Absent, Year by Year*

Teacher Presence	Computation Method (%) Written	Calculator	Mental
	Year 5 (n = 18)		
Teacher Present	56.5	24.5	19.0
Teacher Absent	44.4	37.9	17.6
Chi square:	4.72 (df = 2)*		
	Year 6 (n = 18)		
Teacher Present	63.4	18.5	18.1
Teacher Absent	53.7	31.9	14.4
Chi square:	5.19 (df = 2)*		
	Year 7 (n = 16)		
Teacher Present	46.3	34.9	18.8
Teacher Absent	39.6	39.6	20.8
Chi square:	0.90 (df = 2)**		

*p <.1
**ns

Discussion

These findings raise a number of interesting, and potentially very important, questions regarding the interaction between teacher and students in a classroom. First, it is clear that the students as a group felt constrained to use written methods more, and consequently the calculator less, when the teacher was present. Since the students' decisions altered after the teacher left, it appears that for some reason they felt that the teacher wanted them to use written methods, though they themselves preferred to use the calculator.

The children were not questioned later about their choices, which could have provided valuable data about these constraints. However there is anecdotal evidence of students (a) attempting to conceal their use of the calculator, or (b) making verbal comments about the use of the calculator, that are quite revealing. First, a number of students were observed concealing their use of the calculator. Lucy (not her real name), a high-ability year 7 student, used paper-and-pencil or mental methods for every question in the presence of the teacher. When the teacher had left, however, she hid the calculator buttons with a question card, and while holding her pencil, pressed the calculator buttons behind the card. Thus she attempted to make it appear that she was continuing to use paper and pencil, while actually using the calculator. Another Year 6 student, Julie, casually dropped her hat onto the calculator after the teacher left, and then moved the calculator off the table, and used it below table level, clearly attempting to hide it from the camera. Second, a number of students made verbal comments during interviews indicating that they believed there was a hidden agenda in place. These include comments from Mick that "We're not allowed to [use a calculator]", and Eric's "You can use a calculator, it's your choice, but you get better marks if you're using your head". These actions and comments occurred despite the interviewer twice giving clear permission to use whichever method the student desired. These observations give to support to three hypotheses about the students' decisions. First, students seemed to have the idea that one or more methods were preferable *in the teacher s eyes* to others. Specifically it seems that the students believed that either mental or paper-and-pencil computation was preferable in some way to use of a calculator (as evidence, see Eric's

comment above). Since using a calculator is easier, quicker, and more accurate than using written computation, the test in the students' eyes would appear to be about the correctness of the computational choice and perhaps about whether they would be "lazy", or "cheat" (see Hembree & Dessart, 1992, p. 30).

A second conclusion that may be drawn is that at least some children would have made different choices, had they been totally free to do so. Calculator use rose in the absence of the teacher, and students were observed attempting to use the calculator without being seen to do so. Thus it appears that they *wanted* to use the calculator but felt compelled not to. It may be presumed that for some students, at least, their sense of "correct behaviour" prevented them from choosing to use the calculator.

Third, it may be surmised that these children are used to "playing the game" of school, and will quickly and often unobtrusively alter their behaviour to meet certain (real or imagined) criteria. As mentioned earlier, the school in question fosters an environment in which there is a clear expectation that students will follow teachers' directions with little questioning. This may have been a significant factor in the findings, though it is likely that other schools' environments are similar in this respect. The task put to the students in this study was quite straightforward: Answer a series of familiar questions using any of three available methods. In fact some students (using the calculator for all or most questions) answered all the questions in five or six minutes, including time for the interviewer to give instructions. However, this apparently simple task was made much more complicated for some students by the evident inferences made by them, that various "unwritten rules" were in place, and that they should do their best to behave according to these rules, to maximise their "score". This bears out observations reported elsewhere (Morgan, 1989; Silver, 1994; Willis, 1990) that students see mathematics as being about "getting the right answer", and not about making sense or developing skills that are useful outside the classroom.

Conclusions

Implications for Teaching

The history of teaching of mathematics has been one of majoring on the mechanics of carrying out written algorithms, rather than the sensible use of a variety of mathematical tools to solve real world problems. This has produced generations of people who believe that mathematics is about using a single standard method for each type of question, about answers rather than processes, about treating mathematical questions as separate from and largely unrelated to the real world.

This study showed some of today's students behaving in this manner. The participants were observed trying to "play the game" of mathematics, to guess what the teacher *really* wanted, to do mathematics according to what they thought was *expected*, rather than to what was *sensible*. When students have learned that mathematics is unrelated to real life, that it is not supposed to make sense, then their mathematics education has been seriously flawed, and in a manner that impoverishes their future prospects for mathematical thinking.

The evidence described above indicates a clear need for openness and honesty from teachers when discussing computation. The question of how to solve a mathematical problem, including the means of computation to be used, is associated with issues of correct behaviour and student and teacher expectations. Thus it is important that teachers are candid with their students about the issue of how to make decisions in this area. It is important that teachers include in their teaching of computation activities that require their students to make choices between calculator, mental and written methods of computation, and openly discuss factors that influence such decisions. As R. E. Reys (1994) put it, teachers need to develop a "broad conception of computation" (p. 2).

The question of the place of written, mental and calculator computation methods continues to receive attention in the literature (Price, 1997; R. E. Reys, 1994; Sparrow, Kershaw & Jones, 1994); this study shows that we cannot assume that primary students will make sensible decisions in this area without assistance.

Recommendations for Further Research

Two strands of further research are suggested by this study. First, further work is needed in the area of computation in general, and students' use of various methods in particular. The advice regarding the use of technology in computation notwithstanding, much work still needs to be done to answer a number of important questions about the implementation of that advice. A list of many such questions was provided by Robert Reys (1994). Some examples:

> How should computational alternatives (mental computation, estimation, written algorithms, calculators) be developed? When should computational alternatives be introduced? . . . How are wise choices of computational alternatives developed? Do students know when mental computation is appropriate? (p. 5)

The second area of research that warrants further investigation is that of teacher presence, and its effects on students' thinking and decision-making. Answers to questions such as these could reveal important details about the interaction between teacher, student and learning:

- Do teachers' critical notions about mathematics match those of their students? How does the presence of a teacher influence students' problem-solving activity?
- Do students see themselves as being in charge as they answer mathematical questions? How do students construct their role, and that of the teacher, in such activities?
- How does a teacher's attitude to conflict and disagreement in the classroom influence students' attitudes to and learning of mathematics? How is a "mathematical community" (NCTM, 1991, p. 3) developed?

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The Effects of a Computer Algebra System on the Learning of, and Attitudes towards Mathematics amongst Engineering Students in Papua New Guinea.

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This paper reports on a study which investigated the effects of a computer algebra system, *Derive*, on the learning of, and attitudes towards mathematics amongst engineering students in Papua New Guinea. The study compared the traditional course approach to an experimental approach integrating *Derive*. The findings indicate no statistically significant differences in achievement performances between control and experimental groups, nor in students' attitudes towards mathematics. However, positive indicators of enjoyment, motivation and usefulness of mathematics observed with the experimental group are considered educationally significant.

Introduction

Only a moderate amount of literature is available relating to research in mathematics education and the learning of mathematics in the South Pacific region, and Papua New Guinea (PNG) in particular. In PNG, with computers only relatively recently finding their way into the education system, a paucity of research (Arganbright, 1989; Seldon, 1984; Sheridan, 1985) exists pertaining to the use of computers in the teaching and learning of mathematics. No literature was available on any studies or findings relating to the use of a computer algebra system (CAS) in the country, or on students' attitudes towards the use of computers in the learning environment.

World-wide, research in connection with the use of computer algebra systems in the educational setting has been steadily growing since the early nineties (e.g., Berry, Graham & Watkins, 1993; Hillel, Lee, Laborde & Linchevski, 1992; Hunter, Marshall, Monaghan, and Roper, 1995; Judson, 1990; Mayes, 1995; Palmiter, 1991). The majority of these researchers have performed control experiments with CASs and, while observing improvements in student performance and attitudes, they have not generally shown their findings to be statistically significant. They are usually cautious in their conclusions, stating that the improvements cannot be attributed only to the use of the CAS and suggest extensions to their studies.

Some published research has shown that PNG students have problems with language in mathematics (Clarkson, 1987; Jones, 1982, Roberts, 1989), and Audrey and Michael Wilson (1983) demonstrated that about one quarter of PNG students entering tertiary studies were at Piaget's 'abstract reasoning' level, while another quarter was still at the 'concrete material' level. The remaining half of the Wilsons' sample was at a transitional stage of thinking abstractly when on familiar problems, but turned to a concrete approach when confronted with new ideas. Students completing grade 12 in PNG are thus, from a curriculum point of view, approximately 1.5 to 2 years behind that their counterparts in Australia and New Zealand. The Foundation programme at the PNG University of Technology (UNITECH) attempts to bridge the gap from grade 12 to the commencement of certificate, diploma and degree studies. As a result, the foundation courses tend to be heavy in content, and many topics are covered superficially and with little depth. The Foundation Year Engineering (FYE) mathematics course thus tends to be structured around procedural knowledge and which emphasises routine methods. Concepts of generalisation, proof and logical reasoning have to be developed very gradually, and are usually met with frustration and confusion by many students. In this

course, students work with locally written modules and tutorial groups consist of 15 to 20 students. Although attempts are made at stretching students' minds to apply learnt concepts, they find difficulty in independently extending their thinking beyond the familiar. Students need to be given 'thinking' assistance to help them to recognise patterns and generalisations. Roberts (1989, p. 196), commenting on his observations of grade 12 school leavers in PNG has said that, on entry to tertiary studies, only a minority of students are "ready to proceed to more formal deductive mathematics" and "problems and situations which stress logical thinking and generalisation should be presented" to the bulk of students.

The author of this paper, as coordinator of the FYE Mathematics Programme at UNITECH, was familiar with the computer algebra system, *Derive*, and felt that with its interactive and visual approach, it could provide students with 'thinking assistance' in recognising patterns and generalisations, as well as being a medium of visual and interactive learning where the teacher's role of a verbal communicator was reduced. Much of the emphasis in the learning would thus be centred around student interaction with *Derive* and student-to-student discussion in a language of their choice. Tutors would also be present for further assistance. It was also hoped that this mode of learning would encourage more student motivation and help to develop improved attitudes towards mathematics. These expectations formed the basis and underlying motive for this research. The objectives of the *Derive* sessions were seen as:

- reinforcing the learning that was taking place. Worksheets would be staggered by two weeks, that is, students would work through material on the computer which would have been presented in lectures about two weeks earlier. Thus, students would have had a good opportunity to try fairly routine and regular problems with pencil, paper and calculator in their classroom tutorials before reviewing similar work in a computer session.
- extending some of the learning to real-world application situations where the computations could be taken care of by *Derive* and students could think a little more about the structure of the problem.
- helping to develop an investigative attitude to mathematical problems.
- motivating students' interest in the mathematical concepts they were learning
- helping to improve students' attitudes towards mathematics.
- making students more aware of the need for, and use of, correct notation and meaningful interpretation of the symbols used.

With these objectives, the role played by *Derive* was seen as pedagogical in nature, with the emphasis on helping to develop and encourage student thinking, which in turn could aid conceptual understanding.

Aims of the Study

The overall aims of the study were twofold:

- to determine the efficacy of computer algebra systems in aiding and improving the achievement in mathematics of engineering students at the PNG University of Technology.
- to determine whether the use of computer algebra systems could assist in improving the attitudes that students have towards mathematics.

'Achievement in mathematics' here means the student's performance in the tests and examinations taken as the part of the normal Foundation Mathematics programme. As these assessment measures would involve, in the main, procedural knowledge and routine applications, they could not be interpreted as reliable indicators of a student's conceptual knowledge of the mathematics involved.

Methodology

The research was concerned with a comparison of the traditional lecture/tutorial-based approach with an experimental approach which integrated the computer algebra system, *Derive*, into the traditional programme. The subjects of the research were 186 Foundation year students, and these were divided into experimental and control groups. Due to a variety of constraints, full experimental control in the allocation of students to class groupings was not possible and random assignment and equivalence of groupings could not be assumed. The design of the research was therefore quasi-experimental using a non-equivalent control group design (Campbell & Stanley, 1966). In addition to the quantitative measures, the design further involved a qualitative aspect based on written comments made by students on the use of *Derive*. The students' achievement in mathematics, as determined by their performance in tests and examinations, and their attitudes towards mathematics, as measured through questionnaires, were the dependent variables in this research.

A pilot study was conducted with the 1994 intake with the view to develop and trial worksheets for the computer sessions, determine logistics, familiarise staff with *Derive*, observe students and obtain feedback from them. The main research was conducted over two semesters in the following year, with the experimental groups having one of three regular weekly tutorials replaced by a computer tutorial. All students did a common review exercise at the beginning of the first semester.

To minimise bias, as far as possible, an experimental group and a control group were assigned to a same tutor. Under the staffing and time-tabling restrictions, it was possible to do this with three experimental/control pairs. The main analysis was centred on these six groups.

Researcher-designed questionnaires based on Sandman's (1980) attitude subscales were administered to obtain students' views on the use of *Derive* and to determine their general attitudes towards mathematics. The items were designed by the researcher specifically for the FYE students, with wording on the items being kept simple, unambiguous and pertinent to the PNG context. The comments from students with respect to working with *Derive* would provide a qualitative aspect to the research. It was hoped that the analysis of objectively measured variables could be given further interpretation through the use of qualitative measures. The students' comments relating to *Derive* would also provide vital feedback towards improving and modifying future approaches with computer tutorial sessions.

Results

A one-way analysis of variance for unequal sample sizes performed on the achievement scores on the common review exercise determined that the experimental and control groups could be considered to have the same means and be treated as statistically equivalent (Computed $F(1,176) = 0.7010 < F_{0.01} = 6.63$). Bartlett's test (Walpole, 1990) further indicated that both experimental and controls groups belonged to populations with equivalent variances ($b = 0.9986 > b_2(0.05; 88, 89) \approx 0.9778$).

As three experimental/control pairs of groups shared the same tutor and sample sizes were unequal, it was decided to use the tutor as a 'blocking' factor and implement a two-factor non-orthogonal or unbalanced design (Keppel, 1991). The two factors involved in this design were the treatment and the tutor. While the focus of the investigation would be on the treatment, the second tutor factor would help to facilitate the study of the treatment factor. The non-orthogonal analysis of variance performed on student mean achievement scores indicated that treatment has not produced any significant effect on achievement after two semesters at a 5% level of significance ($F(1,109) = 1.570 < F_{0.025} \approx 5.176$), setting $\alpha = 0.025$ in order to compensate for the slight bias associated with this test. This analysis further indicated that the tutor had not produced any

significant effect on achievement after two semesters ($F(2.109) = 0.783 < F_{0.025} \approx 3.824$), and that there was no significant interaction between the tutor and the treatment ($F(2,109) = 0.315 < F_{0.025} \approx 3.824$).

The total scores on the attitudes questionnaire were also analysed using a non-orthogonal analysis of variance. This analysis also showed that the treatment had not produced any significant effect on students' attitudes towards mathematics at the end of the second semester ($F(1,90) = 0.4965 < F_{0.025} \approx 5.22$). The analysis also indicated that the tutor had not produced any significant effect on achievement after two semesters ($F(2,90) = 1.3993 < F_{0.025} \approx 3.865$), and that there was no significant interaction between the tutor and the treatment ($F(2,90) = 0.8722 < F_{0.025} \approx 3.865$).

Similar analyses were carried out with respect to five attitude subscales: enjoyment in mathematics; self-concept in mathematics; anxiety towards mathematics; motivation in mathematics; general value/usefulness of mathematics. While no statistically significant differences were found in relation to these subscales, the control group tended to have a slightly higher mean on all subscales except on 'general value/usefulness of mathematics'. This subscale related mainly to how students perceived mathematics as a subject. It may be possible that working with *Derive* has brought some realisation that mathematics is not just a matter of learning rules and applying them, but rather one of reasoning and logical thinking.

Responses and optional written comments made by the experimental students on additional questionnaire items pertaining to the use of *Derive* indicated quite a favourable and positive reaction to working with *Derive*. 81% clearly felt that the computer tutorials helped with improving their understanding of the mathematics; 85% agreed that a computer package such as *Derive* should be used in FYE mathematics; 62% stated that working with *Derive* had made them more careful in their reading and writing of symbolic algebraic expressions; 81% indicated that they had a more positive attitude towards mathematics as a result of the computer tutorials. It is noteworthy that 84% of the students wrote down optional comments. As there was no compulsion for students to write down anything or place their names on the questionnaire, it was interpreted that their comments were frank and honest and reflected genuine concerns, suggestions or appraisals.

A generally good and positive attitude comes through on reading through students' comments. Words such a 'pleasure', 'helpful', 'interesting', 'enjoyable', 'fascinating', are quite frequent. The students have quite clearly been excited with the chance to use *Derive* in their learning. Students were writing these comments at the end of the semester, by which time it could be taken that much of the excitement due to the novelty of working with computers would have diminished considerably. Many commented that working with *Derive* had helped with their understanding, and clarified the mathematical ideas. There was much support for computer interaction in learning. Many students expressed the desire for more availability and access to computers in order to work with *Derive* in their spare time. With a high percentage of students having little or no prior interaction with computers, it was not surprising to see frequent comments relating to general computing skills. There was also a desire present to know more about computers. One student stated, "*Using derive ... also aids the student to be comfortable when using a computer without fear as is always evident with such a pro-primitive society*". The student, clearly aware of the way the new technology is impinging on PNG society, is recognising the user-friendly nature of a computing package as an introduction to working with computers.

From other comments, it was also evident that a few students had misunderstood the role of *Derive* in their sessions. These students were almost certainly simply following the instructions on the worksheets, pressing keys, and so on, and merely observing results obtained on the screen. Such students expressed concern that the computer was merely giving them answers and not showing them the intermediate steps. They saw *Derive* as a glorified graphical calculator and ceased to do any thinking beyond their actions with the keyboard. The exercises in the worksheets were designed to create

an investigative approach to the mathematics, while at the same time reinforcing ideas met in lectures. This clearly was not having its effect with some students. Other students were concerned about the effect that using computers may have on their own "*natural computer*", namely their own thinking and analytic processes. These students, too, were most likely to be preoccupied with obtaining correct answers and less concerned with the processes involved. While there were a few negative aspects, the overall picture, in relation to the comments made by the experimental group students, has been one of much encouragement to the researcher lending support to the argument that the results could be considered as educationally significant.

Discussion

These results demonstrate that the attempt to integrate *Derive* into the FYE mathematics programme *in its current state* is not very likely to help students with their achievement scores. Despite experimental students stating that *Derive* brought more understanding, enjoyment and motivation, it would seem that these had little effect on the students' performance in tests and examinations. Currently, students are tested on the performance of routine algorithms and reproduction of memorised procedures, with little emphasis on testing for conceptual understanding. If *Derive* did enhance experimental students' understanding, then this suggests that the present assessment of student achievement in the FYE mathematics course needs to be reviewed.

The findings from this research have also shown that despite the experimental students showing much eagerness to work with *Derive*, their enthusiasm has made no difference in their attitudes towards mathematics in general, as a subject. The same students, however, do see more value in mathematics and have a better appreciation of it as a logical system.

Much of the researcher's initial concern in regard to students' anxiety in working with computers was also dispelled by this research. PNG students have shown themselves to be enthusiastic and readily adaptable to this new medium of learning. Computer tutorials virtually ran themselves, with students taking charge almost as soon as the sessions started. They demonstrated no difficulty in starting up *Derive* and setting up the necessary screen windows for the tutorial sessions. Students were observed by their tutors to be often actively involved in experimentation with the computer and in serious and searching discussion with their peers. Much of this discussion was in pidgin English. This interaction with mathematical ideas being thrown back and forth provided further indication that possibly more 'thinking' around the concepts involved was taking place, and more so than in the normal classroom tutorial sessions. Students in the experimental groups also displayed little desire to leave the computer room when a tutorial session ended and were usually quite engrossed with the mathematics.

This research was principally exploratory in it aims. It was concerned with the application of a computer algebra system to a learning environment, and as such, there were naturally some preferred expectancies and outcomes. While *Derive* has not made any statistically significant impact on students' achievement in mathematics, the results of this research have highlighted the role of *Derive* as a catalyst for stimulating enthusiasm and bringing excitement to the students' learning of mathematics.

This research has implications for students, teachers, instructors, and researchers, both at the secondary and tertiary levels in PNG. There are also implications for the PNG national curriculum. Results of the study should provide some motivation to instructors of mathematics, and other related subjects, to begin to explore the pedagogical possibilities that a CAS can offer in the instruction of their students. It has shown that students in PNG respond very favourably to the new technology and can be motivated by it. There is every indication that PNG students in the upper secondary school would respond equally enthusiastically to working with a software package such as *Derive*. These students would, at the same time, be gaining a gradual and much-needed exposure to computers through their learning process. A computer algebra system will offer these students the ability to shift quickly between numeric, graphic and symbolic representations enhancing concept development.

Limitations

Limitations to this research were in the areas of existing pertinent literature, instrumentation, sampling, facilities, and constraints of time. Almost all literature reviewed relating to the design of the research were based in countries with very different cultural and socio-economic backgrounds to PNG. Findings reported by these studies were incorporated into the present research based on their relevance to this study. As much care as possible was taken to ensure the validity and reliability of the findings from this research. Instruments were researcher-constructed due to the lack of existence of similar instruments in PNG research and were designed to specifically address the situation as perceived in PNG. The randomness of selection of subjects for the research was also compromised in part by the preselection of student groupings by the university's admissions section. Availability of computer rooms dictated the frequency and scheduling of the *Derive* tutorials. Electricity power cuts are a fairly frequent occurrence at UNITECH and on several occasions disrupted a *Derive* session. The heavy demand for use of computer laboratories made re-scheduled bookings difficult. Staffing movements caused further disruptions to a smooth running programme.

Conclusion

This research has shown that a computer algebra system has the capability of
- making students more enthusiastic about learning mathematics
- holding the students' attention and making them think more about the mathematics
- encouraging students to experiment with mathematical ideas
- helping students to see mathematics more as a logical system.

Students in PNG have responded well to the experiment of integrating *Derive* into their mathematics learning, and this paves the way for further thought, experimentation, and research which can bring about improvements to the present system. This study has highlighted a need for a reassessment of, not so much what is presently being taught in the FYE mathematics course, but how that mathematics can best be taught to the benefit of the students. With changes in the emphasis on learning objectives towards conceptual outcomes, there is every indication that a computer algebra system, such as *Derive,* has a successful role to play through its integration with the mathematics programme. Its role as a pedagogical tool and a means of exposing students to much needed computer awareness provides an exciting and fruitful opportunity for advancing education in PNG.

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A Report of Research into Student Attitude and Teacher-Student Interpersonal Behaviour in a large Sample of Australian Secondary Mathematics Classrooms

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This paper reports on research using the *Questionnaire on Teacher Interaction (QTI)*, and provides validation data for the first use of the QTI with a large sample of mathematics classes. The effect of teacher-student interpersonal behaviour on the students' attitude towards their mathematics class was investigated and the dimensions of the QTI were found to be associated significantly with student attitude scores. The paper also describes how mathematics teachers could use the QTI as a basis for reflecting on their own teaching.

Most mathematics teachers believe that good relationships with their students are important. But are the students' perceptions of teacher-student interpersonal behaviour the same as their teachers? Is there a difference in mathematics teachers' perceptions of their actual teacher-student interpersonal behaviour in the classroom and what they perceive to be ideal? What associations are there between student attitudes and teacher-student interpersonal behaviour in the mathematics classroom?

The purposes of this paper are to outline a convenient questionnaire designed to assess teacher-student interpersonal behaviour and to report its use in answering such questions as these. The paper describes various forms of the Questionnaire on Teacher Interaction (QTI) and reports its use in past research. Associations between QTI scales and students attitudinal outcomes in mathematics classrooms are made and further validation data for the QTI are provided. Finally, the paper describes how mathematics teachers could use the QTI as a basis for reflecting on their own teaching.

The Questionnaire on Teacher Interaction

International research efforts involving the conceptualisation, assessment and investigation of perceptions of psychosocial aspects of the classroom environment have firmly established classroom environment as a thriving field of study (Fraser, 1994; Fraser & Walberg, 1991). Recent classroom environment research has focused on science laboratory classroom environments (McRobbie & Fraser, 1993), constructivist classroom environments (Taylor, Dawson & Fraser, 1995) and computer-assisted instruction classrooms (Teh & Fraser, 1994).

Researchers in The Netherlands have extended this research by focusing specifically on the interpersonal relationships between teachers and their students as assessed by the QTI (Wubbels, Créton & Hoomayers, 1992; Wubbels & Levy, 1993). The Dutch researchers (Wubbels, Créton & Holvast, 1988) investigated teacher behaviour in a classroom from a systems perspective, adapting a theory on communications processes developed by Waltzlawick, Beavin and Jackson (1967). Within the systems perspective of communication, it is assumed that the behaviours of participants mutually influence each other. The behaviour of the teacher is influenced by the behaviour of the students and in turn influences the student behaviour. Thus, a circular communication process develops which not only consists of behaviour, but determines behaviour as well.

With the systems perspective in mind, Wubbels, Créton and Hooymayers (1985) developed a model to map interpersonal teacher behaviour using an adaptation of the work of Leary (1957). In the adaptation of the Leary model, teacher behaviour is mapped with a Proximity dimension (Cooperation, C - Opposition, O) and an Influence dimension (Dominance, D, - Submission, S) to form eight sectors, each describing different behaviour aspects: Leadership, Helpful/Friendly, Understanding, Student Responsibility and Freedom, Uncertain, Dissatisfied, Admonishing and Strict behaviour. Figure 1 shows typical behaviours for each sector. The Questionnaire on Teacher Interaction (QTI) is based on this model.

The items of the QTI belong to eight scales, each consisting of six items and corresponding to one of the eight sections in the model. Examples of items are "This teacher is friendly"(Helping/Friendly) and "This teacher gets angry unexpectedly"(Admonishing). The scores for each item within the same sector

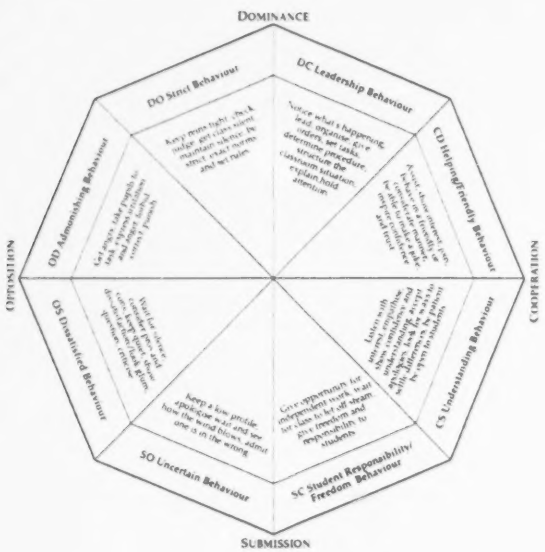

Figure 1. The model for interpersonal teacher behaviour

are added to obtain a total scale score. The higher the scale score the more a teacher shows behaviours from that sector. Scale scores can be obtained for individual students, or can be combined to form the mean of all students in a class.

An Australian Version of the QTI

The original version of the QTI developed in the early 1980s in The Netherlands had 77 items (Wubbels, Créton, & Hooymayers, 1985). Later, an American version of the QTI was developed which had 64 items (Wubbels & Levy, 1991). The Australian version of the QTI described in this paper, is more economical and has 48 items which are answered using a five-point response scale. This version of the QTI is available for use by mathematics teachers to gather their own perceptions and the perceptions of their students about their mathematics classrooms.

One advantage of the QTI is that it can be used to obtain the perceptions of interpersonal behaviour of either students or teachers. When the QTI is administered to both mathematics teachers and their students, information is provided about the perceptions of teachers and the perceptions of their students of the interpersonal behaviour of that teacher. The information obtained by means of the questionnaire includes perceptions of the behaviour of the teacher towards the students as a class, and reflects relatively stable patterns of behaviour over a considerable period. Similarly, teachers can be asked for their perceptions of their own behaviour or the behaviour that they consider to be ideal. The wording of the questionnaire is varied slightly when used to obtain

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teachers' self-perceptions and ideals. For example the question "This teacher talks enthusiastically about his/her subject", becomes "I talk enthusiastically about my subject" in the teacher self-perception version, and "This teacher would talk enthusiastically about his/her subject" in the teacher ideal version. These latter two versions are also available from the authors.

By using these three separate forms of the QTI it is possible to collect data on students' perceptions of teacher-student interpersonal behaviour, teachers' perceptions of their actual teacher-student interpersonal behaviour in the classroom and what they perceive to be ideal? These three sets of data can be represented graphically for ease of analysis by participants.

Past uses of the QTI

The QTI has been shown to be a valid and reliable instrument when used in The Netherlands (Wubbels & Levy, 1993). When the 64-item USA version of the QTI was used with 1,606 students and 66 teachers in the USA, the cross-cultural validity and usefulness of the QTI were confirmed. Using the Cronbach alpha coefficient, Wubbels and Levy (1991) reported acceptable internal consistency reliabilities for the QTI scales ranging from 0.76 to 0.84 for student responses and from 0.74 to 0. 84 for teacher responses.

Wubbels (1993) used the QTI with a sample of 792 students and 46 teachers in Western Australia and Tasmania. The results of this study were similar to previous Dutch and American research in that, generally, teachers did not reach their ideal and differed from the best teachers as perceived by students. It is noteworthy that the best teachers, according to students, are stronger leaders, more friendly and understanding, and less uncertain, dissatisfied and admonishing than teachers on average.

When teachers described their perceptions of their own behaviours, they tended to see it a little more favourably than did their students. On average, the teachers' perceptions were between the students' perceptions of actual behaviour and the teachers' ideal behaviour. An interpretation of this is that teachers think that they behave closer to their ideal than their students think that they do.

Variations in the students' appreciation of the subject and the lessons have been characterised on the basis of the proximity dimension: the more cooperative the behaviour displayed, the higher the affective outcome scores (Wubbels, Brekelmans & Hooymayers, 1991). That is, student responsibility and freedom, understanding, helpful/friendly and leadership behaviours were related positively to student attitudes. Uncertain, dissatisfied, admonishing and strict behaviours were related negatively to student attitudes. Overall, previous studies have indicated that interpersonal teacher behaviour is an important aspect of the learning environment and that it is related strongly to student outcomes.

Australian Applications of the QTI

In one of the first uses of the QTI in Australia (Fisher, Fraser & Wubbels, 1993), associations were investigated between teachers' perceptions of their work environment, using the School Level Environment Questionnaire (SLEQ), and students' and teachers' perceptions of their classroom interactions (Fisher & Fraser, 1990) . Results from this study indicated that relationships between SLEQ and QTI scores generally were weak, thus suggesting that teachers believed that they had considerable freedom to shape their own classrooms regardless of their school environment.

Methodology

This study is distinctive in that it is centred on students in mathematics classes, whereas previous research using the QTI has focused largely on students in science classes. The study involved students in grades 8, 9 and 10 mathematics classes in Australia and was composed of 405 students in 9 schools with their 21 teachers.

Associations between students' perceptions of their interpersonal relationships with their teachers and their attitudinal outcomes were examined in this study. The 48-item version of the QTI (Wubbels, 1993) was used to gauge students' perceptions of student-teacher interpersonal behaviour and student attitudes were assessed with a seven-

item Attitude To This Class scale, which was based on the the Test of Science-Related Attitudes [TOSRA] (Fraser, 1981).

Using the scales of the QTI as independent variables, associations were computed with attitude to the class. Simple correlations were calculated between each QTI scale and each student attitude. Also a multiple regression analysis, involving the whole set of QTI scales, was conducted to provide a more conservative test of the association between each QTI scale and attitude when all other QTI scales were mutually controlled.

Results

Validity of the QTI

Table 1 provides some cross-validation information for the QTI when used specifically in the present sample of mathematics classes. Statistics are reported for two units of analysis, namely, the student's score and the class mean score. As expected, reliabilities for class means were higher than those where the individual student was used as the unit of analysis. Table 1 shows that the alpha reliability figures for different QTI scales ranged from 0.62 to 0.88 when the individual student was used as the unit of analysis, and from 0.60 to 0.96 when the class mean was used as the unit of analysis. The values presented in Table 1 for the present sample provide further cross-validation information supporting the internal consistency of the QTI, with either the individual student or the class mean as the unit of analysis.

Table 1
Internal Consistency (Cronbach Alpha Coefficient) and Ability to Differentiate between Classrooms of the QTI

Scale		Alpha Reliability		ANOVA Results
		Student	Class	Eta^2
DC	Leadership	0.86	0.93	0.43*
CD	Helping/friendly	0.88	0.94	0.29*
CS	Understanding	0.88	0.96	0.36*
SC	Student responsibility/ freedom	0.69	0.79	0.23*
SO	Uncertain	0.78	0.87	0.29*
OS	Dissatisfied	0.83	0.91	0.28*
OD	Admonishing	0.84	0.89	0.36*
DO	Strict	0.62	0.60	0.14*

*$p < .001$ n = 405 students in 21 mathematics classes

Another desirable characteristic of any instrument like the QTI is that it is capable of differentiating between the perceptions of students in different classrooms. That is, students within the same class should perceive it relatively similarly, while mean within-class perceptions should vary from class to class. This characteristic was explored in mathematics classrooms for each scale of the QTI using one-way ANOVA, with class membership as the main effect. It was found that each QTI scale differentiated significantly ($p<.001$) between classes and that the eta^2 statistic, representing the proportion of variance explained by class membership, ranged from 0.14 to 0.43 for different classes.

To investigate further validation data for the QTI, interscale correlations were used to test the validity of the circular two dimensional model for interpersonal behaviour upon which the QTI is based. The two dimensional model would be validated if the interscale correlations were highest between adjacent scales and lowest between scales opposite to one and other on the model. Table 2 reports interscale correlations for the student version

of the QTI with the student as the primary unit of analysis and serves to further validate the circumplex nature of the QTI in that, with a few minor exceptions, the highest correlations are found in scales adjacent on the two-dimensional model (eg. between Helping/Friendly and Understanding behaviour for example) and the lowest correlations between scales that are opposite to one and other on the model (eg. between Helping/Friendly and Dissatisfied behaviour for example).

Table 2
Means for QTI scales for grades 8,9 & 10 secondary mathematics classes

Scale	Hfr	Und	Sre	Unc	Dis	Adm	Str
Lea	0.73	0.75	-0.17	-0.67	-0.54	-0.48	-0.06
Hfr		0.83	0.17	-0.50	-0.66	-0.62	-0.32
Und			0.10	-0.56	-0.67	-0.69	-0.24
Sre				0.38	0.03	-0.11	-0.27
Unc					0.53	0.50	0.06
Dis						0.72	0.46
Adm							0.56

n = 405 students in 21 mathematics classes

Associations between Interpersonal Teacher Behaviour and Student Outcomes

Table 3 reports the results for associations between students' perceptions of teacher-student interpersonal behaviour and students' attitudinal outcomes when the data were analysed using both simple and multiple correlations. Whereas the simple correlation (r) describes the bivariate association between attitudinal outcome and a QTI scale, the standardized regression weight (ß) characterises the association between attitudinal outcome and a particular QTI scale when all other QTI dimensions are controlled.

An examination of the simple correlation (r) figures in Table 3 indicates that there were eight significant relationships ($p<.05$), out of eight possible, between student-teacher interactions and the student attitudinal outcome; this is 20 times that expected by chance alone. An examination of the beta weights reveals four out of eight significant relationships ($p<.05$), which is ten times that expected by chance alone.

The simple correlation (r) figures indicate statistically significant associations between the students' attitude to class and all QTI scales. The beta weights show that some of these associations retain their significance in a more conservative test with all other QTI scales controlled. In classes where the students perceived greater leadership and helpful/friendly behaviours in their teachers, there was a more favorable attitude towards the class. The converse was true when the teacher was perceived as strict and dissatisfied.

Table 3
Associations between QTI Scales and Students' Attitudinal Outcomes in terms of Simple Correlations (r) and Standardized Regression Coefficients (§).

QTI Scale	Strength of Environment – Outcome Association Attitude to Class	
	r	*§*
Leadership	0.53**	0.24**
Helpful/friendly	0.64**	0.19*
Understanding	0.61**	0.13
Student responsibility/freedom	0.15**	0.07
Uncertain	-0.35**	0.07
Dissatisfied	-0.58**	-0.15*
Admonishing	-0.54**	-0.06
Strict	-0.40**	-0.18**
Multiple Correlation, *R*		0.71**

*$p < .05$ **$p < .01$ n = 405 students in 21 mathematics classes

Conclusions

This study confirmed the reliability and validity of the QTI when used in secondary school mathematics classes. Generally, the dimensions of the QTI were found to be significantly associated with student attitude scores. In particular, students' attitude scores were higher in classrooms in which students perceived greater leadership and helpful/friendly behaviours in their teachers' interpersonal behaviours. If mathematics teachers want to promote favorable student attitudes to their class, they should ensure the presence of these interpersonal behaviours. Conversely students' attitude scores were lower in classrooms in which students perceived greater dissatisfaction and strictness in their teachers' interpersonal behaviours.

Mathematics teachers can make use of the QTI to monitor students' views of their classes, investigate the impact that different interpersonal behaviours have on student outcomes, and provide a basis for guiding systematic attempts to improve this aspect of their teaching. Furthermore, the QTI could be used in assessing changes that result from the introduction of new curricula or teaching methods, and in checking whether the mathematics teacher's interpersonal behaviour is seen differently by students of different genders, abilities or ethnic backgrounds. Further research may wish to consider the impact of learning environment measures on the achievement of students in mathematics classes.

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The three selves of the prospective primary school teacher of mathematics: An Australian case study

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The paper considers the way that a cohort of prospective primary school teachers engage with a mathematics education subject in their initial teacher education program. They appear to do so through the medium of three selves. This metaphor of the three selves is explained, and the implications of using this metaphor to promote positive interactions in mathematics education subjects are discussed.

Introduction

Prospective school teachers enter their initial teacher preparation courses with various experiences, beliefs and attitudes, many of which have developed as a result of their personal histories (Ball, 1989; Holt-Reynolds, 1994). The challenge for teacher educators responsible for mathematics education in initial teacher education courses is to help students make these beliefs and attitudes explicit, so that students may examine them and investigate their implications (McDiarmid, 1989; Holt-Reynolds, 1994; Schuck, 1996). It appears that mathematics education subjects in initial education courses do not always succeed in challenging students' thought about the nature of mathematics and the teaching of mathematics (McDiarmid, 1989; Ball, 1988). One of the reasons for this lack of success could be that the engagements between teacher educators and student teachers, or between the material of the courses and student teachers, are in fact, engagements with an inappropriate "self" of the student teacher. In what follows, I shall demonstrate that student teachers often have three selves operating in their attempts to make meaning of teacher education courses, and that for engagement with these student teachers to be effective, the appropriate selves must be addressed.

As a result of her research with prospective secondary school teachers enrolled in a content area literacy course, Holt-Reynolds (1994) suggests that student teachers "invent Self-as-Teacher based on what Self-as-Student believes" (Holt-Reynolds, 1994, p. 3), that is, prospective teachers constantly evaluate pedagogical ideas in terms of their own anticipated reaction to the ideas, had they been students.

This paper expands on the notion that student teachers respond to the beliefs of their "selves-as-students", by suggesting a model in which three aspects of student teachers' beliefs interact. The paper describes a study of the beliefs and attitudes of student teachers towards mathematics, and the teaching and learning of mathematics. In particular, it examines one of the findings of the study which suggests that many student teachers engage in three different ways with their mathematics education courses. The paper describes this phenomenon in detail and discusses the implications of this finding for initial teacher education courses in mathematics education.

The study

Participants and background

The study focused on a group of students, all within their first year in an initial primary school teacher education program at an Australian university. In their first semester, students were required to enrol for an orientation subject which covered the major learning areas encompassed by the primary school syllabus. The aims of the mathematics component of the curriculum orientation subject were to improve attitudes to mathematics; to broaden students' views of the scope of primary school mathematics; and to model some of the current approaches to teaching in the primary school.

In their second semester, students studied the first of a four subject sequence in mathematics education. In this subject, Mathematics Education 1, students covered three major areas of curriculum. The first dealt with learning theories in the area of mathematics education; the next section dealt with the historical development of our number systems and a discussion of the characteristics of some ancient numeration systems; and the last section dealt with topics from number theory. A problem solving

approach was used and a variety of mathematical models were suggested to develop new concepts. Cooperative learning was also encouraged. In this subject, emphasis was on concept development; the links between areas of higher order mathematics and primary school mathematics; the connectedness of mathematics; aspects of the history of mathematics which highlighted cultural influences on the development of mathematics; and development of understanding of the nature of mathematical activity, both by examining such activity as it has occurred in the past, and by engaging the students themselves in mathematical activity. It also aimed to increase students' competency in mathematics and to help students become aware of current theories about the learning of mathematics.

The students were all members of two tutorial groups out of a total of four groups in the first year cohort. There were approximately 100 students enrolled in the first year of the primary teacher education course. Fifty of these students were in the two groups participating in the study. Seventeen of the students in the study had been out of school for more than three years, the remainder had completed their school education within three years of attending university. The two groups were initially chosen because of my access to them as both researcher and lecturer in the first part of the study.

The majority of students had studied mathematics in their HSC, with over 60% at the 2 Unit or higher level. Consequently, these students could not be seen as innumerate in any way.

Methodology

Data for this study was collected by a variety of means. In phase I of the data collection, paired interviews were used. In these interviews the fifty first year students were asked to each formulate a set of five questions which would elicit from another class member their current attitudes towards mathematics and mathematics teaching and also indicate how these had been affected or influenced by past mathematical experiences. Students then answered each others' questions and results were audio taped.

The next phase of data collection involved obtaining responses from the same group to a questionnaire in which most questions were open-ended. The questions were developed from issues arising from the first phase of data collection and so were well grounded. In the third phase, in-depth interviews were conducted with eight of the student teachers at the end of their first year, in order to provide triangulation and to furnish further detail. All students in the two classes were involved in the first set of interviews and in the questionnaire; the eight students chosen for the second set of interviews were selected using the following two criteria: (1) an equal number of mature age students and recent school leavers and (2) a mixture of students in terms of their demonstrated ability to reflect on their teaching and learning. Finally, in phase four, four mathematics education lecturers were interviewed using an interview schedule developed from issues arising in earlier phases. For the purposes of this paper it was primarily the data collected in the third and fourth phases that was most informative, as it dealt with students' experiences in the teacher education subjects.

Data was analysed using the qualitative analysis software package NUD•IST. Data was coded according to a number of categories that emerged from a reading and re-reading of the data.

Results

The data was extremely rich and informed a number of different areas. This paper focuses on one result in particular. It became apparent from the way students talked or wrote about their learning in phases two and three, that they were viewing their own learning of mathematics, and their learning about pedagogical issues in mathematics education from certain perspectives. I have expanded on Holt-Reynold's model of "Self-as-Student" informing "Self-as-Teacher", discussed above, and consider the model with reference to a primary mathematics education context. The data in my study reveals that the student teachers reflected on, and informed their learning through the perspectives of three "selves". I have called these three selves "self-as-student-learning-to-teach", "self-as-primary-school-student" and "self-as-teacher". In Holt-Reynolds' (1991) study, the

student teacher decided whether Self-as Teacher would benefit from implementing a particular practice learned in the teacher education course, by listening to the responses of Self-as-Student undergoing the imagined activity. My data indicated that, for prospective primary school teachers, this inner conversation is strongly allied to the students' orientation to studies. I have classified students' orientations to their studies by using the framework suggested by Gibbs, Morgan and Taylor (1984). Gibbs et al suggest that there are four major orientations to study at university: academic, vocational, personal and social. Further, each orientation can be seen as extrinsic or intrinsic. My results indicated that students were very strongly vocationally oriented. A vocational orientation is described by Gibbs et al as follows: if students are intrinsically vocationally oriented, they are interested in the course as a preparation for their future career; if extrinsically vocationally oriented, they have chosen the course because the qualification is necessary for their future career. Consequently, because of the strong vocational orientation of the students, all subjects were judged for their value in preparing students to become good teachers.

In order to make this judgement in the mathematics education subject sequence, the students would assess the mathematics content from the viewpoint of self-as-primary-school-student, and this assessment allowed self-as-teacher to make a decision as to the value of the topic. If self-as-teacher would use that topic in her/his classroom, then the topic was assessed as worthy of time and effort and was regarded as a useful topic to study. If self-as-teacher could not envisage teaching that topic, the topic was regarded as worthless, and discarded. Self-as-teacher made the decision about the topic by consulting self-as-primary-school-student. If self-as-primary-school-student could cope with the demands of the work, and the topic agreed with self-as-teacher's notions of what mathematics should be taught, the topic was given the validity that it required, in order to be learnt by the student teacher.

Consequently, the data suggests that for many prospective primary school teachers, there are three selves that are consulted: self-as-primary-school-student, self-as-teacher, and self-as-student-learning-to-teach. From the discussion above it can be seen that self-as-primary-school-student tests the content in order to advise self-as-teacher. If self-as-teacher is convinced of the worth of the content, then self-as-student-learning-to-teach finds value in the content in the mathematics education subject.

The following extracts from the data all illustrate the constant use of the filters of self-as-primary-school-student, self-as-teacher and self-as-student-learning-to-teach. Maria, in the phase three interview (all names are pseudonyms to ensure anonymity), was asked whether she would have studied the mathematics education subjects if they had not been compulsory. The conversation continued (the italics are added for emphasis):

> Oh yeah, I'd definitely do it, ...because I'm not very strong in maths I'd do it *just so I can build up a strength.* Because I mean that's something that they really need to know and you want to teach it well, especially mathematics.
>
> Question: And did you feel that you got any help in that direction in the first year mathematics, Mathematics Education 1 in particular?
>
> Answer: Um, oh I learnt a few new little skills and ideas *but I didn t really use them on prac*, I mainly used the syllabus, like that really helped more, my supervising teacher went by the syllabus so I used that, but some things, ..yeah some things I found interesting.

Maria first said that she felt it was important for her to build up her mathematics, yet when asked whether the Mathematics Education 1 subject had helped her achieve that aim, she answered a different question: was the subject content useful on the teaching practicum? So Maria appeared to see building up subject content knowledge as purely developing content knowledge of the current NSW primary school syllabus, as indicated by the NSW K-6 Curriculum Document. Here self-as-teacher is evaluating the content of the subject Mathematics Education 1 by judging its worth as material to be taught on the practicum. Self-as-student-learning-to-teach is then advised as to its value.

Another student, Terry, in her phase three interview, discussed studying the ancient numeration systems in Mathematics Education 1. It should be noted here that the ancient numeration systems were taught to students for a variety of reasons: to help students see what made a numeration system efficient in a particular context; to show how mathematics developed in a socio-cultural context and was dependent on the needs of the society in which it was situated; and to isolate the underlying principles of our place value system. Further, study of different numeration systems was recommended in the Discipline Review of Teacher Education in Mathematics and Science (DEET, 1989). However, Terry did not seem aware of these reasons:

> ... and the second semester I didn't think was as practical, we did all those different numeral systems and I just really didn't see where they were used as much. But I didn't think it was ... I didn't think primary school ... I don't think I would have understood it, when I was in primary school.
>
> I don't know, I don't know why we did that. I didn't know why, I didn't really see the point. I didn't mind it, because I'd never done that, but I just didn't think ... I didn't think, I wouldn't do it with my primary school class.

Terry evaluated the worth of the topic by assessing it as self-as-teacher. To do this she first put herself into her self-as-primary-school-student position and decided she would not have understood the concepts. This then linked to her assessment as self-as-teacher: it was not a useful topic. Consequently, she did not see any reason for being taught the topic in Mathematics Education 1 - if a topic was not going to be taught to primary school students, then self-as-teacher advises self-as-student-learning-to-teach that there is no value in studying this topic at university.

It was apparent from the text that Terry seemed quite unaware of any other reason for studying the topic in the mathematics education sequence. It is possible that the teacher educator who took Terry's class for this subject had not felt it was necessary to convince the students of the value of the topic, but might have regarded the intellectual challenge inherent in the study as sufficient reason for studying the systems. This belief highlights the need for teacher educators to address self-as-teacher in their mathematics education classes. The following comment from one of the other students in this class illustrates this:

> We really didn't know why we were doing it and so a lot of the good was lost, it wasn't being reinforced, it wasn't being consolidated, it wasn't being tied in with the base ten or with children.

Further the students were reinforced in their belief as self-as-teacher, that studying the ancient numeration systems was not of any value, as it could not be used in the school. Supervising teachers on the practicum dismissed any lessons students might have prepared on the ancient numeration systems as irrelevant and wasteful of time:

> ...we were actually doing all the Egyptian [numeration system] and everything when we went out on our first prac and I thought this will be terrific you know, we really can tell the children all about the history of mathematics and they can understand it all more and they can get really excited about it too.
>
> And of course, the teacher said "Oh, are you doing that rubbish? That's just absolutely ludicrous, you know. We'll get on with this, there's too much to cover in the curriculum without all that" and so that was squelched...
>
> That reinforces something in my mind, it reinforces that what we're doing at university is so totally unrelated to what happens in reality in schools, where do you tie them both in? That's what it reinforces. (Gail, Phase three interview).

The student's self-as-teacher accorded the supervising teacher a very high status and seemed to derive valued information from the supervising teacher as well as from self-as-primary-school-student. As a result, the student's self-as-teacher received confirmation that the topic was not going to be useful: a "real teacher" had indicated that it had little

value in the primary classroom. This confirmation for self-as-teacher, together with the fact that the topic was not evaluated for its usefulness in any other context but that of the primary school classroom, led to a perception by self-as-student-learning-to-teach that the topic was without any value.

On the other hand, some topics were seen as worthwhile by self-as-student-learning-to-teach, because self-as-teacher could see value in them. A study of number patterns was one of these for Terry;

> The patterns were actually not too bad because I thought that they were more practical. I thought, you know, you see patterns in things and I thought that was useful for children to be able to look at something and see a pattern ... I think that's useful.

Students were also asked questions about their strategies for studying mathematics at university, in the first year mathematics education subject. As indicated above, where students saw the value of the topic they were studying, they made some effort to understand it and to investigate it at depth. The value of the topic was judged from the perspective of the self-as-teacher. It was judged to be worth studying if it was perceived as a topic that would help the student become a better teacher of mathematics at primary school level. The following extracts from the data illustrate how students studied Mathematics Education 1, by first filtering the content through their selves as primary school student and teacher, and then by choosing the most appropriate learning strategy available to them for the particular assessment. To some extent when the material was not considered useful, motivation came from extrinsic vocational orientations; the qualification was needed and so the subject was studied sufficiently to achieve a passing grade. It should also be noted that students did not always have a very large repertoire of appropriate learning strategies for studying mathematics content. Much of their learning in the past had comprised only rote learning and reproduction of facts and formulas; other learning strategies were often lacking.

For Terry, self-as-teacher saw that most of the topics had value for her teaching and so she prepared herself quite thoroughly:

> ... because I knew that it would be needed, not just for the exam, but to do that topic, and it might help me understand that topic more or have suggestions on how to teach it or what children find hard in that topic or whatever they had to say about it.

On the other hand Aaron exhibited extrinsic motivation and this affected the way he studied the subject:

> ... I thought "God, I've got to pass this exam" and then I thought "*!?* it, I'm not going to spend, you know, too much time on it". No, I thought it was enough to understand enough to pass. That's pretty well the way we worked, I think. None of us were really that concerned about getting high marks, we just wanted to pass, which is, I suppose, the wrong attitude, but at that stage of the year that's really all we felt like.

The reason that Aaron exhibited extrinsic motivation rather than intrinsic motivation is because his self-as-student-learning-to-teach could not see the value of the work when judging it as self-as-primary-school-student and self-as-teacher.

> ... especially for us, because a lot of us had trouble understanding it, and I thought, you know if we don't understand it, how are these kids going to understand it. Then we got to this stuff and we just didn't understand it, nothing, nothing clicked and we thought "Oh, what the hell, none of us are going to teach this.

Self-as-primary-school-student considered whether the work was conceptually accessible, and advised self-as-teacher that it was too difficult. Self-as-teacher then discarded the topic and advised self-as-student-learning-to-teach not to spend time on that topic as it had no value in the primary school classroom.

Finally, the teacher educators seemed to be aware to some extent of these internal conversations and expressed concern about the stance held by self-as-teacher.

> I think that it's always a great worry that our students ... are still making these kinds of comments: "Will we be teaching this?" rather than "What's the relevance?". If they asked you what the relevance of it was, that's a good question, and you can discuss that, but when they ask "Will we be teaching this?" and you have to say no, they might say "Well, what's the point?", which is a pity. (George, teacher educator, phase 4)

Discussion and conclusions

The very strong vocational orientation of the students led them to filter all content of the mathematics education subject through their three selves. This process led students to select those mathematical topics that were both highly accessible to self-as-primary-school-student and also appeared to match self-as-teacher's vision of appropriate mathematics for the primary school classroom. Consequently, any mathematics that did not explicitly address these criteria tended to be neglected by the majority of the students, as they were not prepared to devote any cognitive effort to the study and understanding of what appeared to be irrelevant topics. At the same time, students were reluctant to spend valuable time learning topics that did not fit the needs of self-as-teacher, and wanted, instead, to spend that time learning how to teach topics that self-as-teacher was nervous about explaining to children, such as long division. These feelings led to a devaluing of the mathematics education curriculum by many of the students.

This highlights a difficulty that arises from self-as-primary-school-student constantly evaluating the topics. Prospective teachers often have a very narrow view of the mathematics that could or should be done by primary school students. Past experiences have often closed students' minds to the possibilities that exist in primary school mathematics, and also encourage prospective teachers to place ceilings on the sorts of activities that might be done in a primary school classroom. The selves-as-primary-school-pupils are, in fact, projections into the past by the students themselves. Their experiences in the primary school classroom have led them to accept as authentic activities, that self-as-teacher would teach, either activities similar to their past experiences, or activities that promise to be highly enjoyable, regardless of whether learning occurs or not. As a result, self-as-student-learning-to-teach expects the topics covered in the mathematics education subjects to be either extremely familiar, or fun to do. For the latter, the many unsuccessful experiences that some of these students have had in the past suggest that the cognitive demands of the mathematics must not be extensive if students are to enjoy the topic, as their lack of confidence is always near the surface, and will emerge at the first sign of difficulty. These views can create conflicts between the aims that student teachers have for their mathematics education subjects and the aims of the teacher educators.

There are implications of these internal conversations, and the consequent valuing of mathematics education subjects, for mathematics educators in teacher education programs. It becomes essential for the teacher educator to address the three selves of the student teacher rather than merely bemoan the fact that they seem to exist. If self-as-teacher and self-as-primary-school-student exert such a strong influence on self-as-student-learning-to-teach, then they too must be invited to participate in the discourse on learning and teaching. This participation should occur in choosing the curriculum for mathematics education subjects, so that topics are chosen which have clear applications to primary school classrooms and are also easily accessible to the majority of student teachers. Much of the primary mathematics curriculum can be used as a starting point for the study of mathematics that is connected, rich and deep, and at the same time has clear relevance for self-as-teacher and accessibility for self-as-primary-school-student. Further, the reasons for which teacher educators have chosen particular topics to be studied need to be clear to the student, and these reasons need to be linked to the aims held by self-as-teacher and self-as-primary-school-student in order to motivate self-as-student-learning-to-teach.

Consequently, it is necessary for teacher educators to be aware of the internal conversations of student teachers, and to negotiate a curriculum with the three selves of

the student teacher, that both student teacher and mathematics educator believe is capable of achieving their diverse aims.

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In symmetry is the left the same as the right or is the right left out?

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This study was concerned with finding what characteristics of data, such as direction of skewness, degree of skewness and degree of kurtosis, affected students' ability to use histograms and boxplots for detecting non-symmetry in the parent poplation. The study found that while there was no consistent difference between boxplots and histograms in the proportion of students detecting non-symmetry in the parent population, the direction of skewness did have a significant effect, with more students detecting skewness when the data was displayed in a left-skewed orientation than when the same data was displayed in a right-skewed orientation. This result is consistent with research reported in the psychological literature where many, but not all, studies have shown an over emphasis on the left hand field of view for normal subjects. Other findings of the study are given and suggestions for further research made.

Introduction

Over the years there has been much discussion of the importance of visualisation and use of diagrams in mathematics. Presmeg (1991) has looked at the influence of the teacher on students' use of visualisation. Polya (1957) considers that drawing a diagram should be the first step in problem solving yet Vinner (1989) states that in calculus students are reluctant to draw a diagram.

In statistics Tukey (1977) states that "the greatest value of a picture is that it forces us to notice what we never expected to see"; a view confirmed by Wainer (1992) with his illustrations of vulnerable parts of aeroplanes and sources of typhoid infections in Britain.

Until recently, with the exception of work done by Vernon (1946,1951) and subsequently criticised by MacDonald-Ross (1977), very little work has been done on how students use and understand the graphical displays used in statistics. Recently, with Chance and Data forming a stream in the mathematics curriculum in many countries, this situation is changing.

Bright and Friel (1996) have looked at how children in grades 6 and 8 interpret stem and leaf plots and make connections with histograms and in separate paper have tried to construct a theory of graphicacy.

Before embarking on any formal analysis of data that they have collected students in statistics are encouraged to display their data. That is, they are recommended to create some sort of visual representation of their data. This enables the detection of unexpected relationships, unusual structure in the data and outliers - an outlier being a value unusually far from the main batch of data. In statistics, diagrams are not an alternative way of looking at data (as might be the situation in calculus) but an intergral part of the process. Data displays, for instance, allow for the possible detection of non-symmetry in the parent population. Symmetry, or more specifically normality, is a common assumption for many statistical tests. However, very little work has been done on what students actually see.

The present study, based on statistical displays - histograms and boxplots - is aimed at determining what factors, such as direction of skewness, degree of skewness and degree of kurtosis, influenced students' decisions as to whether or not a sample could have come from a symmetric population. It also aimed to determine which of boxplots or histograms was better for showing genuine departures from symmetry in the population. Boxplots use a rectangular box to identify the middle fifty percent of values in the sample and also highlight the median and extreme values. Histograms give more detail about the internal variation of the values.

Method

One hundred and twenty-five undergraduate students enrolled at an Australian University participated in the study. There were 54 first-year, 34 second-year and 37 third-year students. All students had successfully completed a basic one-semester course in statistics, with second- and third-year students having completed two or more courses.

Subjects were each presented with 32 graphical displays, each of which was generated using the statistical package Minitab. The displays were based on eight simulated data sets each was presented in a left- and a right-skewed orientation (the right-skewed display being the mirror image of the left. The distributions were chosen to ensure the samples covered a range of skewness and kurtosis values. The coefficients of skewness ranged from 0.06 to 1.3 in absolute value; the kurtosis coefficients (m_4/m_2^2) ranged from 2.4 to 12.9. The 16 right-skewed displays, in increasing order of skewness as measured by the associated p-value are shown in Figure 1.

The displays were randomly arranged on the questionnaire with four figures per page. Four versions of the test paper were used; the versions differed in the order in which the questions were presented. All participants were provided with written instructions which included illustrative examples of samples drawn from symmetric and asymmetric populations. The importance of being able to determine symmetry or otherwise was spelt out and this was highlighted by examples which portrayed the existence of sampling variability. Students were required to decide whether each sample could have come from a symmetric, left skewed or right skewed population. There was no time limit, but most students took between 10 and 20 minutes to complete the test.

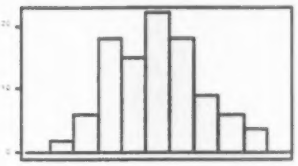
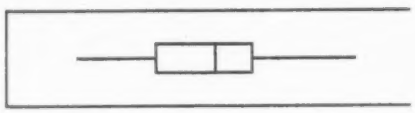
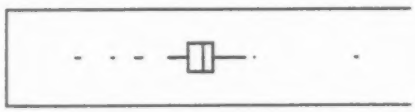
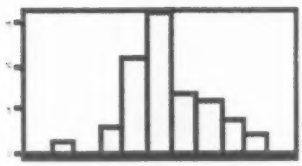
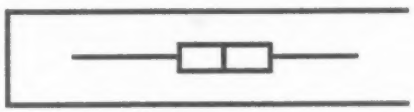
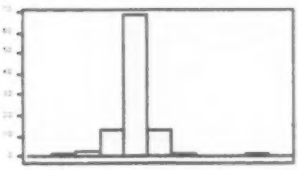
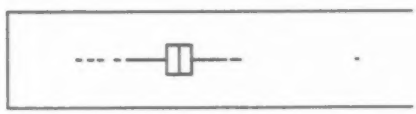
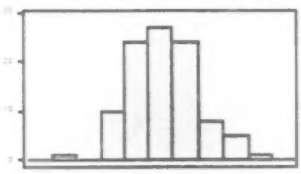
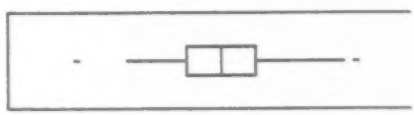
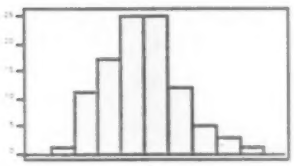
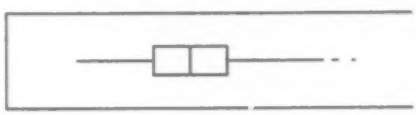
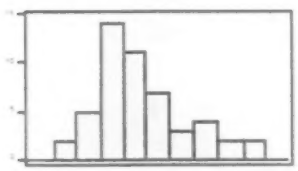
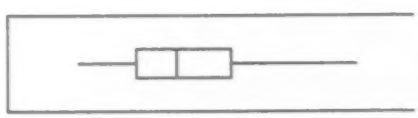
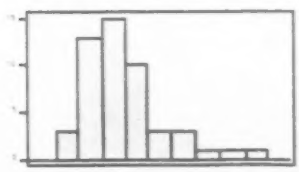
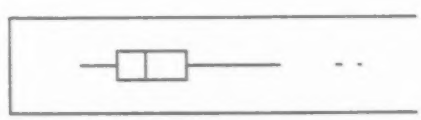
Analysis

Skewness measures and p-values

The validity of subjects' assessment of skewness can be evaluated by correlating observed outcomes with expected outcomes based on some objective measure of skewness. The skewness of the data sample, not the population from which it was sampled, was the measure used. The sample measures define a characteristic of the display (boxplot or histogram) available to the subject.

The measure of symmetry used was David and Johnson's (1956) statistic $J = \frac{\hat{\eta} - M}{\hat{\phi}}$ where $\hat{\eta}$ is the average of two symmetric percentiles (the sample minimum and maximum were used), $\hat{\phi}$ is the interquartile range, and M is the sample median. A p-value, or probability of obtaining a larger value of the test statistic if the population was

Figure 1. Right-skewed displays

Distribution	P-value	Histogram	Boxplot
a: n = 100	0.69		
b: n = 50	0.47		
c: n = 50	0.46		
d: n = 100	0.34		
e: n = 100	0.31		
f: n = 100	0.15		
g: n = 50	0.09		
h: n = 50	0.07		

symmetric, was obtained using a bootstrap procedure (Efron, 1979). The bootstrap procedure is based on simulations using the original data. Small p-values are indicative of non-symmetrical populations.

Analysis

On inspection of the data it was apparent that some students were confused at to whether the naming of the direction of skewness was determined by the direction of the tail or the position of the 'hump'. Therefore it was preferable to consider whether or not they considered it symmetrical. For each of the 32 displays the percentage of subjects stating that the sample could have come from a symmetrical population was obtained. These percentages, together with their associated p-values, are displayed in Table 2. These percentages were plotted against the calculated p-values and showed a weak relationship.

Table 2. Percentages of responses : Skewed

Observed (Fitted) Values

Distribution	Sample	P-value (J)	Histogram	Boxplot
a:	100	0.69		
right-skewed			2	39
left-skewed			6	34
b:	50	0.47		
right-skewed			25	28
left-skewed			29	50
c:	50	0.46		
right-skewed			18	18
left-skewed			52	10
d:	100	0.34		
right-skewed			20	57
left-skewed			43	66
e:	100	0.31		
right-skewed			10	11
left-skewed			8	17
f:	100	0.15		
right-skewed			48	36
left-skewed			59	63
g:	50	0.09		
right-skewed			91	90
left-skewed			98	94
h:	50	0.07		
right-skewed			99	98
left-skewed			100	98

A three way analysis of variance of a transformation of the proportion saying the data were skewed was performed. The factors were the direction of skewness, distribution and display type (histogram or boxplot). The transformation was chosen to achieve normality of the response variable.

The analysis of variance found that direction of skewness ($F_{1,15}$=9.52, p=0.008), distribution ($F_{7,15}$=105.80, p=0.000) and interaction between distribution and display ($F_{1,15}$=10.12, p=0.000) were all significant. Subjects showed a greater tendency to say that the displays indicated skewness when the data were skewed to the left than when the same data were displayed in the right-skewed orientation. This was particulary

pronounced for distributions that were not highly skewed. Display type was not significant but the interaction with distribution indicated that for some distributions the boxplot gave a greater indication of skewness whereas for others the histogram gave the greater indication of skewness. Further detail of the statistical aspects of this study can be found in Heller et al (1996).

Discussion

This emphasis on the left corresponds to results obtained in psychology. Patients with a lesion on the right hemisphere of the brain, when asked to find the middle of a horizontal line which was placed centrally in front of them, marked the centre to the right of the midpoint. The condition was known as 'neglect' as they were neglecting the left hand part of the display. However, normal subjects when asked to perform the same task marked the middle towards the left. Even though this was not a clinical condition, by analogy with the previous work, it was known as 'pseudoneglect'(Bowers & Heilman, 1980). Possible causes of pseudoneglect have been investigated in relation to line bisection such as hemisphericity of style, (Roig & Cicero,1994), direction of reading (Chokron & De Agostini,1995; Chokron & Imbert,1993), left-right-handedness (Sampaio & Chokron, 1992), hand used (Dellatolas et al, 1996), scanning direction (Brodie & Pettigrew,1996) and sex of subject (Laeng, Buchtel & Butter, 1996; Roig & Cicero, 1994). A similar favouring of the left side was found in a cancellation task using spatial stimuli but was not found with verbal stimuli (Vingiano, 1991). However, Luh (1995) did not find a similar effect in a number of asymmetry tasks. The differences found by Chokron et al would indicate that a repetition of this study with only right handed subjects in a country where initial reading direction was from right to left would be worthwhile.

If the results are indeed due to an overemphasis on the left-hand part of the display this would suggest that even though students have trouble naming the direction of skewness they are influenced in their conclusions by the direction of the tail. Even when the p-value for rejection of the null hypothesis of symmetry was about 0.50 (figures b and c), approximately one-quarter of the students thought the sample had come from a skewed population. It therefore behoves teachers of statistics to give students more examples from symmetric distributions to illustrate the variability that can occur.

The study gives a cautionary warning that until we investigate we can not be certain what students are seeing nor how they are interpreting it

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A Further Development of a Theoretical Framework for Research into Values in Mathematics Education

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Following a previous attempt to develop a theoretical framework for research into values in mathematics education, a further attempt is made using the three domains of values, culture and mathematics education. The valuing process is examined and the distinction between beliefs and values defined. The four dimensions of culture are linked with the valuing process and the elements in the teaching/learning act to construct a framework which could be the basis for further research.

In the immediate wake of the release of the results from the Third International Mathematics and Science Study (TIMSS), NSW politicians and business and community leaders deplored the fact that Australia in general, and New South Wales in particular, did not have results which placed them in the top bracket of nations taking part in the study. Their reaction and that of the students and teachers responding to the survey of beliefs included in TIMSS brought into sharp focus once again the issue of beliefs and values in mathematics education. Those quickly deciding that Australia must do better were making statements which indicated they had not considered some of the factors that might have affected the performance of Australian students such as the values placed on the subject by the culture of the students concerned or the type of test item that was used.

On a previous occasion the writer (1995) sought to develop a theoretical framework for research into values in mathematics education, on that occasion prompted by a concern to link previous research into problem solving (1992) and other research into beliefs concerning mathematics education (Southwell & Khamis, 1994). On reflecting on these results, it seemed that the critical role played by beliefs or perceptions on a person's awareness of problem solving strategies and acceptance of resolutions which conflict with those beliefs, needed further examination. Some attempt was made to identify commonly held beliefs concerning the subjects' perception of mathematics as a discipline, themselves as mathematicians and themselves as teachers of mathematics. While it does not seem that that particular research added greatly to general understanding of this aspect of mathematics teaching, it did prompt the exploration of strategies that might alleviate the anxiety and stress associated with some aspects of mathematics education.

From these general concerns, the writer was prompted to further explore the issue of beliefs and values in mathematics education and several questions were raised. What value does the educational system, community members, parents, teachers and students place on mathematics in the curriculum? What are the beliefs that underpin that value? What values are developed or clarified through study of mathematics or through teaching mathematics? What values support or derive from research into values associated with mathematics education?

The responses recorded in the TIMSS report on the beliefs of teachers and students are cause for great alarm, e.g. forty five per cent of Australian male students and forty eight percent of female students believe that "to do well in mathematics, you need lots of natural ability". Over half the Australian sample believe that "To do well in mathematics, you need lots of hard work studying at home". These responses imply that students have a narrow view of mathematics as a discipline. It appears that they see mathematics as a rigid, rule-governed set of facts and formulae that have to be memorised and applied. It also implies that these students see mathematics as a difficult subject for which a person needs to have innate talent in great degree and to have to work hard to get a good result. Such a belief might well result in students seeing mathematics as a chore which has to be endured rather than an exciting, challenging subject which will be of enormous benefit to them. These general trends are not different from previous results by Schoenfeld (1989). The responses to the items in TIMSS dealing with teaching practice indicate that the majority of classroom teachers follow a fairly "traditional" style. One

particular result of concern is the 90% response to the practice "We work from worksheets on our own". This suggests an almost total absence of group work or collaborative learning. The almost total reliance on worksheets which may be inferred from this result is most alarming.

The teacher survey focussed on two aspects of teaching. The first was teachers' preferences about teaching as a career and the other was beliefs about appreciation of their work. To the first, half the teachers surveyed said they would change their job if they were able, only one quarter of the teachers perceived any appreciation of their work from society and 64% of the teachers of the lower grade and 57% of the teachers of the higher grade felt they were appreciated by the students. It seems obvious that these beliefs are not unrelated but they do indicate a state of affairs which calls for exploration and remedy. Research into the role of values in mathematics education and the outcomes that result from it are essential fields of interest for research.

The previous framework put forward by the writer (1995) was built on the model provided by McLeod (1992) and took into consideration the definition of values given by Hill (1991). It examined the field of research into values in mathematics education from three different levels; namely, conceptual, task and output levels. These corresponded to the three domains of concepts and constructs, research methodology and classroom implications. There are some problems associated with this model and for that reason, the model has been re-examined and an alternate model suggested. This current model begins with Hill's definition and incorporates the cultural influences as well as the central features of the mathematics teaching/learning act.

Values in Mathematics Education

The writer has for some time been convinced that beliefs and values are directly related but are not the same as is implied in some of the literature. She sees the difference between the two concepts as being largely consistency and intention. A person might espouse certain beliefs but, when challenged in some way, might take action contrary to those professed beliefs. Or the other hand, a person who holds certain values, albeit based on the same beliefs, will act consistently in the face of any pressure. This is to say that beliefs are generally a subset of one's values.

Hill's definition (1991) of valuing combines three different dimensions; the cognitive, the affective and the volitional as follows:

> The three elements (of valuing) are:
> 1. A value can be described by a statement expressing a person's belief in an idea, having to do particularly with judgements of worth or obligation. Thus it has a rational or 'cognitive' element, and the belief statement involved is often referred to as a 'value judgement';
> 2. The intensity with which individuals believe certain ideas, especially those affecting the priorities they attach to certain kinds of experience, such as the moral, aesthetic and physical, indicates that there is an emotional or 'affective' element to valuing, and this is often (but not always) what people have in mind when they speak of 'attitudes';
> 3. Because one's most deeply held beliefs dispose one to act in certain ways, there is a volitional element, which leads us to refer to such beliefs as 'dispositions' or 'commitments'. We tend to make choices which are consistent with our value systems, that is, to live by the beliefs and values to which we attach highest priority. (Hill, 1991, p. 4)

These three elements can be depicted as in Figure 1. They are interconnected and interdependent.

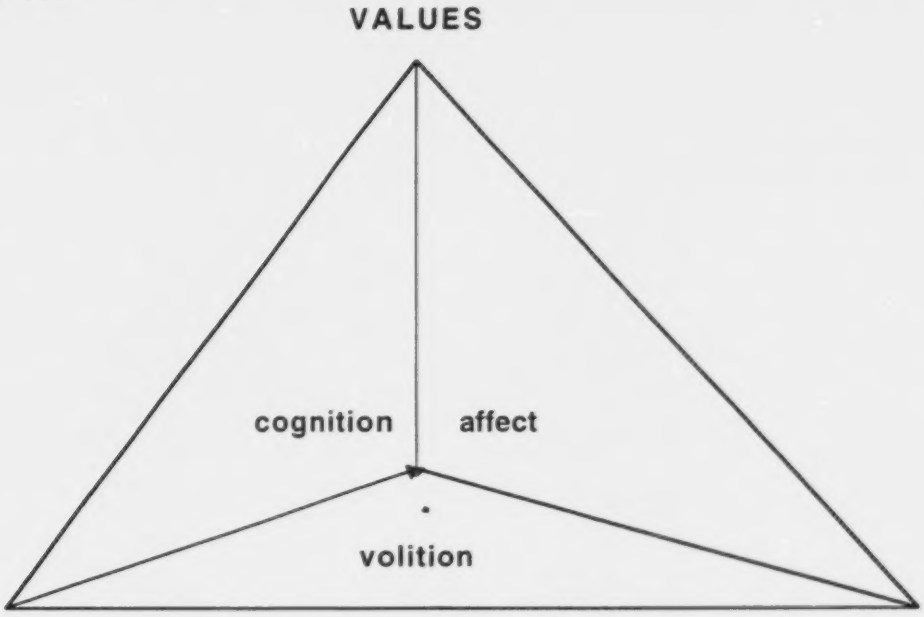

Figure 1. Three elements of Valuing

In Figure 1, values are denoted by a triangle, the three vertices of which represent the three dimensions cognition, affect and volition. They interact and unite to constitute the values a person holds and lives by.

Research in this area is regaining interest though most studies have dealt with beliefs or some other affect such as attitude (Aiken, 1970), or disposition (Brahier and Speer, 1995) rather than values.

This description of Hill's nicely ties together the relevant concepts of emotion, attitudes, beliefs, values, disposition and commitment and helps to clarify the distinctions between the closely related ideas.

Building on his description, Hill (p. 10) suggests that values education should seek to develop firstly a knowledge base which includes the traditions which have helped to mould current cultural values. Values education should also enable students to empathise with 'the perceptions and feelings of people who have been strongly committed to these traditions' and to develop skills in critical appreciation of values. Values education should also aim to encourage students 'to develop a concern for the community and the care of its members.'

What has this to do with mathematics education? What possible contribution can mathematics education make to necessary changes in values and beliefs which will enable it to contribute to national development and growth? The evidence is there in the more obvious application of values education to science and technology education, but does values education have same relevance to mathematics?

The writer maintains that attitudes towards mathematics on the part of students, teachers and parents, beliefs about mathematics and values placed on mathematics all need to change fairly radically if there is going to be a corresponding change in the value governments and the community place on the subject. The change could start from the classroom but change from that direction will be slower than if governments and the community raise the status of mathematics education in the eyes of the public. Teacher education can contribute greatly to an overall change in attitudes, beliefs and values, but if a formal examination system implies contrary values or the school executives do not keep ahead of international trends in the learning and teaching of mathematics and insist on novice teachers teaching in a routine, stereotyped way, no progress can be made.

Values are closely related to beliefs but are more complicated and encompassing. Both beliefs and attitudes can change. Beliefs are relatively stable but they can and do change with age as a person becomes more mature, has different experiences and enjoys different states. Other beliefs and attitudes which are not dependent on maturation or natural growth and development could well be more resistant to change. Some radical re-discovery or re-experience of the concept involved is usually necessary for such a change in beliefs and attitudes to take place.

An example of such a change in belief is usually necessary for primary teachers in relation to the use of calculators or computers in the classroom. The author has found, however, that a change in attitude towards calculators is harder to achieve than that towards computer usage. In one particular year, considerable resistance towards the use of calculators in the primary school among final year teacher education students was detected. They believed that students who used calculators would immediately become lazy, forget their multiplication tables and take no interest in understanding algorithms. To counter these arguments, each member of the class was asked to find a partner to work with and each pair of students was given a sheet of paper on which were written five or six key points promoting the use of calculators in schools. After reading the points, one person in each pair was asked to play the role of the teacher, while the other became an anxious parent who came to school to complain about his/her child being given a calculator in mathematics. The parent was armed with negative points and the teacher had to argue against the parent in order to justify what he or she had done. After a few minutes, the pairs were assigned new roles. The student who had been the parent became the teacher and the student who had been the teacher became a member of the school executive from whom the teacher tried to obtain permission to buy a class set of calculators. After these two role plays, all the students were asked to write a summary of what they had learnt and what they had experienced. It was interesting and encouraging that most of the students in the class had changed their beliefs about calculators and their attitude towards the use of calculators in the classroom. This was a simple activity to organise and monitor with quite astounding results.

Culture and Mathematics Education

Culture is another term which seems to be used in a number of different ways. White (1959, cited by Bishop, 1995) claims that 'the functions of culture are to relate man to his environment on the one hand, and to relate man to man, on the other' (p. 8). He goes on to categorise the components of culture under four headings:

> "ideological - composed of beliefs, dependent on symbols, philosophies;
> sociological - the customs, institutions, rules and patterns of interpersonal behaviour;
> sentimental - attitudes, feelings concerning people, behaviour;
> technological - manufacture and use of tools and implements."

White claimed that these four categories are connected and that the technological component is the one that drives the other three. The arrows from the technological domain are to denote the impact of that domain on the other three.

The social institutions of people are obviously dependent on the technology of those people. This is clear from the earliest history of man when hunting, food production and similar processes were not merely technological in nature but were also social events. Even in modern times, in many western societies, cooking, gardening, harvesting and even barn-construction are all social events. This is even more so in some less developed countries. The impact of some of the modern technologies is sometimes deplored, in that a valued social time, such as washing up in western societies, has been taken over by the dishwasher.

The ideological or philosophical component is derived from the technological component also, in that changes in technology create changes in ideology. The use of modern farm machinery gives rise to a different philosophy to that arising from the use of hand ploughs or those driven by oxen or buffaloes. Usually the latter are the plight of small subsistence farmers whereas the latest machinery indicates a more capitalist venture.

The sentimental component also is affected by the technology available, though less strongly than the other two components. Possible examples of this are the attitudes that have emerged with the development of such technologies as genetic engineering, information technology and the atomic bomb.

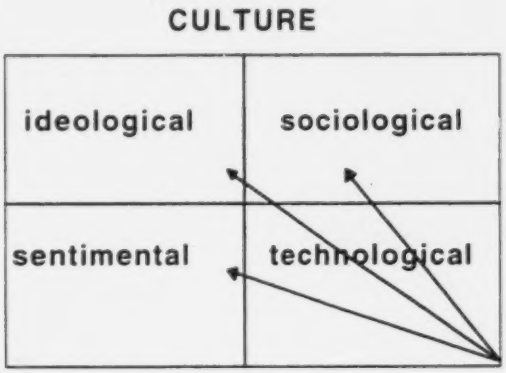

Figure 2. White's four dimensions of culture

In Figure 2, a rectangle divided into four parts is used to denote the four dimensions of culture as set out by White (1951). Although this particular representation implies a distinctiveness for the four dimensions, they are joined together by common boundaries and together they form a whole.

Delors (1996), in his report for UNESCO, introduced the notion of the four pillars of education. While he was relating these specifically to developing countries, the same statements could be made about developed countries. The aims of education must be *learning to live*, *learning to do*, *learning to be* and *learning to live together*. He places great stress - more than previously - on the last two. These two, learning to be and learning to live together, are those that enhance the humanness of the whole function of education.

It may be that governments see their national development in terms of economic growth and world trade and place little value on the maintenance of their unique culture and on the education of their citizens. Systems are designed to follow the bidding of governments, while communities and teachers are not always aware of the issues involved. Their main concern is the difficult business of living from day to day. The real importance of education for national development is not recognised and consequently education becomes a luxury rather than an essential element of a government's program. It is alarming that it appears that education is regarded more highly in developing countries than in the developed ones. A change in attitude and beliefs is required in these developed countries and this will lead in turn to greater value being given to education in national development.

Science and technology education are more highly regarded in many countries than is mathematics education. Conferences on science and technology, including values education, have been held at various times over several years. Why is it that mathematics is not so universally highly regarded?

The Mathematics Teaching/Learning Act

It is with mathematics in the classroom that this development continues. The mathematics teaching/learning act involves in turn three main elements. These are the students, the teacher and the curriculum. The students are generally taken to mean learners and teachers the instructors, though it is increasingly recognised that teachers are also learners and students are also teachers. Curriculum is used in the sense of being not only the content of the subject but also the methodology, environmental aspects and contexts. To explore the teaching/learning act, research is needed, not only on each of these elements, but also on the interactions between two or all three of these elements

since the interactions in the situation are typical of the essential characteristics of the situation. These elements and interactions can be denoted as in Figure 3.

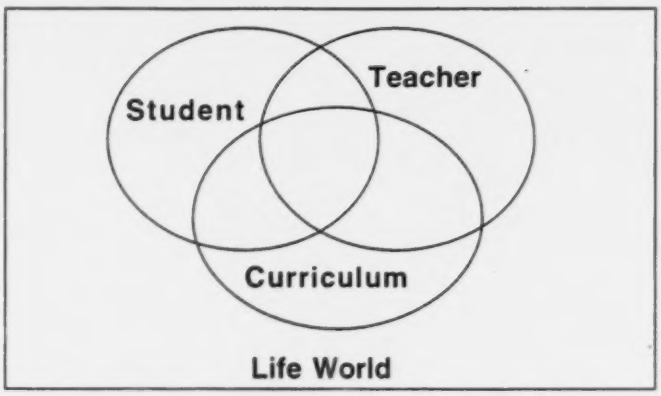

Figure 3. The Teaching/Learning Act

Figure 3 uses circles within a Venn diagram to denote the three key elements in the teaching/learning act. The three circles represent respectively the student, the teacher and the curriculum in its fullest sense. The intersection of these three sets denotes the teaching/learning act itself.

Research in this area is well founded and is approached through a variety of methodologies, including qualitative as well as quantitative methods.

Range of Research

As indicated above, very little research seems to have inquired into values in mathematics education. There is, however, a growing number of studies which investigate various aspects of beliefs and attitudes. As well as the study by Aiken (1970) mentioned previously, there have been studies more closely directed towards specific topics such as Callaghan and Watson's study (1995) on teacher attitudes towards chance and data, Taylor's (1995) on attitude to mathematics of adults returning to study using a distance mode and Kyeleve and Williams' study (1996) on the attitudes of teachers towards mathematical modelling.

Studies on beliefs related to various issues are somewhat more numerous. They include Anderson's (1996) study on beliefs concerning problem solving, Buzeika's (1996) work on teachers' beliefs and practices, Gervasoni's inquiry into students' and teachers' beliefs concerning helpful teaching practices in mathematics and Goos's (1996) study on metacognitive knowledge, beliefs and classroom mathematics. Two interesting cross-cultural studies are those by Yong (1993) who analysed differences in attitudes between performance in mathematics and science of middle grade students from African-American, Mexican-American and Chinese-America students, and Dunn (1997) whose work was on the beliefs about the mathematics instruction of preservice elementary teachers..

A variety of methodologies has been used in research into beliefs, attitudes and values. They range from the survey instrument administered to groups or individuals, such as that given by Anderson (1996) to studies similar to that of Brahier and Speer (1995) in which a number of different sets of data was collected and used. Brahier and Speer observed the children first, then had the student fill out a survey. This was followed by individual interviews and this was followed in turn by parent telephone interviews. Other methodologies used in this area are life histories (Taylor, L, 1995), reflective journals (Bobis and Cusworth , 1995) and lived experience anecdotes (Carroll, 1995).

The Outcome

When the three domains of values, culture and mathematics education are linked in some way, a structure emerges against which research into values in mathematics education could be initiated, monitored and evaluated. Such structure is shown in Figure 4.

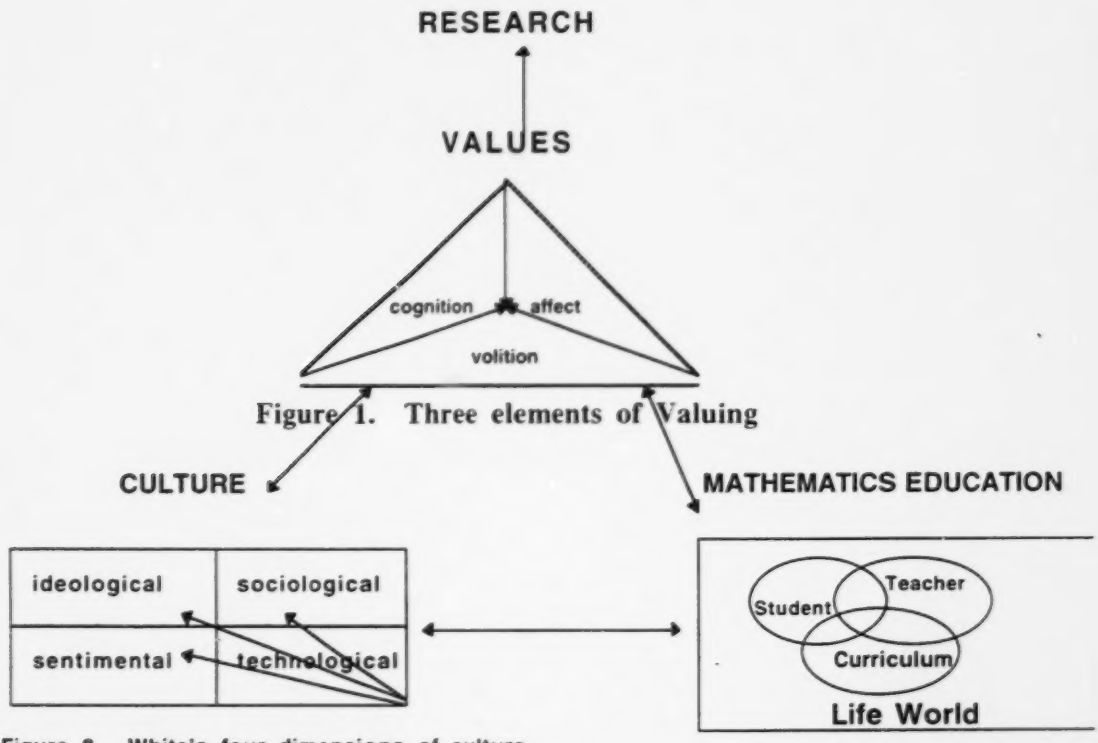

Figure 1. Three elements of Valuing

Figure 2. White's four dimensions of culture

Figure 3. The Teaching/Learning Act

Figure 4. The Accumulated Framework

The domain of values is placed at the top and central part of the diagram, and interlinked with culture, which in turn is linked with mathematics education, further linked with values. The interdependent aspect of the three domains is given by the arrows between each of the domains. The resultant of the interconnection of these three domains is research into values which makes sense of mathematics teaching in a multicultural society.

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Should all children use calculators? A Western Australian Survey.

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Edith Cowan University.

To what extent are calculators used in primary school classrooms? Data sources are scarce and somewhat dated for judgements to be made. This paper outlines some of the findings of a recent survey undertaken in Western Australian primary schools. Data related to the issue of young children using calculators will be discussed. Initial findings suggest that integrated calculator use is rare with most use being extra to mathematics learning and trivial in nature

Introduction

There are very few issues in mathematics teaching over the past twenty five years which have caused as much debate among teachers, parents and the community as the use of calculators in primary schools. Letters to newspapers, magazine articles and public comment often express the opinion that the apparent decline in the ability of young people to calculate even simple sums is a direct result of the use of calculators in schools. Anecdotal evidence in the form of reports from teaching practice students, personal observation in schools and comments from teachers suggests there is, in fact, very little calculator use in primary classrooms. Where a calculator is used it is for checking work - an electronic answer book - or for trivial activities, for example to make words appear on the display. No concerted, planned, integrated use as outlined by the *Statement on the Use of Calculators and Computers for Mathematics in Australian Schools* (AAMT, 1996) or the previous *National Statement on the Use of Calculators for Mathematics in Australian Schools* (AAMT, 1987) has been observed at this anecdotal level. So if there is little or no calculator use in primary mathematics classes what is the cause of this apparent decline in numeracy standards? Debate on this is left to another time.

One is left to wonder about the actual use of calculators in primary classrooms. Blane and Willis (1986) reported on the use of calculators by 12 to 14 year old children in Australian schools. They noted "information on this topic (calculator use) is relatively sparse" (p.8). From a search of the relevant literature it seems that there is very little if any current information about calculator use in primary schools in Australia.

Literature Review

Very few surveys of calculator use in primary schools have been attempted. A search of the literature covering all current listed documents revealed the following surveys:

- Reys, B. J., Reys, R. E., Barger, R., Hauck, J., Morton, L., Reehm, S., Sturdevant, R. & Wyatt, J. W. (1990). *Calculator use in Mathematics teaching in Missouri schools: A 1990 status report.* A report of a survey conducted by the University of Missouri for the Missouri Department of Elementary and Secondary Education. This report provides an indication of the situation in one state in the U.S.A.
- Reys, R. E., Bestgen, B. J., Rybolt, J. F. & Wyatt, J. W. (1980). Hand calculators: What's happening in schools today? The Arithmetic Teacher, 27, 38-43. This article outlines the results from a study carried out in Missouri a decade earlier.
- Groves and Stacey (1994) refer to two Australian Surveys - one carried out by Ferres in 1981 and another as Part of the Calculators in Primary Mathematics Project in 1990.

In 1986, Blane and Willis presented a *Report on the UNESCO Pilot Project on the Applications of Calculators to Mathematics Teaching in Australia* in which they

considered the question of calculator use by 12 to 14 year olds. As part of that study, questionnaires were sent to a large number of schools across Australia. In summary the authors stated that "despite clear statements from State and Territory Education Departments in Australia, there still appears to be a great deal of confusion about whether, when and how calculators should be used in Australian mathematics classrooms" (1986, p.6). They went on to say that "there have been some surveys on the use of calculators in school mathematics classes in Australia, but few have been published, and information on this topic is relatively sparse" (1986, p.8).

Groves and Stacey (1994) compared the results of two surveys carried out almost ten years apart. Their key findings are reported below:

> "Results indicated a remarkable shift in favour of the early introduction of calculators since an earlier survey was carried out ten years earlier (Ferres, 1981). In the 1990 survey, 75% of teachers supported calculator use in kindergarten to grade 3, compared with a mere 7% in 1980. These attitudes, however, did not necessarily translate into practice, as 58% of K-3 teachers admitted to rarely or never using calculators in their classrooms." (p.2)

Reys and Reys et al (1990) also compared changes in the use of calculators over a decade. They surveyed both primary and secondary teachers within the state of Missouri. They found that the changes which had occurred were most pronounced in the secondary school. Teachers still primarily viewed the calculator as a tool for checking answers produced by written computation. Some teachers even expressed the belief that they had to spend more time teaching written computation to make sure that proficiency with the written algorithm didn't decline as a result of the increased access to calculators in the home.

Research Questions

The survey was designed to provide data to answer the following questions.

1. To what extent are calculators being used in primary schools?
2. For what purposes are calculators used in the primary classroom?
3. What is the attitude of teachers toward the use of calculators in the primary school?
4. What impediments are there to the use of calculators in the primary school?

Procedures

In Term 3 of 1995 a survey of primary school teachers in Western Australia was conducted. Because of the size of the State and the scattered distribution of schools a questionnaire was used to gather initial data. The two researchers were aware of the typical low response rate often attributed to the use of questionnaires as well as the apparent mismatch between attitudes and actions which is highlighted by Groves and Stacey (1994). This is a problem with most surveys, particularly those that canvas attitudes and opinions. The questionnaire was based on the earlier work of Reys & Reys et al (1990) and used many of the questions which were used in their interviews.

Questionnaires, with reply paid envelopes, were sent to 787 primary schools across Western Australia. These included all state primary and independent primary schools in Western Australia. Responses were requested from those teachers with classes at either Year 1, Year 3, Year 5, or Year 7 levels. Teachers with composite classes eg. Year 3/4 were asked to reply as well. If there was more than one class at each year level then all teachers for that year were asked for their comments. The responses were to be anonymous on the part of the teacher; however, schools were requested to supply their post code so later clustering by region could be attempted.

It is not possible to provide a response rate as single and multiple responses were received from schools. There is no way of knowing whether schools had one or more classes at each of the targeted year groups. However, 1297 responses were received, quite a substantial response taking into account the industrial unrest in state schools at the time.

Data from the surveys were transcribed to a computer data base ready for analysis and comparison with similar data from previous surveys. Responses were tabulated and where appropriate a simple percentage calculated. Comments to the particular question were clustered under thematic headings.

Findings

For the purpose of this paper only aspects of data related to a single survey question will be discussed. A more complete account may be found in *Calculator Use in Western Australian Primary Schools: A survey* (Sparrow & Swan, 1997).

The first question on the survey and the one for discussion in this paper is shown in Table 1.

Table 1. *Should all students use calculators in primary school?*

No response*	No students	A few students	Some students	Most Students	All Students
12	4	8	107	238	928
1%	0.3%	0.6%	8.25%	18.4%	71.6 %

* It should be noted that while we received 1297 responses not all teachers responded to every question.

The overwhelming response to this question was 'yes' with almost three-quarters of the teachers concurring with 'A National Statement on the use of Calculators for Mathematics in Australian Schools'(1987) recommendation that, "ALL students use calculators at ALL year levels (K-12)" (p.1)

This data with its high level of positive response would suggest expressions of support for calculator use being consistent across the range of schools - country, large, small, metropolitan, and independent. One must, however, temper this statement with the realisation that the responses may have come from a minority of teachers within the total school staff. Reys & Reys et al (1990) from their interview data gained a 94% positive response. It must be noted, however, that they did not include the word *all* in their question.

Looking beyond the statistics it was noted that when teachers responded that *all students* should use calculators in primary school they were not referring to free access to calculators but rather limited use as decreed by the teacher. The following quotes taken from the comments section on the survey revealed, in fact, a rather guarded response to this question. A cross section of comments from teachers responding "all students" is given below;

> Only in senior classes and only occasionally.

> But to varying degrees. More able children can use calculators to mark and correct work, puzzles, games and to do sums above their level.

Note the qualifying remarks in the above comment. The teacher refers to more able children and also outlines various uses for the calculator. There is a reluctance on the part of teachers to allow students to determine when and how a calculator should be used.

Several themes became evident when reading the more detailed evidence from the responses given by participants. These are summarised below along with responses that typify the nature of the theme.

Delayed Starting Points: Many teachers referred to various stages below which children should not use calculators. These tended to cluster around the end of year one and after year four:

> I don't think they are necessary below grade 3
> Only senior grades - when tables are automatic.

Overeliance: Some concern was raised that children might become reliant on the use of calculators:

> Should be made available but not under all conditions. Shouldn't learn to be totally reliant on them.

> All students need the knowledge to use one but it shouldn't be used instead of their brain and pen and paper.

Understanding Will Suffer: In particular teachers were worried that 'understanding' would suffer as a result of using calculators. Basic number facts and particularly the tables were mentioned as prerequisites to using a calculator at any great length:

> Until there is understanding of the process I don't believe calculators should be used.
>
> They need to learn the basic number facts before they can use a calculator
>
> I believe all children should be exposed to calculators but I don't believe they should use them consistently. I believe the constant use of calculators weakens an individual's mental maths ability.

Terms such as 'understanding' and 'process' were often used by teachers when responding to questions in the survey. It was clear from the variety of uses of these words that teachers' interpretations of these terms vary widely.

Calculators as Checking Devices: Many teachers relegated the use of calculators to a checking device:

> Can be useful for checking written calculations - AFTER solving without calculator first.
>
> Calculator use should not replace using the algorithms but should be primarily for checking work in the earlier years.

Formal Calculator Lessons: Formal calculator lessons were mentioned by several respondents. The responses seem to indicate that these lessons were separate from what was going on in mathematics lessons at the time:

> I don't allow children to use calculators during lessons. However, I take the children for an intensive course on the function and use of the calculators as part of my maths program.

Real Life Use: Some teachers indicated that they felt calculators should be used because they had now become entrenched in 'real life'. Comments, such as the following, were typical of those in this category:

> Every child will be exposed at one time or another to calculators out there in the 'real world'. Our job is to equip them with life skills.
>
> People do not use pen and paper maths when they leave school. They should be familiar with how to use a calculator.

Some respondents also referred to the need to be familiar with calculators because high school teachers assume primary students can use calculators.

Positive Comments: Some positive comments regarding the use of calculators were received. These are highlighted below:

> All students should have their own calculator as part of their essential equipment.
>
> From cavemen people have looked for easier ways to calculate number problems. Children should be able to use whatever technology is available to them to demystify mathematics and make it available to all.
>
> Calculators are now the pen and paper of mathematics. The idea of students still doing long algorithms (or complex ones) on paper is dead and not relevant to the real world.

Reys and Reys et al (1990) also found that while many respondents thought children should use calculators in school they wanted to place conditions on that use. The

main restriction was that children should know their basic facts or table facts prior to using calculators.

Conclusions

From the statistical data it seems that most teachers are in favour of all students in the primary school using a calculator. When one looks beyond the data to the comments section a different picture emerges. Use is much more guarded, delayed or under the guidance of the teacher, highlighting the mismatch of actions and beliefs noted earlier. Where they are used it is often for checking calculations rather than for fundamental approaches to learning mathematics.

Many teachers see the calculator in a very restricted way, viewing it only as a machine to do sums. The notion that it might be used as a teaching and learning machine was rarely mentioned. The calculator is considered as an extra, an add-on to the normal day-to-day mathematics teaching. Thus, it is used to check answers, to play with, to do fun things and in some cases for specific calculator lessons. It makes no impact on the algorithmic, procedural teaching of the four rules and rote learning and testing of tables. This is possibly due to a lack of awareness as to how the calculator can be used to help children learn mathematics or a philosophical stand-point which equates mathematics with precise replication of taught procedures for calculations.

There seems to be no planned use of calculators in the programs of many teachers. If calculators are used it is for trivial, incidental or "fun" activities. The impression given is that the calculator is not important, and that using it is tantamount to cheating and far too easy. Mathematics should be serious, mystical, hard and boring. Some teachers, however, are incorporating the calculator into their planning and finding worthwhile and interesting ways to use it to help children understand the mathematics they are completing.

Teachers who use calculators to help their children learn and understand mathematics appear to have been exposed either to research-based ideas or have undertaken planned activities with their children and have seen the benefits of calculator use.

It appears from the survey that the teaching of standard algorithms and the rote learning of tables is the main emphasis in Western Australian primary schools. Calculators, where they are used, apart from minor exceptions, are for checking and trivia. The rumours and media hype regarding the alleged inability of children to do simple calculations in their head cannot be attributed to the widespread use of calculators in primary schools.

Development work in schools with the CAN project (Duffin, 1996) and the Victorian Calculator project (Groves and Cheeseman, 1995) has shown that calculators provide a rich source of teaching and learning and that children benefit from their integrated and sensible use. Major reviews of research studies (Hembree & Dessart, 1986, 1992; Suydam, 1982) have reported the fact that calculator use has no detrimental effects on the mathematics achievement of children. In fact the reviews point to the opposite conclusion, that is they improve children's ability with calculations and problem solving.

Data from the first of sixteen questions in the survey offers insights into the research questions, particularly numbers 1 to 3. Question 4 relating to the impediments to calculator use is more difficult to answer from the evidence presented here. One impediment that does appear strongly is the attitude and belief of the teacher. Until this generally reserved and negative attitude is changed there will be no concerted, integrated use of calculators in primary classrooms in Western Australia.

Returning to the original question in the title of this paper -Should all children use calculators?- the answer appears from the evidence presented in response to question one of our survey to be a resounding yes BUT...

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Developing tasks to assess mathematical performance

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There is a need to find ways to assess performance of students in mathematics which also provide some assurance to the community of the quality of the work of the students. As part of a larger project which sought to develop assessment tasks, we found that tasks could be used to assess the level of students' mathematical knowledge, that teachers could make holistic judgments on students' responses. We also argue that it is necessary to use a range of tasks to assess mathematical performance on particular aspects of the curriculum.

Introduction

In many English speaking countries there is considerable attention to assessment of students in mathematics, and particularly in using such assessment as measures of accountability within school systems. Within Australia, most States and Territories have implemented large-scale data collection on student performance in mathematics through compulsory tests.

The model used in these large scale assessments has been typically that of the standard achievement test, with response types often being constrained by a requirement for electronic correction. However, there is growing dissatisfaction with this form of assessment. Some criticisms have centred on limitations of this form of testing mathematical understanding (e.g. Clements & Ellerton, 1995). Other criticisms have been directed at the ineffectiveness of large-scale testing in supporting curriculum planning and informing instruction (Clarke, 1996).

It seems that there is a need to identify assessment tasks which satisfy community demands for accountability and which are also educationally meaningful and valid. One approach which has the potential to do this is called performance assessment. This paper describes some characteristics of performance assessments and presents some data from a project which sought to identify tasks which could support teachers in assessing mathematical performance.

Performance assessments

Performance assessments may take the form of practical tasks, interviews with students, project work or specially designed written tasks. Judicious use of performance assessment tasks taken over time is intended to assist teachers to make overall judgments of whether a student is performing at, below, or beyond a given level. This focus on performance assessment is a direct outcome of recent work in North America and other countries (see for example National Council of Teachers of Mathematics, 1989; Hong Kong Department of Education, 1996) on curriculum frameworks and assessment standards.

Performance assessment is generally linked to a statement of a standard derived from content frameworks developed by state or national agencies or by professional associations, such as the Curriculum and Evaluation Standards for School Mathematics (National Council of Teachers of Mathematics, 1989). A program of performance assessment typically contains two further elements: performance descriptions and annotated work samples. In the words of the New Standards Project (1995, p2), the former are "descriptions of what students should know and the ways they should demonstrate that knowledge and the skills they have acquired", and the latter are examples of student work chosen "for their capacity to illustrate the meaning of the performance descriptions together with commentary that shows how the performance descriptions are reflected in the work sample".

The nature of performance assessment is well summarized by the following description:

> Performance assessment requires that students actively respond to questions. Success in producing a response depends on reasoning, problem solving, and communication skills that students bring to bear on the problems that are posed. In contrast, success in traditional tests depends substantially on the ability of students to recall knowledge. In the course of performance assessment, students are

> required to apply their knowledge to situations to closely resemble "real life" circumstances. Performance assessment tasks require that students respond by completing one or more complex tasks or activities, such as developing an argument, performing an analysis, formulating a policy, or solving a problem that included several component problems. (Wisconsin Center for Education Research, 1993, p. 7)

One of the key aspects of performance assessments is the ways in which they differ from standard achievement tests. In the use of standard achievement tests, links between assessment and teaching are difficult to identify. The progressive nature of assessment over time marks a further key distinction, as does the inclusion of a wide range of types of assessment tasks, ranging from practical tasks, projects, observations of student performance, oral interviews, discussions, portfolios of selected samples of students' work, as well as specific assessment tasks (Hong Kong Department of Education, 1996). Further, the importance of teacher judgment in the interpretation of assessment marks an obvious break with standard achievement testing. What is also needed is a way to satisfy the legitimate community interest in the standard at which the students are performing.

Defining standards of performance

Proponents of global tests presumably justify their use by claiming that the tests measure performance against a clear standard, they are free from teacher bias, and are reliable. The inference is that teacher assessments, while being useful for informing teaching, are not robust in terms of objectivity or consistency. Performance assessment have the potential to address such concerns directly. One of the advantages of performance assessments is that they provide teachers with tools to define, measure and report the standard of performance of the students. Sometimes referred to as benchmarks, targets or indicators of achievement, outcomes, or pointers, these standards describe the knowledge, skills and understanding that students are intended to exhibit at a given stage or level of schooling. Different levels of students' achievement are described by reference to such standards.

For example, in the Victorian Curriculum and Standards Framework - Mathematics (CSF) (Victorian Board of Studies, 1995), the curriculum is described in six major strands: Space; Number; Measurement; Chance and data; Algebra, and Mathematical tools and procedures. Each strand is then further elaborated by Substrands. The Number Strand, for example, is considered in terms of the Substrands*: Number, Counting and Numeration; Mental Computation and Estimation; Written Computation; Applying Numbers; and Number Patterns and Relationships.* For each substrand, seven levels of performance are given for reporting student achievement over the eleven years of schooling covered.

For example, in the *Applying Number* Substrand levels 3 and 4 read as follows:

> Level 3: Representing and solving problems involving up to two of the four operations, including situations involving whole numbers and simple fractions of objects, money, and other quantities within the student's experience. (Level 3 describes what the majority of students might be expected to achieve by the end of Year 4.)
>
> Level 4: Choosing and using appropriate operations to solve problems which may involve whole and fractional numbers and more than one operation (Whole-number multipliers and divisors only). (Level 4 describes what the majority of students might be expected to achieve by the end of Year 6.)

In the CSF these statements are illustrated by indicators of performance sometimes referred to as outcomes. The statements also describe an increasing complexity of performance from one level to the next. A similar approach is taken in the Targeted Oriented Curriculum (Hong Kong Department of Education, 1994), whereas the NCTM Curriculum and Evaluation Standards for School Mathematics (1989) uses a broader framework for describing standards and the curriculum. Thirteen standards are used involving mathematical content and thinking for Grades K - 4, fourteen standards for Grades 5 - 8, and the same number for Grades 9 -12. The fourteen standards relating to Evaluation are intended to "help teachers better understand what students know and make meaningful instructional decisions ... As the curriculum changes, so must the tests" (p. 189). Performance standards have several important features which distinguish them from behavioural objectives. First, performance standards are intended to reflect key emphases in the development of the mathematics curriculum, and their achievement is to be taken as

evidence of student progress through that curriculum. Behavioural objectives were developed to indicate a sequence of learning, usually in tiny chunks, and are not necessarily linked to a curriculum or program of teaching. Performance standards reflect a tendency to view learning and achievement in more complex terms:

> The tasks must have multiple objectives and require higher order levels of thinking than is demanded by most pencil-and-paper examinations. (Foster, 1991, p. 35)

In the New Standards Project (1995), for example, where for the High School years performance descriptions are elaborated in eight areas of the mathematics curriculum, a single assessment task might be designed to elicit performances from more than one area. Embodying more complex statements of achievement, performance assessments have a different level of aggregation from items on a standard achievement test. Several performance assessments, considered together, can provide a guide to the level of achievement of a particular student or group of students in a given area of mathematics. In other words, teachers make judgments on student responses to rich tasks designed to elicit different levels of achievement. If it can be established that teachers do make such judgments validly and reliably, then performance assessment will serve the same accountability requirements as mandatory tests, as well as providing better instructional support.

Preparing tasks for assessing mathematical performance

This is the report of some data which were collected as part of the Exemplary Assessment Materials Project which produced assessment tasks to support individual teachers in monitoring performance of their students in mathematics (Beesey, Clarke, Clarke, Stephens & Sullivan, 1997)[ii]. The project prepared 200 tasks which teachers could use to assess student achievement of specific aspects of content as specified within curriculum frameworks, which give insights into how students approach mathematical tasks, and which provide opportunities for students to give high level responses, and which are rich teaching activities as well. The tasks were a mix of extended, medium length or short tasks drawn from the Number, Measurement, Algebra and Chance and Data strands of the curriculum.

Data were collected on a variety of aspects of the assessment tasks in order to establish whether the tasks were suitable and whether the recommended scoring was manageable. In particular, the data collection sought to determine whether:

- the tasks were able to be completed by students at the levels at which they were posed;
- the tasks were able to be compared with each other;
- teachers could make valid and reliable judgments on the performance of the students at the tasks as a whole and whether responses of the students were consistent across the tasks.

Task administration

All tasks were piloted to adjust wording and difficulty level. For this data collection, the tasks were grouped together into components of particular Substrands of the mathematics curriculum. For example, the tasks which addressed Written computation were completed by the same students.

The tasks within each CSF Strand were completed by at least two complete classes of students. The responses to the tasks were scored by their teacher and analysis done on that scoring. For one sub-strand in each strand of the curriculum a member of the research team scored one sub-strand of tasks independent of the scoring done by the teacher.

Scoring the tasks

The tasks generally focused on a particular aspect of mathematical content. Each tasks consisted of several related questions, often in increasing order of difficulty, set in a particular context. The project took the view that providing a direct numerical scoring (e.g. 3 out of 5) was not an appropriate way for recording information on student responses to the tasks. A more global approach was used to assist teachers to form on-

balance judgments of the student's performance on the task as a whole. Each of the tasks were scored on a range from 1 to 4 using the following general rubric:

Score	Summary/Description (Used for Content Strands: Space, Number, Measurement, Chance & Data, Algebra)	Mathematical Tools and Procedures Equivalences
Goes Beyond	Fully accomplishes the task, but uses methods and/or makes interpretations significantly beyond those specified for this level.	Strategies /mathematical communication /reasoning significantly beyond those specified for this level.
4	Task accomplished. Central mathematical ideas clearly demonstrated and understood.	Appropriate plan. Clear communication of strategies and mathematics used.
3	Substantial progress towards completing the task; indicative of understanding of relevant knowledge, concepts and skills, but some key ideas may be missing.	Some evidence of planning; some communication of strategies and mathematics used.
2	Attempt at the task makes some progress; partial but limited grasp of the central mathematical ideas; reeals gaps in knowledge, understanding and/or skills.	Little evidence of effective strategies/communication/ reasoning.
1	Little progress or understanding evident.	Ineffective strategies.

In addition to the rubric, teachers were give about a particular task, including comments on the particular mathematical focus, guidelines for administration, and samples of student work.

Results

Data were collected and analyzed across a range of strands and Substrands of the mathematics curriculum. Just one aspect of these data are presented to illustrated the style of tasks, the data collected, and form of analysis. Data are presented here on one substrand of the Number in the CSF, *Number Counting and Numeration*, for Level 4, which is applicable to upper primary students (ages 11/12). The specific descriptor for this substrand at the level was:

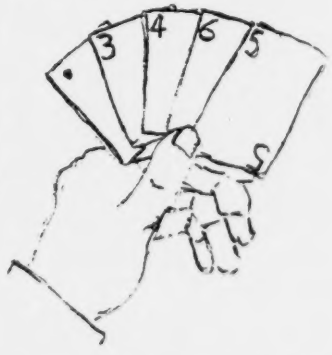
> Counting, ordering, estimating and describing with large numbers and common and decimal fractions. Using place value (thousandths to millions and beyond)

The following is one of the tasks from this substrand presented to show the format in which the tasks were posed.

Five cards

You are playing a game using cards with numbers and decimal points. When it is your turn, you can choose to use all five of your cards, or just 4, or just 3, or just 2 or just 1.

You have these 5 cards:



1. Using these cards, what is the smallest possible number you can make?
2. Using these cards, what is the largest possible number you can make?
3. What numbers can you make that are between 3.5 and 5? (Give as many answers as you can. Remember that you can use as few or as many cards as you wish.)
4. Circle the number which is closest to 5.4?

   5.3 5.46 5.6

   5.364 5 5.463

5. One of your friends asks you to explain the best way to decide which number is closest to 5.4. How would you explain how to work out the number closest to 5.4?

Note that the task includes some closed items including some which require analytical thinking, an open-ended item, and an item which requires communication of thinking processes.

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Three other tasks from this substrand are also part of the data presented. *Fraction Boxes* sought information on different forms of fraction representations, *Favourite Foods* was about the application of fractions, decimals and percentage to a menu context, and *Ordering* compared numbers written in different forms, such as common fractions, decimals and percentages.

The tasks were completed by 4 classes of grade 5/6 students. All tasks were scored on a scale from 1 to 4. A score of 1 indicates the student showed no evidence of appreciation of the task requirements, while a score of 4 suggests a high level response with a clear understanding of the task requirements. Fractional scores are possible.

Table 1 is a description of the scores given to the tasks by the teachers of the classes.

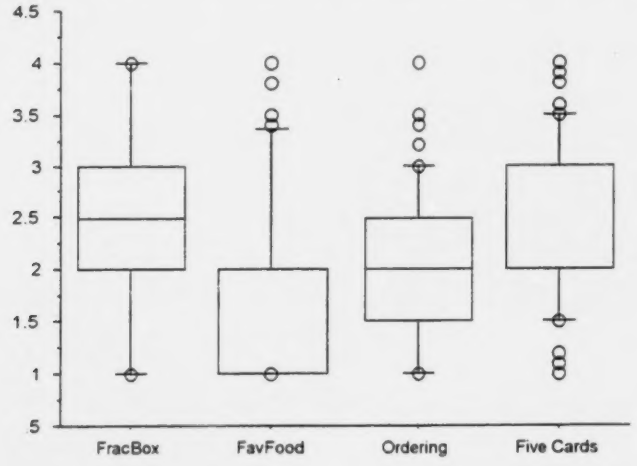
Table 1: Summary of task scores

	Fraction Boxes	Favourite Foods	Ordering	Five Cards
No. of Responses	132	133	127	137
Mean	2.4	1.9	2.1	2.4
Std. Deviation	1.0	0.9	0.8	0.8

Our first goal of was to determine whether the tasks could be completed by students at the level at which the content standard is specified. If the mean scoring for a task was well above 3 to well below 2, it would suggest that the task was either too easy or too difficult. It seems that although *Favourite Foods* is slightly more difficult for the students, the tasks seem to be within an acceptable range.

While the above summary gives some indication of the difficulty and spread, it is not so clear how the tasks can be compared with each other. Table 2 uses an alternate format, in which box plots compare the task scores (in this case with the teacher and researcher scores combined). Each box shows the middle 50% of the scores, and the whiskers show the spread from the 10th to 90th percentile. Basically it is a graphical representation of the spread of scores.

Table 2: Comparison of task results (teacher and researcher combined).



These box plots provide a clear indication of the comparative difficulty of the tasks and the spread of responses. The median of *Favourite Foods* is 2 and *Five Cards* is 3. The responses of the students to *Five Cards* and *Fraction Boxes* were scored higher than those of the other tasks, with a greater spread of the scores for *Fraction Boxes*. It seems that, overall, these tasks were able to be completed by students at this year level, with at least some students giving comprehensive responses. The tasks are able to be compared with each other.

The project also sought to explore whether teachers could make consistent judgments on the performance of students on the tasks. The researchers reported on the ease with which the teachers scored the tasks, and interpreted the rubric and additional information. As

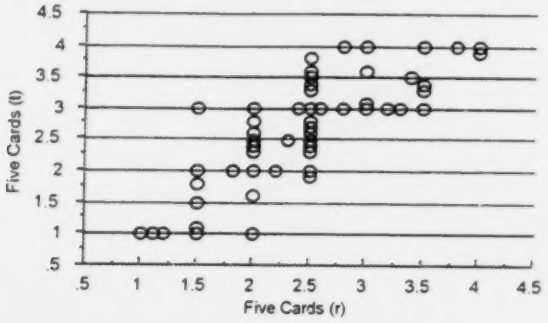
another measure of the validity of the judgments of the teachers, for a selected substrand a researcher scored each task independently of the teacher scoring. Table 3 presents the correlation between the teacher and researcher scores for each of the tasks discussed.

Table3: Correlation of teacher and researcher scores

Task	Correlation Value	Number of Scores
Fractions in Boxes	.93	132
Favourite Foods	.87	133
Ordering	.90	127
Five Cards	.80	137

All of these are highly significant but it is difficult to interpret the educational significance of such figures. To illustrate the strength of the relationships, Table 4 is a scattergram containing teacher and researcher scores on the *Five Cards* task, the lowest correlation figure.

Table 4: Scattergram of teacher and researcher scores for Five Cards.



This is clearly indicative of a strong relationship between the two scorings. There are quite few instances of marked divergence between the scores. On balance, the degree of association suggests that teachers and researchers applied the rubric to the individual student responses in similar ways.

A further aspect of the data collection was to explore whether students give consistent responses across the tasks. It is difficult to find a way to analyze, present or interpret the range of responses given by individuals to the related tasks. Basically we are interested in the extent to which the scores given to individual students over the four tasks are spread. To present this information simply, the number of students whose scores ranged from 0 to 1 were counted, and likewise for 1 to 2, and 2 to 3. Table 5 shows the number of students with scores within these ranges.

Table 5: *Range of scores of individuals over the four tasks*

Range of individual's scores	Frequency
$0 < x \leq 1$	57
$1 < x \leq 2$	54
$2 < x \leq 3$	34

The first impression is that these scores are spread quite widely. One possible inference is that the tasks do not provide reliable measures of achievement.

Another interpretation is possible. Even though the tasks the *Number, Counting and Numeration* substrand, they cover a broad range of outcomes drawn from ordering of decimals, ordering of fractions, comparison of different representation of fractions, and interpretation of the meaning of fractions. It may be that this spread in scores illustrates both the need to use a range of tasks to assess achievement even within a narrow component of the curriculum and also the need to exercise care in forming judgments from responses to a single question.

Summary

This has been some concern expressed about the educational value of mandatory testing. It seems useful to explore whether systems can be devised which have positive instructional implications and which also provide a measure of the overall standards of achievement. Data are presented here from a project which sought to compile assessment tasks and to develop a process by which teachers assess students' achievement on mathematical tasks by making holistic judgments on performance which are practical, meaningful, and valid. It is suggested that teachers are able to apply the scoring rubric to the tasks, that the scoring allows the tasks and student performance to be evaluated, and that independent judgments can be validly made. It appears that there is variation in the responses of individual students. This makes the need for performance assessments more critical. More research need to be undertaken into the type of tasks which provide the best source material for teachers' assessment of the mathematical performance of their students, and how best to enhance the quality and consistency of teacher judgments in using performance assessments.

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Links Between Early Arithmetical Knowledge And Early Space And Measurement Knowledge: An Exploratory Study.

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This paper focuses on better understanding children's early space and measurement knowledge. Five Year 1 students from each of two schools and five Kindergarteners from one of those schools, were given an initial and final assessment in arithmetic, space and measurement. Students' advancements in arithmetic and space were not accompanied by advancements in measurement. Arithmetically more able students were found to be more able on space and measurement tasks and there was great variation in students' space and measurement knowledge.

Whilst research has provided useful models of student's early arithmetical knowledge and its development (see Wright, 1994; 1996) little research related to early geometric knowledge seems available. Historically the most influential research in this area has been the work of Pierre and Dina van Hiele (van Hiele, 1959; van Hiele, 1986) and Piaget and Inhelder (Gruber & Voneche, 1977). The van Hieles postulated five levels of cognitive development in geometric thinking: recognition, analysis, ordering, deduction and rigour. Piaget believed that children's organisation of geometric ideas follows a definite and logical rather than historical order. Initially the child constructs topological relations, such as closure, connectedness and continuity. Projective and Euclidean relations are developed later (Gruber & Voneche, 1977). Both the theories of Piaget and the van Hieles emphasise the role of students in actively constructing their own knowledge and stress the importance of challenging the learner's thinking. Both theories have been influential in the teaching of geometry.

During the last decade school mathematics has changed from a focus on procedures to an emphasis on patterns and relationships (Wheatley, 1990; National Statement, 1990), the construction of which is largely dependent on spatial sense (NCTM, 1989). Wheatley (1990) likens spatial sense to imagery and maintains that "spatial sense is indispensable in giving meaning to our mathematical experience" (p. 15). He deplores the mathematics program typical of most US elementary schools where the study of geometry is reduced to the identification and naming of simple geometric shapes and emphasises the importance of taking account of students' spatial sense, both in teaching and assessment (p.15).

In more recent times there has been an increased interest in research on students' spatial thinking, although most studies appear to have focussed on older groups of children or on very specific areas of children's spatial thinking. For instance, Wheatley and Cobb (1990) presented children with shape-covering tasks to determine the level of their understanding of area; Mitchelmore (eg 1994) focussed on the development of specific spatial concepts, such as angles and parallels; Pegg & Davey (1989) and Pegg(1992) seem largely concerned with descriptors of students' spatial understanding; and Leeson (1995) conducted two studies involving kindergarteners' spatial construction of three-dimensional shape. Leeson found that young children were capable of significant geometrical thinking and suggested that many children are under-challenged in geometry in their first year of school. Thorpe (1995) who views spatial relationships as having "wider parameters than the traditional subject of geometry" (p. 517) investigated pre-school students' learning and use of spatial concepts from an interactive perspective whilst she was an observer in the students' normal setting.

Owens who has worked extensively in the area of assessment of students' spatial concepts (eg 1992a; 1992b; 1994) studied a broad range of children's spatial understanding. She devised a group 'pencil and paper' test which assessed children's ability to recognise, reproduce and analyse shapes (1992). Owens maintained that "despite the emphasis on qualitative approaches to the research the relationship between spatial abilities and spatial concept development is not yet clear" (1992, p.2). Owens,

Mitchelmore, Outhred and Pegg (1996) report a study by Brodie (1994) in which 138 teachers were surveyed in regard to the Space strand in Mathematics K-6. It was found that teachers' attitudes towards the strand were less positive than to the other strands; less time was spent on the strand; theoretical knowledge of students' spatial learning was poor; and their understanding of the scope of geometry was rooted in their own school learning.

There appears to be little research available on students' early measurement knowledge. As well, one could argue that much early measurement knowledge is not distinct from and might be subsumed under early arithmetical knowledge. The theory of the development of early arithmetical knowledge by Steffe and Cobb (1988) for example, includes notions such as units and units of units, which seem also relevant to the development of early measurement knowledge. Given that measurement is an important topic of early mathematics curricula, its inclusion in this study is warranted.

The present study was carried out in response to a perceived need to gain a deeper understanding of the spatial and measurement knowledge of students in their first or second years of formal schooling. An important aim is to extend to the assessment of students' early spatial and measurement knowledge, an interview-based approach to the assessment of early arithmetical knowledge. Additional aims of the project include the documentation of students' advancements in spatial and measurement knowledge over the course of a school year and a comparison of students' arithmetical, spatial and measurement knowledge. One question addressed by this study is: What spatial and measurement knowledge is possessed by students in Kindergarten or Year 1 in NSW? Research techniques developed in the Maths Recovery project (see Wright, in press) were used in this project. These include (a) student assessment based on individual interview in which the student is presented with tasks which are problematic for them; (b) routine videotaping of assessment interviews for subsequent analysis; and (c) an explicit focus on students' strategies and meaning as demonstrated in the course of the assessment interview.

Assessment Schedules

During the first half of 1996, assessment schedules in early space and measurement were developed. These schedules were informed by the national and NSW curriculum documents (A National Statement on Mathematics for Australian Schools, 1990; Mathematics K-6,1989), the work of one of the authors (Leeson) and current research into space and measurement. The two assessment schedules were trialled in April with ten students, five from each of two schools, and modifications were made. The final versions of these schedules each required approximately 45 minutes for completion. The space schedule comprised 20 activities and measurement 19.

The space schedule covered three broad areas of students' spatial knowledge: two dimensional (2D) space; three dimensional (3D) space and position. Within the area of 2D space, students' abilities to name, recognise, draw, construct, classify and identify were assessed. Students' understanding of symmetry and tiling with seven-piece tangrams also formed part of this assessment schedule. In the area of 3D space, students were required to sort, classify and name objects. They were also required to construct 3D shapes from a 2D representation, match pictorial representations of the nets of shapes with solids, determine the number of cubes in a structure from its pictorial representation, and provide explanations of their solutions. Simple grids were used to assess students' understanding of position. Students were required both to place an object in a specified position on the grid and to describe the position of objects placed onto the grid by the interviewer.

The measurement schedule involved the use of materials (such as straws, popsticks, cubes, boxes, various containers, balloons, and balls) and informal units. Children were required to estimate, measure, compare and order length, perimeter, area, capacity, volume, mass and time.

Method

Fifteen students from local schools, who were not part of the trial phase, participated in the study. There were five Kindergarten and five Year 1 students from one school (School A), and five Year 1 students from a second school (School B). Each student was given an initial assessment early in the third school term and a final assessment toward the end of the fourth (final) term. The assessments consisted of three interviews - arithmetic, space and measurement. The arithmetic interview was one which was developed and used in earlier research (eg Wright, 1994; Wright, Stanger, Cowper & Dyson, 1996). Analysis of the arithmetic interview includes determination of student's stage of early arithmetical learning, viz: Stage 0 - Emergent; Stage 1- Perceptual; Stage 2 - Figurative; Stage 3 - Initial Number Sequence; Stage 4 - Intermediate Number Sequence; or Stage 5 - Facile Number Sequence (see Wright, 1994, pp 26-28).

On each of the spatial and measurement schedules it was decided that comparisons could more readily be made if some numerical value could be assigned to each student's performance. Analysis of the space and measurement assessments involved writing protocols, ie descriptions of each student's solutions as well as student's and interviewer's statements. Subsequent analysis involved assigning a numerical value to the student's solution of each task. The value was determined according to task-specific criteria. For instance, Activity 3 in the space assessment was made up of three subtasks and involved students selecting, from a total of 19 shapes those which were 'like' one held up by the interviewer and then explaining the likeness. There were two circles, four squares, two hexagons, three rectangles and eight triangles to choose from. The interviewer first held up a square, then a rectangle, then a triangle. The maximum score available for this activity was six. One point was allocated for selecting all the correct shapes in each sub-task and one point was awarded for each appropriate explanation. For example, Jack was allocated five out of six for Activity 3. He correctly selected all the squares and rectangles and gave satisfactory explanations for his selections, but only selected three of the eight triangles. He explained that the three chosen triangles were like the shape held up by the interviewer because they "all have three sides and three corners". Jack was awarded four points for selecting all the squares and rectangles and offering appropriate explanations. He was given one point for his explanation and zero for his selection because he had not selected all the triangles, although his explanation should have encompassed all of them. When asked why the remaining five had not been chosen Jack replied that they "had really long sides and two short ones". A similar scoring system was employed in assigning numerical values to each students' solution of each of the measurement tasks.

In this manner each student was allocated a score for their performance on each of the space and measurement tasks. These scores were summed and this led to an aggregate score for each student's performance on the initial and final space and measurement assessments. The maximum possible score was 70 on the space assessment and 48 on the measurement assessment.

Results

Table 1 shows the total scores of the 14 students who were given the initial and final space assessments. The table also includes the grade level - Year 1(1) or Kindergarten (K)- and school of each student, together with each student's total score, expressed as a percentage of the maximum possible score of 70. Table 2 lists the total scores of the 13 students who completed the initial and final measurement assessments. This table also includes each student's school, grade level and score as a percentage of the maximum possible score of 48. Table 3 shows a comparison of students' early arithmetical, early space and early measurement knowledge The table lists each student's grade level, Stage (Wright, 1994) on initial and final interviews, and each student's total score in space and measurement for the initial and final interviews. In Table 3 students have been ordered according to their Stage in the initial interview.

Table 1
Initial and final scores on space assessments.

Name	School	Grade	Initial Assessment (max. 70)		Final Assessment (max. 70)	
Beth	A	1	66	(94.3)	68	(97.1)
Jack	A	1	62	(88.6)	68	(97.1)
Jessica	A	1	60	(85.7)	60	(85.7)
Jessamy	A	1	59	(84.3)	65	(92.8)
Keenan	B	1	55	(78.6)	59	(84.3)
Jason	B	1	54	(77.1)	59	(84.3)
Andrew	A	1	54	(77.1)	66	(94.3)
Isaac	A	K	43	(61.4)	43	(61.4)
Kellie-Anne	B	1	42	(60.0)	53	(75.7)
Hollie	A	K	39	(55.7)	50	(71.4)
Carmin	A	K	38	(54.3)	*	*
Chloe	A	K	37	(52.9)	39	(55.7)
Michelle	B	1	37	(52.9)	46	(65.7)
Mark	B	1	35	(50.0)	45	(64.3)
Dwyatt	A	K	35	(50.0)	46	(65.7)

Note: *Carmin left the school prior to the final interviews.
Scores are from a possible total of 70.

Table 2
Initial and final scores on measurement assessments.

Name	School	Grade	Initial Assessment (max. 48)		Final Assessment (max. 48)	
Jack	A	1	40	(83.3)	38	(79.2)
Andrew	A	1	39	(81.3)	39	(81.3)
Jason	B	1	37	(77.1)	37	(77.1)
Jessamy	A	1	36	(75.0)	39	(81.3)
Jessica	A	1	36	(75.0)	31	(64.6)
Hollie	A	K	33	(68.8)	30	(62.5)
Kellie-Anne	B	1	33	(68.8)	32	(66.7)
Carmin	A	K	32	(66.7)	n/a	n/a
Keenan	B	1	31	(64.6)	30	(62.5)
Mark	B	1	30	(62.5)	30	(62.5)
Isaac	A	K	29	(60.4)	30	(62.5)
Michelle	B	1	28	(58.3)	36	(75.0)
Dwyatt	A	K	24	(50.0)	28	(58.3)
Chloe	A	K	21	(43.7)	16	(33.3)
Beth	A	1	#	#	37	(77.1)

Note: # Beth was unavailable for the initial measurement assessment.
Scores are from a possible total of 48.

Table 3
Initial (A1) and final (A2) Stage and scores on space and measurement assessments.

Name	Grade	Number		Space		Measurement	
		A1	A2	A1	A2	A1	A2
Jessamy	1	4	5	59	65	36	39
Beth	1	3	5	66	68	n/a	37
Jack	1	3	4	62	68	40	38
Andrew	1	3	3	54	66	39	39
Jason	1	3	4	54	59	37	37
Jessica	1	3	3	60	60	36	31
Kelli-Anne	1	3	3	42	53	33	32
Michelle	1	2/3	3	37	46	28	36
Keenan	1	2	3	55	59	31	30
Mark	1	2	2	35	45	30	30
Carmin	K	1/2	*	38	*	32	n/a
Dwyatt	K	1/2	2	35	46	24	28
Isaac	K	1	1	43	43	29	30
Holllie	K	1	1	39	50	33	30
Chloe	K	0	1	37	39	21	16

Note: * Carmin left the school prior to the final interviews.
Students' scores have been ordered by their Stage as determined from their initial assessments.

Summary points from the results.

1. Great variation is evident in students' spatial knowledge, not only across school years but within school years as well. Thus, Dwyatt, a Kindergarten student gained a much lower score (35) on his initial space interview than Beth, a Year 1 student (66). Similarly, Mark (35), who was also in Year 1, scored low in comparison to Beth. This trend was also evident in students' scores on the measurement assessment. For instance, Jack, a Year 1 student, scored considerably higher than Chloe, a Kindergarten student, and Michelle who was in Year 1.

2. Most students (twelve of 14) showed advancements in spatial knowledge over the four month period. The two students who did not advance scored the same total in the initial and final space assessments.

3. Six of the 14 students showed improvements of more than 10% of the possible maximum in their final space assessment scores.

4. Only 4 of 13 students showed improvements in their overall score on measurement, and one of these improved by one point only. Three students attained the same score and six attained lower scores, two showing decreases of five points. Moreover, the increases shown by students were small and only one student improved by more than 10% of the possible maximum on her final interview.

5. Students' overall scores, when expressed as a percentage of the maximum possible score, were higher on both initial and final space assessments than their initial and final measurement scores. For instance scores on the initial space assessment ranged from 94.3% to 50%, whereas scores on the initial measurement assessment ranged from 83.3% to 43.7%.

6. Most students who were more able arithmetically showed advancements between all initial and final interviews

7. Generally speaking the students who were more able arithmetically were also more able on the space and measurement assessments. For example, Jessamy, who was assessed at Stage 4 from her initial number interview, showed a similar high ability with spatial and measurement tasks in her initial and final assessments.

8. The student (Chloe) who was assessed at Stage 0 scored low in her initial and final space assessments and attained the lowest scores of 13 students in her initial and final measurement assessments. Her gains in spatial knowledge were very low and she scored considerably lower in her final measurement assessment than in her initial measurement assessment.

Discussion

The lack of improvement in students' scores on measurement may have been due in part to the inclusion of estimation and conservation tasks in the measurement assessment. Five of the 19 tasks involved student estimations of length, perimeter or area, and three of the tasks involved students' abilities to conserve. Points were awarded for giving exact estimations, close estimations, and in some instances for just giving an estimate. Students were given one point if they were able to conserve or zero if they were not able to conserve. Some student's estimates varied from initial to final assessments, as did their ability to conserve. For instance, five students scored higher on Task 2 in the initial assessment than they did in the final assessment because their initial estimates were more accurate. Similarly, only two of 13 students conserved consistently on Task 13 in the initial and final interviews. It is conceivable that students' ability to conserve may not be firmly established at this age. Similarly students' estimation skills, particularly in relation to perimeter and area, may still be in a formative phase for these age groups.

The apparent link between high-scoring students' numerical ability and their spatial and measurement ability warrants further investigation. Similarly, the case of Chloe raises some interesting questions. Is there a tendency for children who are low-achievers in early arithmetic to show correspondingly low levels in space and measurement? Thus there may be a case for extending maths recovery kinds of programs to encompass all areas of mathematics and not just arithmetic.

Though not dealt with in this paper, of interest to the authors are the kinds of strategies students used in task solution. Several different student strategies were noted and it appears that a larger study may result in the identification of levels in children's thinking in the areas of space and measurement.

Following this exploratory study work needs to be done in refining the assessment schedules for early space and measurement. For instance, the value of a few tasks which were easily accomplished by all students is questionable. Tasks need to be designed so that they are both challenging and discriminatory.

Other factors warranting consideration are possible gender differences and the effects of different schools on the participants.

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Cooperative learning in secondary mathematics: A quantitative review

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This paper is a quantitative review of research on cooperative learning in secondary mathematics. Results from the analysis demonstrate that cooperative learning has an overall positive effect in the cognitive domain as well as the social and affective domain. The paper also identifies several potential moderators of the effects and makes implications for further research reviews in the field.

Introduction

During the past few decades there has been a growing interest among researchers in cooperative learning and it is being accepted as an important teaching methodology (Cohen, 1994; Lou et al., 1995). International associations, such as The International Association for the Study of Cooperation in Education (IASCE) have been formed to provide a forum for researchers and educationists to share ideas about cooperation in education (Slavin, 1985).

The use of cooperative learning in mathematics classrooms is a related area which has also attracted considerable interest (Davidson & Kroll, 1991). Curriculum developers and educational agencies such as Board of Studies and National Council of Teachers of Mathematics (NCTM) explicitly recommend the use of cooperative learning strategies in secondary mathematics (Board of Studies, 1995; NCTM, 1991).

Despite an overwhelming acceptance of cooperative learning among researchers and educational organisations, the strategy is not used frequently in schools (Bossert, 1988; Stebler & Reusser, 1996). In addition, a closer look at the literature shows that cooperative learning is more often used in subject areas like English than in mathematics. To some extent this discrepancy can be attributed to teachers' beliefs about cooperative learning; often teachers are unsure of the cognitive and metacognitive productivity of cooperative learning even if they believe in its productivity in affective and social domains. The awareness that cooperation has positive effects in affective and social domains is not enough to motivate teachers to use this technique in their class rooms. They are more willing to use the strategy if they feel confident that it will promote student achievement and be effective in cognitive domain (Stebler & Reusser, 1996).

The research literature on cooperative learning is not easy to interpret and analyse, as studies conducted in this field have a diversity of aims, subject matter, topics, grade levels and study designs. Several attempts have been made to synthesise this overwhelming volume of literature (e.g. Bossert, 1988; Johnson, Maruyama, Johnson, Nelson & Skon, 1981; Slavin, 1990). However, often these research reviews have been very broad in scope and hence may not have provided teachers with the specific information they need for restructuring their classrooms. A need therefore arises to synthesise that subset of literature which may interest secondary mathematics teachers.

This paper reports the results of a pilot study for a comprehensive review of research on cooperative learning in secondary mathematics. It is a synthesis of the quantitative data which could be converted to an effect size (ES). The paper also examines the pros and cons of several methodological approaches.

Method

Operational definition

In this study, cooperative learning is used in a broad sense. It refers to learning that takes place when two or more students work in small groups and it involves some form of resource sharing, reward sharing or both among the group members. The definition includes what is referred to as peer tutoring, collaborative learning and small group work. It encompasses both heterogenous grouping as well as homogenous grouping.

Collecting the empirical studies

In accordance with Slavin's (1986) best-evidence synthesis method, this study used well-justified *a priori* inclusion criteria for empirical studies. It included all the studies which: (a) examined the effects of cooperative learning in mathematics on secondary (grade levels 7-12) students; and (b) were conducted on normal progress students even if from different socio-economic backgrounds or with varying academic potential such as low achievers or high achievers.

The synthesis excluded the empirical studies which: (a) were conducted on special need students such as academically handicapped students or gifted students; (b) were conducted in special educational programs, such as remedial classes, community college, adult education program or university preparatory courses; (c) examined a form of cooperative learning which did not agree with the operational definition; (d) were not reported in English; (e) focussed on other factors and in which cooperative learning was very incidental; (f) focussed only on teachers' attitudes to cooperative learning and reported no measure of students' beliefs or achievement variations; (g) were reported before 1981.

Using this selection criteria, electronic searches were conducted on ERIC (Education Resources Information Center), AUSTROM (Australian Social Science, Law and Education Database on CD-ROM), PsycLIT and APA (Psychology and related subjects by American Psychological Association), Maths Sci (Mathematics and Mathematically related research by American Mathematical Society), SSI (Social Sciences Index), SSCI (Social Sciences Citation Index). Owing to financial constraints, no attempts were made to find the studies from the DAI (Dissertation Abstract International). Consequently, 29 studies were selected for the analysis.

Data collection and analysis

Each selected study was closely examined to fill in the relevant information in a coding form and to compute the Effect Size (ES). Slavin's (1986) formulae were used to compute the ES with a slight modification; in each denominator, the control group standard deviation was replaced by the pooled standard deviation. The extent to which the effects were pooled within a study or across studies was guided by Slavin's (1986) best-evidence synthesis method. Within each sub category, the mean of the appropriately weighted effect sizes (as recommended by Hedges & Olkins, 1985) rather than the median ES (as recommended by Slavin, 1986) was taken as the pooled ES.

Study outcomes within each category were analysed for homogeneity to determine if a single ES was a good representation of the studies (Hedges & Olkins, 1985; Johnson, 1989). Significant within group inconsistencies were observed in each sub category. Hence, the pooled ES was not taken as the conclusive result. An outlier diagnosis was performed to account for the within group heterogeneity. If the largest outlier had major methodological difference from the other studies, it was eliminated from further analysis.

Outlier diagnosis was followed by categorical model testing. This involved dividing the findings into sub categories on the basis of a study quality and testing these sub categories for within group and between group homogeneity to quantify the extent to which the study quality moderated the effect. In general, DSTAT, a software for meta-analysis, was used for computing and averaging the effects and for carrying out the outlier diagnosis and the categorical model testing (Johnson, 1989).

Results and Discussion

Overall effect of cooperative learning

This pilot study quantitatively synthesised the primary research literature on the effects of cooperative learning in secondary mathematics. *Appendix A* lists the findings extracted from different studies. The mean sample size weighted ES ($\underline{d}+$) in the cognitive domain was +0.63, a value close to the values reported by Johnson et al. (1981), Slavin (1990) and Lou et al. (1995). Cooperative learning was also found to promote the retention capacity of students ($\underline{d}+ = +0.50$). In general, most of the previous broadly based reviews agreed that cooperative learning enhances the student achievement. This

study adds another specific dimension to the sweeping generalisation. It demonstrates that cooperative learning has overall positive effects on achievement in secondary mathematics.

In the social and affective domain, the effect was measured on variables such as learning goal, intrinsic motivation, performance goal, extrinsic motivation, efficacy, social satisfaction, mathematics value and mathematics cost. The mean sample size weighted ES ($\underline{d}+ = +0.25$) was significantly positive. This concurs with the findings of Johnson & Johnson (1985), Slavin (1990), Davidson & Kroll (1991) and Lou et al. (1996).

Further, cooperative learning method is also preferred by students. An analysis of findings from the Learning Preference Style for Students (LPSS) administered to Sydney students revealed that in general, students prefer cooperative rather than competitive or individualistic goal structure ($\underline{d}+ = +0.29$). This result confirms similar assertions made by Barnes & Owens (1992).

Potential moderators of the effect

The study also investigated potential moderator variables of the ES. The presence or absence of equivalence between experimental and control conditions, duration of the study, form of cooperative learning method used, source of the outcome, and grade level of the students were the qualities which significantly moderated the ES.

Equivalence of experimental - control conditions: The mean weighted effect on achievement for the findings from studies in which both the experimental and control groups were subjected to equivalent conditions ($\underline{d}+ = 0.76$) was significantly higher than those for which the conditions were non equivalent ($\underline{d}+ = 0.55$), $\underline{Q}_B(1) = 4.04$, $\underline{p} < .05$. This is in contrast with the finding of Lou et al. (1995). They found that studies which subjected the experimental and control group students to similar treatments yielded lower average ES than those which differed in their treatments. Hence they inferred that to a large extent the positive effect on achievement could be attributed to the non equivalence of the experimental and the control groups, a claim which is challenged by these results.

To account for this difference in results, the aims, methods and scope of Lou et al. (1995) were examined. A closer inspection revealed that they had studied the effects of "within-class grouping" rather than cooperative learning. They also appear to have included some studies where the small learning groups were not cooperative. In addition, their review was not restricted to secondary mathematics only. These marked differences in the aim and scope of the previous review could have been responsible for the difference in findings.

Duration of the study: In the cognitive domain, the mean weighted ES for the findings from studies which lasted up to 20 weeks ($\underline{d}+ = 0.75$) was significantly higher than those which lasted from 21 to 40 weeks ($\underline{d}+ = 0.55$), $\underline{Q}_B(1) = 3.87$, $\underline{p} < .05$. This is in agreement with Johnson et al. (1981) who concluded that studies of shorter duration produced relatively higher effects on achievement. Perhaps studies of shorter duration show higher ES due to the novelty of the cooperative learning technique. This Hawthorn effect wears out with time in the studies of longer duration. In the social and affective domain, the effect of study duration could not be examined on account of the limited number of findings.

Source of outcome: The source of outcome significantly moderated the effect on achievement, $\underline{Q}_B(2) = 15.50$, $\underline{p} < .05$. The mean sample size weighted ES for the findings based on researcher made tests ($\underline{d}+ = 0.77$) was significantly higher than those based on teacher made ($\underline{d}+ = 0.53$) or standardised ($\underline{d}+ = 0.12$) tests. One possible explanation could be that larger researcher bias was introduced in studies which used researcher made testing instruments. Effect of source of outcome in the affective domain could not be studied as all the findings were based on researcher made tests.

Form of cooperative learning: The form of cooperative learning used for the study also significantly moderated the effect on achievement, $\underline{Q}_B(4) = 19.29$, $\underline{p} < .05$. The mean sample size weighted ES for the findings using Slavin's Student Team Learning (STL) methods ($\underline{d}+ = +0.79$) was significantly higher than those using AGO ($\underline{d}+ =$ 0.+68), MTC ($\underline{d}+ = +0.49$), the structural approach ($\underline{d}+ = +0.24$) or 'Groups of four' method ($\underline{d}+ = -0.06$). This concurs with the assertion of Newmann and Thompson

(1987) and Slavin (1990) that Student Team Learning methods are more effective than other cooperative learning methods.

Slavin (1990) suggested that his Student Team Learning methods were more effective because they use individual accountability and team rewards. In order to test Slavin's hypothesis, the findings were divided in two categories: those which used team rewards and individual accountability and those which did not. No significant difference was found in the mean sample size weighted ES of the two categories, $Q_B(1) = 1.40$, $p > .05$. This supports the case made in chapter 2 that the higher effects of the STL methods could be attributed to several factors other than team rewards and individual accountability. For example, learning while playing could make learning more interesting. Again, it is hard to say that all the studies in the latter category did not use any trace of team rewards: the teachers might have verbally recognised the group's collective efforts (Bossert, 1988). Also, Slavin's STL methods such as STAD, TAI are older and more commonly used than some of the new methods such as MTC and the scripted cooperative learning. It is possible that both researchers and teachers were more comfortable with using the STL methods.

Grade level: The mean sample size weighted ES for grade 11 students ($\underline{d}+ = +0.55$) was significantly higher than that of grade 7 students ($\underline{d}+ = +0.11$) on LPSS. This cumulative effect was computed by pooling the 8 findings extracted from a study conducted by Barnes & Owens (1982). The original study specified the direction of the effect but did not report its magnitude. In the present thesis, the primary data was reanalysed to answer the research hypothesis. This in turn added another dimension to the results of the original study. It not only confirmed the direction of the effect but also computed its magnitude.

Other variables: Two other variables, grouping criteria and the number of students in each cooperative group, tended to moderate the effect although not at 95% level of confidence. Cooperative learning tended to be more effective when each cooperative group was composed of heterogenous ability groups rather friendship groups. Likewise, the effect was higher when four students worked in each group as against dyadic groups. Several other variables which were examined included the year of reporting, publication status, student ability, ethnic breakdown, socio economic background, location, and the topic of instruction. Many of these qualities demonstrated variation in within group effect of different sub categories although the difference was not at 95% significance level. Perhaps more potential moderators could be diagnosed by increasing the number of studies included in the analysis.

Limitation of the Study

The small number of empirical findings included in the analysis appeared to be a limitation in the diagnosis of many potential moderators of the effects. In a meta-analysis, the chances of detecting significant heterogeneity between the sub categories tends to increase with an increase in the size of each sub category (Slavin, 1986). In this analysis, several sub categories showed between group heterogeneity even if not at 95% significance level. It appears that increasing the number of empirical studies could result in the finding of some more mediating variables of the effect.

Implications for further Research

The overall effect of cooperative learning was significantly positive in the cognitive domain as well as the social and affective domain, even though the individual empirical findings were significantly inconsistent. The overall productivity of cooperative learning suggests that the area is worthy of future research. The inconsistencies between individual empirical findings caution against the generalisation that cooperative learning is effective in all learning situations in secondary mathematics. Future research should focus on identifying the specific factors which hinder or enhance the effects.

The study has worthwhile implications for methodological decision making in related future research. It confirms several strengths and limitations of the meta-analytic procedures. First, the meta-analytic procedures bring all study findings to a common metric. This practice gives appropriate weight to each finding by largely eliminating the

effect of subjective interpretation of the original result in an empirical study, thereby reducing the chances of Type I error. For example, Owens & Barnes (1982), in their paper, concluded that female students demonstrated significantly higher preference for cooperative learning style when compared to their male counterparts. In this thesis, a reanalysis of their primary data, confirmed the tendency towards the effect but the effect was found to be insignificant at 95% confidence level, thus challenging the claims made by authors of the original study.

At times, a reviewer finds empirical studies in a related area which contain the data with a potential of addressing the reviewer's research hypothesis, even though they do not directly do so. Meta-analytic procedures enable a reanalysis of the original data to address the reviewer's research hypothesis. For instance, Owens & Barnes (1982) concluded that grade 11 students when compared to grade 7 students demonstrate higher preference for cooperative as well as competitive learning modes. Using meta analytic procedures, the data was reanalysed to address an aim of this study which was to quantify the difference in relative preferences of grade 11 and grade 7 students for the two learning styles.

The meta-analytic procedures described in this study appeared to comprehensively cover most cases of computing and averaging the effects in a variety of empirical findings, with varied study designs and reported statistical data. When the literature consists of a large number of studies, the meta-analytic procedures would appear, on the basis of the analysis carried out for this study, to be sufficiently robust to objectively identify the overall trends. The meta-analytic procedures also provide a method of identifying variables that mediate the overall trends.

In general, the software DSTAT appeared to be sufficiently powerful for computing the effect sizes from a wide range of statistical data. It was also found to be efficient during the analysis stage which involved computation of the standard deviations, the pooled effect sizes and identification of the potential moderators of the effect. It appeared to have no operational problems.

Despite the obvious advantages of the meta-analytic procedures in analysing quantitative data, reviewers on cooperative learning are recommended not to rely solely on these procedures. A significant number of studies in cooperative learning are qualitative studies, case studies on two or three students, or studies on interaction analysis of students working in small groups which fall beyond the scope of meta-analytic procedures. In future reviews, following the principles of best-evidence synthesis, a meta-analysis of quantitative data should be supplemented with a rich literature review of the data not covered in the scope of a meta-analysis.

Conclusions

Cooperative learning has an overall positive effect in the cognitive domain as well as the social and affective domain in secondary mathematics. The presence or absence of equivalence between experimental and control conditions, duration of the study, form of cooperative learning method used, source of the outcome, and grade level of the students were the qualities which significantly moderated the ES. The small number of empirical findings included in the analysis appeared to be a limitation of this thesis.

Results of this study indicate that there is still scope for further research into cooperative learning in secondary mathematics classrooms. On the basis of the analysis carried out in this thesis, meta-analytic procedures are found to have several strengths. They bring all the findings on a common metric which reduces the effect of subjective interpretations of the primary data. Meta-analytic procedures make provision for a reanalysis of the primary data in an empirical study to change the focus of the original study. This, at times, enables a reviewer to include even those studies from related areas that do not directly address the reviewer's research hypothesis.

The meta-analytic procedures described in this study appeared to comprehensively cover most cases of computing and averaging the effects in a variety of empirical findings, with varied study designs and reported statistical data. These procedures appear sufficiently robust to objectively identify the overall trends and the moderating variables, even for a large number of studies. The DSTAT software used in this study appeared to meet most of the computing requirements of a quantitative review. It is recommended that

in future reviews, following the principles of best-evidence synthesis, a meta-analysis of quantitative data should be supplemented with a rich literature review of the data outside the scope of a meta-analysis.

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Appendix A: Samples used for extracting the effect sizes

Finding	G	D	Design	O	d+
Reid (1992)	7	40	Coop Vs Trad	C	+0.49
Duren & Cherrington (1992)	7-8	4	Coop Vs Trad to promote r[illegible]ation Post-test 3 months after instruction	C	+0.50
Johnson (1984A)	8	35	Coop (Experimental 1 gp) Vs Con (M)	C	-0.11
Johnson (1984B)	8	35	Coop (Experimental 1 gp) Vs Con (F)	C	-0.02
Terwel et al. (1994)	7-11	40	Coop Vs Trad	C	+0.68
Berg (1993)	11	8	Coop Vs Trad (Treatment period)	C	+0.24
Stacey (1992)	9	-	Administered a timed test to students in experimental (Coop)and Con (Ind)	C	-0.04
Leighton, Slavin & Davidson (1989A)	7	5	Team practice, a Coop with no reward sharing (T B),Vs Con (T A)	C	+0.84
Leighton, Slavin & Davidson (1989B)	7	5	STL, a Coop method with reward sharing (T C), Vs Con (T A)	C	+0.85
Thomas & Sherman (1986)	10	5	Coop Vs Ind	C	
Nichols & Hall (1995A)	10	9	Coop (T gp 1) Vs Trad (T gp 2) [during first 9 weeks]	C	+0.48
Nichols & Hall (1995B)	10	9	Coop (T gp 2) Vs Trad (T gp 1) [during next 9 weeks]	C	+0.59
Nichols & Hall (1995C)	10	9	Coop (T gp 1) Vs Trad (T gp 2) [during first 9 weeks]	A	+0.48
Nichols & Hall (1995D)	10	9	Coop(T gp 2) Vs Trad (T gp 1) [during next 9 weeks]	A	+0.59
Nichols & Miller (1994A)	11-12	18	Coop Vs Trad	C	+0.45
Nichols & Miller (1994B)	11-12	18	Coop Vs Trad	A	+0.51
Townsend & Hicks (1995A)	8	—	Coop Vs Trad (M)	A	-0.00
Townsend & Hicks (1995B)	7-8	—	Coop Vs Trad (F)	A	+0.24
Owens & Barnes (1982A)	7	—	Coop Vs Con (Comp & Ind) (M from school A only)	A	+0.36
Owens & Barnes (1982B)	7	—	Coop Vs Con (Comp & Ind) (F from school A only)	A	+0.15
Owens & Barnes (1982C)	11	—	Coop Vs Con (Comp & Ind) (M from school A only)	A	+0.61
Owens & Barnes (1982D)	11	—	Coop Vs Con (Comp & Ind) (F from school A only)	A	+0.92
Owens & Barnes (1982E)	7	—	Coop Vs Con (Comp & Ind) (M from school B only)	A	-0.15
Owens & Barnes (1982F)	7	—	Coop Vs Con (Comp & Ind) (F from school B only)	A	+0.20
Owens & Barnes (1982G)	11	—	Coop Vs Con (Comp & Ind) (M from school B only)	A	+0.60
Owens & Barnes (1982H)	11	—	Coop Vs Con (Comp & Ind) (F from school B only)	A	+0.24

Symbols used: A: Affective; C: Cognitive; Comp: Competetive; Con: Control group; Coop: Cooperative learning; D: Duration in weeks; F: Female sample only; G: Grade level; gp: group; Ind: Individualised; M: Male sample only; O: Outcome feature; Trad: Traditional learning; T: Treatment.

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Mathematical Misconceptions -- Can We Eliminate Them?

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If students are to successfully tackle tertiary mathematics, one prerequisite is the mastery of a number of basic concepts. Despite the best efforts of teachers, many students develop mathematical misconceptions. Is it possible to eliminate these misconceptions? In this study, a strategy based on Piaget's notion of cognitive conflict was employed for this purpose. The method used was found to be successful in reducing, and in some cases totally eliminating, the frequency of misconceptions exhibited by a class of students.

Introduction

Due to the sequential nature of many branches of mathematics, it is essential that individuals attempting to learn mathematics at a particular level first master the fundamental mathematical concepts and skills which are pre-requisite knowledge for success at that higher level (Swedosh, 1996). Swedosh asserted that mastery of this pre-requisite knowledge depends very heavily on individuals having a thorough understanding of a limited number of basic concepts and having practised certain skills to the point where they can confidently use them. It was also asserted that the understanding of these concepts, as well as the ability to use this knowledge, is so crucial to subsequent learning of mathematics, that it is of great interest to mathematics educators (teachers, lecturers, and tutors) to be able to ascertain information about the types of misconceptions commonly held by students and the frequency with which these misconceptions occur.

An understanding of the basic mathematical concepts which should be acquired in secondary school is crucial to the preparedness of students to undertake tertiary mathematics subjects (Swedosh, 1996). This issue is important from the viewpoints of prospective tertiary mathematics students and of the Victorian Government whose stated policy is that it is committed to increasing participation in post-secondary mathematics education (Victorian Government, 1987) and that educational programs in Years P--12 should provide all students with a sound preparation for further schooling (Ministry of Education Victoria, 1984).

The importance of this issue is also recognised in A National Statement on Mathematics for Australian Schools (Australian Education Council, 1990). The National Statement includes as one of the goals for mathematics in Australian schools the aim that "as a result of learning mathematics in school, all students should possess sufficient command of mathematical expressions, representations and technology to continue to learn mathematics independently and collaboratively" (p. 18).

The significance of this goal is highlighted when one considers that "contrary to a widely held belief, 90% of all HSC students do apply for tertiary entrance. It is reasonable to infer from this that most students see HSC as a preparation for tertiary studies" (Blyth & Calegari, 1985, p. 312). Also, "statistics collected by the Mathematical Association of Victoria (MAV) show that 75% of all tertiary courses require a pass in HSC mathematics" (Blyth & Calegari, 1985, p. 312).

There are a large number of types of mathematical misconceptions, "and a complete list may not even be practical" (Davis, 1984, p. 335). Many authors have written about how a consideration of the mathematical misconceptions which pupils exhibit can be utilised to develop teaching techniques which are directed towards the diagnosis and elimination of these misconceptions (Bell, 1982; Farrell, 1992; Margulies, 1993; Perso, 1992).

It is vital to have a strategy to deal with misconceptions when they do occur. An approach which has been successful in helping many students overcome misconceptions is the "conflict teaching approach", based on Piaget's notion of cognitive conflict. In this

approach, teachers discuss with the learners (students) the inconsistencies in the thinking of the learners in order to have the learners realise that their conceptions were inadequate and in need of modification (Tirosh, 1990). Vinner (1990) supports this approach and states that "there is no doubt that if inconsistencies in the students' thinking are drawn to their attention, it will help some of them to resolve some inconsistencies in a desirable way" (p. 97).

A number of authors (Stavy & Berkovitz, 1980; Strauss, 1972; Swan, 1983) have found the conflict teaching approach to be an effective way of removing and correcting a variety of misconceptions involving aspects of mathematics and physics. It should be noted that there is no attempt in any of the studies reported in the papers by these authors to eliminate misconceptions in the topics considered in our paper, and the subjects being tested by us constitute a vastly different cohort.

In an earlier study, Swedosh (1996) presented information about the nature of mathematical misconceptions which were commonly exhibited by mathematics students at the University of Melbourne (U. of M.) and at LaTrobe University (LaTrobe) upon entry to a tertiary mathematics subject. A list of these misconceptions and their frequencies of occurrence was provided by Swedosh, together with a discussion of why they may have occurred.

In the study reported on in this paper, an experiment to help students overcome their mathematical misconceptions was devised so that the success or otherwise of the strategy outlined above could be tested.

Methodology

The students in a first year lecture group in Applied Mathematics at the U. of M. were given a short test. The test comprised a number of questions similar to those previously posed in the earlier tests at the U. of M. and at LaTrobe (Worley, 1993) and which yielded a high frequency of misconceptions (Swedosh, 1996) as well as some questions which were similar to those which appeared on the list of misconceptions provided in 'Algebraic Atrocities' (Margulies, 1993, p. 41).

The rationale adopted here was to provide students with the opportunity of exhibiting the misconception, if they do in fact possess that misconception. In other words, if the student does have particular misconceptions, the questions posed on the short test are intended to invite responses which exhibit this fact.

After the test had been administered, each question on the test paper of every student was examined carefully to gain information on the misconceptions exhibited. The occurrence of each misconception on each question was recorded (as well as the occurrence of those who attempted the question and who answered correctly) and the frequencies of these occurrences were then compared from question to question. Those questions on which the greatest number of misconceptions had been exhibited then became the focus of this study (the 'focus' questions were those where the frequency of misconceptions exceeded 6%).

After these 'focus' questions had been determined, time was spent in class, using the 'conflict teaching approach', attempting to eliminate these misconceptions and replace them with the correct concept. For example, if a large number of students were to exhibit the misconception that if $\frac{1}{x}-\frac{1}{b}=\frac{1}{a}$, then $x-b=a$ and so $x=a+b$, a suitable approach would be the following : Students would be shown an equation which they knew to be self-evident such as $\frac{1}{2}-\frac{1}{3}=\frac{1}{6}$. They would then be shown that if one applied the same method as above, it would yield the absurd result that $2-3=6$. Having (hopefully) eliminated (or at least strongly challenged) the misconception, the correct concept would then be taught (use of a common denominator in the example above).

A new test was assembled which comprised only the 'focus' questions. This test was administered to the same class three weeks after the teaching session described in the previous paragraph. The reason for waiting three weeks after that session is that it is desirable that students are answering the questions on the new test based on what they

now believe to be the relevant concepts, not based on the memory of what they had recently heard. Students did not know that they would be asked to do this test.

The frequency of occurrence of misconceptions on each of the 'focus' questions was then calculated, and these were compared with the frequencies which were exhibited on the same questions before the teaching session.

If an improvement was found (less misconceptions exhibited), it was considered that it would then be interesting to determine whether this improvement was still evident if short questions were not used, but the need for the use of these concepts was embedded in some longer and less transparent question. This may be the subject of further investigation.

The Sample

The students who were tested were all enrolled in an Applied Mathematics subject in Semester 2, 1996 at the U. of M., having succeeded in a fairly demanding prerequisite mathematics subject in Semester 1 (only about 25% of first year students were permitted to attempt this first semester subject). A total of 103 students sat the first test and 108 students sat the second test. In order to be able to make a meaningful comparison of the results and ascertain whether the intervention of the teaching approach had caused a reduction in the frequency of misconceptions, a sample was selected from these students. It was thought to be important to consider the backgrounds of the students so that the results of this study could be put into context and not be over-generalised. That is, it was important that, if the teaching method was found to be successful with this group, it could be asserted that the method was helpful with these types of students -- an assertion about other types of students might be less appropriate. The students reported on in this study are those students who sat both the pre-test and the post-test, and who had completed Specialist Mathematics 3/4 (SM) as part of their Victorian Certificate of Education (V.C.E.). There were 60 students in this category.

In each of the three Common Assessment Tasks (CATs) in a V.C.E. subject, students are awarded a grade from E to A+ corresponding to a ten point scale from one to ten. Using this scale, the 60 students herein had an average mark for SM CAT 1 (the challenging problem) of 9.38, an average mark for SM CAT 2 (facts and skills) of 8.52, and an average mark for SM CAT 3 (the analysis task) of 8.50. They had an average Tertiary Entrance Ranking (a percentile with the highest possible ranking of 99.95 and with about 23 students for each .05 -- .05% of students in the state is about 23) of 93.56. Those excluded from this study include mature age students as well as interstate and overseas students whose backgrounds were quite different to the 60 students considered herein.

The pre-test

The pre-test consisted of 53 short answer questions. As previously stated, each question was designed to 'invite' the display of a particular misconception, should that misconception exist. Students were given sufficient time for nearly all of them to complete the test (about 30 minutes). An analysis of each question on the submission of each student was undertaken and a count was taken of how many students had the correct answer, how many had exhibited the misconception, how many had another wrong answer, and how many had not attempted the question.

There were 17 questions for which the frequency of misconceptions exceeded 6% and these became the 'focus' questions. These 17 questions are shown below, and the most common misconception(s) are shown to the right of each question :

Simplify expressions 1-5 as fully as possible.

Note that some expressions may not be able to be simplified. If this is the case, simply rewrite the expression in the space provided for the answer.

1. $\dfrac{100!}{98!}$ $2!$

2. $\left(\dfrac{1}{2}\right)^{-3}$ $\dfrac{1}{8}$

3. $3^x \times 3^x$ 9^{2x}; 3^{x^2}

4. $2^x + 2^x$ 4^x

5. $a^b \times a^c$ a^{bc}

Solve equations 6-12 for x:

6. $x^2 = 81$ $x = 9$

7. $x^2 - 4x = 0$ $x = 4$

8. $(x-1)(x^2-3x) = 0$ $x = 1;\ x = 1,3$

9. $x^2 = x$ $x = 1$

10. $\dfrac{x^2-1}{x-1} = 0$ $x = \pm 1;$ $x = 1$

11. $\dfrac{1}{x} - \dfrac{1}{b} = \dfrac{1}{a}$ $x = a + b$

12. $\log_e x = \log_e t - t + c$, c constant $x = t - e^t + e^c$

13. Solve for x: $2x + 4 < 5x + 10$ $x < -2;$ $x > 2$

14. Factorise $(2x+y)^2 - x^2$ $3x^2 + 4xy + y^2$

15. Given that $\sin\dfrac{\pi}{6} = \dfrac{1}{2}$, evaluate $\sin\dfrac{7\pi}{6}$ $\dfrac{1}{2}$

16. Given that $\sin A = \dfrac{3}{5}$ and that $\dfrac{\pi}{2} < A < \pi$, evaluate $\cos A$. $\dfrac{4}{5}$

For question 17, indicate whether the statement is True or False. (circle one)

17. $k\left(50 - \dfrac{x}{5}\right)(80 - 2x) = k(250 - x)(40 - x)$ T / F

True

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Teaching method

The teaching took place two weeks after the initial test. At this time the 'focus' questions had been determined and the teaching method was specifically targetted at those questions (which had the highest frequency of misconceptions). The method essentially involved showing examples for which the misconception could be seen to lead to a ridiculous conclusion, and, having established a conflict in the minds of the students, the correct concept was taught. When teaching the correct concepts, slightly different examples to the 'focus' questions were used (different numbers, etc.) to ensure that students were not able to simply remember what had been shown to them, but had to use the concept correctly. Some examples of what was taught are shown below. In many cases numerical examples were used to point out the inconsistency within the misconception.

$\frac{4!}{2!} = \frac{24}{2} = 12,\ (4-2)! = 2! = 2;$ $\qquad \frac{4!}{2!} \neq (4-2)!$

$3^2 \times 3^2 = 81;\ \ 3^2 \times 3^2 \neq 9^4 = 81^2$

$2^3 + 2^3 = 16;\ \ 2^3 + 2^3 \neq 4^3 = 64$

$x^2 = 9,\ x^2 - 9 = 0,\ (x-3)(x+3) = 0,\ x = \pm 3$

$\frac{x^2-4}{x-2} = \frac{(x-2)(x+2)}{x-2} = x+2$ iff $x \neq 2;$ $\frac{0}{0}$ does not exist.

$\frac{1}{2} - \frac{1}{3} = \frac{1}{6};\ 2 - 3 \neq 6$

$\sin\frac{\pi}{3} = \frac{\sqrt{3}}{2};\ \sin\frac{4\pi}{3} = \sin\left(\pi + \frac{\pi}{3}\right) = -\sin\frac{\pi}{3} = -\frac{\sqrt{3}}{2}$ as $\frac{4\pi}{3}$ is in the third quadrant.

$k\left(10 - \frac{x}{2}\right)(200 - 5x) = k(20-x)(40-x),$ (True or False?)

$k\left(\frac{1}{2}\right)(20-x)(5)(40-x) = k(20-x)(40-x);\ \frac{5k}{2} = k$ which is only true for $k = 0.$

The post-test

The post-test was given to students three weeks after the teaching session and students did not know that there was to be a second test. The post-test consisted of the 17 short answer questions shown in this paper.

Results

The tables below summarise the responses given by the 60 students to the 17 questions shown earlier. Table 1 shows the frequencies of each response given before and after teaching session. Table 2 shows the proportions of each response before and after teaching session for those who attempted the question. In each table, 'Q' is the question number.

Table 1

Q	Correct		Misconception		Other wrong		No attempt	
	Before	After	Before	After	Before	After	Before	After
1	52	58	5	0	2	2	1	0
2	56	60	4	0	0	0	0	0
3	54	56	6	2	0	1	0	1
4	39	47	8	2	12	9	1	2
5	52	57	7	2	1	0	0	1
6	49	60	11	0	0	0	0	0
7	53	59	5	0	2	1	0	0
8	53	57	6	1	1	2	0	0
9	28	57	30	2	2	1	0	0
10	46	44	12	9	2	6	0	1
11	56	57	4	2	0	0	0	1
12	36	54	15	2	4	2	5	2
13	45	50	7	4	4	6	4	0
14	41	55	7	0	10	2	2	3
15	51	51	7	6	2	2	0	1
16	27	50	23	5	4	4	6	1
17	44	54	10	4	0	0	6	2

Table 2

	Of those who attempted the question					
Q	% Correct		% Misconception		% Other error	
	Before	After	Before	After	Before	After
1	88.1	96.7	8.5	0.0	3.4	3.3
2	93.3	100.0	6.7	0.0	0.0	0.0
3	90.0	94.9	10.0	3.4	0.0	1.7
4	66.1	81.0	13.6	3.4	20.3	15.5
5	86.7	96.6	11.7	3.4	1.7	0.0
6	81.7	100.0	18.3	0.0	0.0	0.0
7	88.3	98.3	8.3	0.0	3.3	1.7
8	88.3	95.0	10.0	1.7	1.7	3.3
9	46.7	95.0	50.0	3.3	3.3	1.7
10	76.7	74.6	20.0	15.3	3.3	10.2
11	93.3	96.6	6.7	3.4	0.0	0.0
12	65.5	93.1	27.3	3.4	7.3	3.4
13	80.4	83.3	12.5	6.7	7.1	10.0
14	70.7	96.5	12.1	0.0	17.2	3.5
15	85.0	86.4	11.7	10.2	3.3	3.4
16	50.0	84.7	42.6	8.5	7.4	6.8
17	81.5	93.1	18.5	6.9	0.0	0.0

It can be seen from Table 2 that for every question there was a decrease in the frequency of misconceptions exhibited. Most of these decreases were substantial, but two of them (questions 10 and 15) were slight. In analysing the results, consideration was given to why responses to questions 10 and 15 had not shown the same level of improvement as all other questions. The authors believe that it is possible that question 6 impacted on question 10 . Question 6 required students to give the two solutions $x = \pm 9$ to the equation $x^2 = 81$. The misconception that $x = 9$ dropped from 18.3% before the teaching session to 0.0% after. It is likely that, having learned this concept well, students then used it when solving $\frac{x^2 - 1}{x - 1} = 0$ without realising that if $x = 1$, they would be dividing by zero.

The reason for the responses to question 15 not improving substantially are less clear. It is possible that some misconceptions are harder to eliminate than others, or that students' basic trigonometry is not as solid as other parts of their mathematics, or that this method is not as well suited or as effective for some areas as it is for others. Questions 15 and 16 are quite similar, yet the results for question 16 improved dramatically. Perhaps students did not recognise which quadrant the angle was in.

Limitations of the Study

The major dilemma with any study such as this one is the extent to which the results can be generalised. Questions as to whether the sample was representative of the group to which a teaching method might be applied are important, as some methods are likely to vary in their effectiveness depending on the make up of the target group.

The students considered in this paper were very bright and their mathematics was, overall, very good. It is not certain that the results discussed herein would be as dramatic with students whose backgrounds were not as good. That is, it may be that the high level of ability of the students considered in this paper, meant that they were able to readily see the inconsistencies in their previous thinking, and learn the correct concepts, and that lesser students may not be able to undergo this process as readily.

It is possible that, if students were tested again at some time in the future, some of them may have reverted to their (often long-held) misconceptions. It is also possible that many of these misconceptions may occur again if the concept being considered was embedded in some longer and less transparent question.

Conclusions

The teaching method used was extremely effective and the improvement in the results greatly exceeded the expectations of the authors. It is clear that by first challenging or undermining the misconception held by the students by showing the ridiculous outcomes which can flow from such 'rules', and then replacing the 'damaged' concept with the correct one, mathematical misconceptions can be, to a great extent, eliminated.

It is not clear whether the results would be as dramatic with less able students and particularly less able mathematics students or if the concepts being tested were embedded in more complex questions, but within the limitations of this study, the strategy used to eliminate mathematical misconceptions has been a remarkable success.

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Story weaving : a process for critical collaborative professional development

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This paper describes on-going action research into the collaborative professional development of the authors themselves. It was based on data collected through story telling and weaving. The results suggest that this is an effective tool for self initiated professional development.

We are two people who are working within pre-service mathematics teacher education at Waikato University. Merilyn has teaching experience with children aged 5 to 7 years and has worked in teacher education for 6 years. She is teaching a mixed media distance class (56 students) for the first time this year. Bronwen has teaching experience with children aged 13 to 18 years and has just started to work with pre-service teachers. We had shared ideas and experiences with respect to mathematics education and research informally prior to this. However, when the opportunity arose to work more closely together (we have adjoining offices) we made a commitment to meet regularly to discuss our teaching and learning. These meetings were intended as planned opportunities for professional development from which we could learn about our own teaching and learning (Bell & Gilbert, 1996).

We used an action research framework to explore the process of our learning (Cohen & Manion, 1994). Data has been generated and analysed through progressive story telling. Our reading and prior experiences suggested that story telling was an effective tool for making sense of and researching the complexities of teaching and learning (Bell and Gilbert, 1996; Bruner, 1986; Connelly & Clandinin 1990; Drake, Elliott & Castle, 1993; James, 1996; Taylor, 1996).

> Story telling is important for both listeners and narrators. Stories are richly textured sources of information to others ... the assumptions underpinning the practices described within these stories may easily be questioned, leading to a reflective and potentially enlightening evaluation. (James, 1996)

Why weave stories together?

Through our previous discussions, we were aware we each found teaching to be a moral and an intellectual activity (Ball & Wilson, 1996; Fullan, 1993; Ocean, 1996). As teachers we are committed to helping our students think about mathematics and mathematics education issues. We consider we have a responsibility to strive to improve our teaching and thus provide more effective learning opportunities for our students and subsequently the children they will teach. We both believe in the value of collaboration, caring and connection (Belenky et al, 1986; Noddings, 1986; Gilligan, 1982). We wanted to model and experience the processes we believed in.

> ... we see a dialectic relationship between the teacher's collaborative inquiring relationship with other professionals and his or her similar relationship with members of his or her classroom community of inquiry. (Wells, 1997)

Our collaboration has involved sharing, by telling, the stories of our teaching and learning. More importantly our collaboration has led to a weaving of our stories.

> Through the interaction of our individual narratives, we collaborated in the telling and retelling of stories of the past and in

> the co-creation of stories for the present and future. (Beattie 1995)

Creating a weaving

Our discussions of the prescribed mathematics education readings for a pre-service mathematics education course are presented here as an example of the process of story telling and weaving. This episode took place over a period of four weeks.

> Merilyn gave the selected readings to Bronwen. Bronwen was uncomfortable with the focus of one of the readings on young children's mathematics learning and did not want to use it. Merilyn explained other students had found the reading to be an accessible introduction to the theory of constuctivism.
>
> Two weeks passed. During this time each of our students interviewed a five year old at a school. The purpose of the interview was to find out a child's ideas of number and number naming and to identify some which might be developed. The students provided their child with some learning experiences. Over this time we discussed the learning in our classes and the teaching and learning happening during the students' interactions with their child.
>
> The next part of the course required a focus on learning theory. Bronwen had thought through some strategies she might use to do this. She asked Merilyn how she introduced her students to ways of thinking about theories of mathematics learning and how she linked these to their own experiences. Merilyn explained she used the reading that Bronwen had questioned. She said she presented it to her students as their first example of academic reading in mathematics education. She gave them class time to read, identify and discuss the issues it raised for them. She described the discussions. This approach was not one Bronwen had considered. However, she said she would try it.

We consider our collaboration enabled Bronwen to gain access to a new approach for focusing students on thinking about learning and to gain a feel for how students might respond to it. It challenged Merilyn to analyse why she used this approach with this reading as it was atypical of her teaching style.

> We both introduced the reading in a similar way and found it generated vigorous debate about the nature of and links between learning and teaching mathematics. For example, the students discussed the links between teaching and learning mathematics, the difference between understanding and using rules and the implications of these ideas for teachers.

Our deliberate use of the same teaching strategy with an intention to share enhanced our awareness of the sense students were making while we were in our respective classrooms. Both of us were used to reflecting on our teaching but telling and re-telling the stories of our experiences required us to be more explicit. When we articulated what happened we were sharing our experiences, our doubts and concerns about the contribution we had made to our students' learning. We were also sharing our students' responses. Our joint story was woven from the insights we brought to the story telling and the sense we made together. Our joint weaving is more textured and complex than either of our stories separately, or if they had only been placed side by side.

> As a result of this experience Bronwen modified two other courses to incorporate the reading. She found it generated stimulating discussion on the possibility of older children "inventing" mathematics, the use of calculators and the nature of mathematics.
>
> Merilyn had previously given her distance mathematics education students a hard copy of the reading. She electronically posted specific focus questions to these students. This stimulated a large number of students to debate the nature of learning and teaching mathematics.

Weaving our stories enabled us to learn from and build on each others' experiences. With our students we found we noticed we could weave in more subtle nuances from their comments. This deepened and enriched all discussions. In addition, this extended process confirmed the teaching strategy was able to be generalised. Our sharing worked with mathematical activities as well. This paper is another joint story and the writing of it has helped us become more aware of ourselves as teachers and learners.

What have we woven?

Even though we are experienced teachers we have found story weaving has made the process of professional development more innovative. Through our collaborative action research we have been able to weave an intricate, textured joint story. It has given us the impetus to reflect on, describe and explore the tentative feelings and thoughts we had about teaching and learning in new situations. It encouraged us to dabble in the unknown.

We have learned:

- the power of **critical collaborative reflection on practice** (Lytle & Cochran-Smith, 1992; Drake, Elliott and Castle, 1993)
- the value of setting aside **time** for this (Fullan, 1993; Hargreaves, 1994, Lytle and Cochran-Smith, 1992)
- the importance of **ongoing** reflection for professional development (Fullan, 1993)
- the importance of being **non-judgemental** when sharing (Cowie and Saunders, 1993)
- that together we can create **new possibilities** for action by weaving our individual prior and current experiences.

The interlacing of experience and critical reflection supported by mutual trust, caring, and sharing created more opportunities and more questions (Fullan, 1993; Thayer-Bacon, 1993; Wells, 1997). Some of these are:

- Who can produce knowledge of teaching that is seen as legitimate?
- Who is able to legitimise this knowledge and what criteria do they use?
- What does the knowledge encompass?

These questions have already been raised by researchers investigating teachers' voice and insider/outsider perspectives (Hargreaves, 1996; Elbaz, 1990; Lytle and Cochran-Smith, 1992). None of them seem to have explored the questions in relation to their own professional development and knowledge of teaching. We intend to use our collaborative weaving to explore these questions within our continuing mathematics education research. We would like others to weave with us.

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The Politics of Mathematics Education 1997

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A review of the politics of mathematics education in Australasia was completed in 1996 just as there was a change of government in Australia. In the year since cautious optimism in Australia has become deepening concern as more and more problems confront the mathematical sciences in schools and universities. This paper examines recent political developments in Australia and their effect on the mathematical sciences. Particular attention is paid to mathematics education research and to issues concerning participation.

This paper is not specifically about research, nor is it specifically about mathematics education. It is about the politics of mathematics education research in a very broad sense, and will try and make links with the politics of the mathematics sciences in general, especially issues concerning participation. The term 'mathematical sciences' is being used to encompass all aspects of mathematics and statistics including educational.

In a review of research on the politics of mathematics education, completed as Australia had a change of government, a somewhat optimistic view of the future was taken (Clements and Thomas, 1996). A year later that view has to be challenged. The comments here are specific to Australia but have broader relevance because of the educational standing that Australia does have in the region. The big question is how much longer that standing can be maintained.

The health of the mathematical sciences is dependent on the state of the individual parts although weaknesses in one area can sometimes be compensated for by strengths in others. For example, good teachers will overcome difficulties with a poor curriculum and good teaching in one sector can overcome poor teaching in another. It would appear that weaknesses are now becoming apparent in many areas. It is questionable whether a few pockets of excellence can sustain the vibrant and vigorous discipline needed in the future if Australia is to be part of a region building its economy on science and technology.

The paper summarises the situation in regard to mathematical sciences in early 1996. It then looks at some post-1996 federal election developments including the impact of the 1996 budget. Finally it sketches a view of the situation at the beginning of 1997 and looks to the future.

Mathematical sciences in Australia: Early 1996

Advanced mathematical sciences

At the beginning of 1996 a Strategic Review of the advanced mathematical sciences in Australia was published (National Committee for Mathematics, 1996). It generated a sense of achievement in the mathematical sciences and highlighted the strengths of the discipline in Australia. Although it identified a very fragile base, these concerns were tempered by an in-coming government whose pre-election policies and promises suggested that research and higher education in areas fundamental to Australia's scientific and technological would be fostered and protected. It was assumed the mathematical sciences would be seen as one such area.

Mathematics education research and teaching

Mathematics education researchers had continued to attract a number of research grants but funding for the one key centre at Curtin University was about to cease. Education faculties were being reduced across Australian universities and failure to attract prospective teachers of mathematics was an additional cause for concern as this was also leading to reductions in mathematics education academics. However, course-work higher degrees and professional development programs for teachers were continuing to be developed. Like the advanced mathematical sciences, the overall view of mathematics education research and teaching was a fragile base but with great potential.

Mathematics education in the universities

Spurred by grants to improve teaching and curriculum, requirements for course evaluations including extensive use of student questionnaires on aspects of courses and teaching, the need to attract and keep students from a shrinking pool and many more students who were under-prepared for university study, the teaching of mathematics at universities has received considerable attention. At the start of 1996 one of the major challenges facing university mathematics departments was how to continue this improvement when all students needed access to computers and relevant software but the funding base still implied that students would be taught in large lectures with the occasional tutorial. In terms of analysis and review of teaching, teachers and courses, there is no doubt that university mathematics was under much greater scrutiny than mathematics in many schools.

Mathematics education in the schools

By 1996 it was clear that there were some serious problems in school mathematics. Schools had been inundated by national and state documents which in most cases were an attempt to implement the Profiles (Curriculum Corporation, 1994). They were complex, difficult documents that mixed content and pedagogy and teachers were given very little help in implementing them. What seemed to have got lost in this process was that the Profiles had been recognised as flawed and any attempt to implement them should have be done with careful trialing. In particular, the emphasis on process rather than content in the hands of teachers who were weak in their content knowledge should have been recognised as something that could create problems rather than solve them.

The need to make the mathematics explicit for many students, especially if they have language difficulties, can be a problem if teachers themselves do not recognise the mathematical concepts underpinning the activities they are using. There is growing evidence of participation in education falling (see for example, Messina, 1997) and that achievement levels, especially for disadvantaged groups, may be declining. By the beginning of 1996 it was also apparent that teacher shortages were looming and that the chances of solving the problems of the many primary and secondary teachers who needed to up-grade their content knowledge were unlikely to improve.

The election of the new government, with a commitment to literacy and numeracy was seen as an opportunity to re-assess some of these issues. It was hoped that a broader range of people would be consulted about curriculum and teaching and that examples of 'best practice' from countries other than America and England would be taken seriously. Nobody expected anything other than a very conservative approach with the need for appropriate accountability and standards clearly outlined. What most did not want was for the juggernaut of self righteousness 'experts' from educational bureaucracies in federal and state government departments that rode rough-shod over any criticism under the previous government to be able to continue to control the mathematics curriculum.

Post the election, March 1996: The ministry

Almost immediately after the election the Treasurer found a 'black-hole' and it became apparent that there were going to be budget cuts across the board. There are various ways that budget cuts can be made but without a vision or knowledge of what they are dealing with, especially if the relevant minister is supported by a weak department, some dreadful mistakes can be made and they become very difficult to undo.

The science ministers

The senior minister who has responsibility for the Department of Industry, Science and Tourism (DIST) has seldom been sighted in scientific circles preferring to look after other aspects of his portfolio. He has left science to the minister for science, Peter McGauran who is enthusiastic about his portfolio and communicates well with the science community. Staff in DIST usually seem to know what they are talking about and have expertise in their area.

The education ministers

The senior minister for the Department of Employment, Education, Training and Youth Affairs (DEETYA) is Senator Vanstone who did not want the portfolio. She has kept control of the higher education sector which she appears to actively dislike. Her understanding of her portfolio seems to be limited and she has failed to communicate with most of the educational community.

The junior minister, Dr David Kemp has responsibility for schools and vocational education. As a former academic, many academics with an interest in educational matters have been disappointed to find themselves still excluded from debate about issues in schools. He has managed to keep their respect by his obvious commitment to literacy and numeracy and a certain optimism that he will eventually realise that he may on occasions get very bad advice and that the juggernaut from educational bureaucracies in federal and state government departments needs to be controlled.

DEETYA has had a reputation over the years as an inefficient, inexpert branch of government. The inability of the senior Minister to maintain leadership in DEETYA may indicate that it is unworkable and needs a major re-structuring. The most recently appointed head's "surprise departure" (Juddery, 1997 p.6) to take up a position with the Sydney Olympics saw a senior economic adviser become the new manager. Further, staff cuts have left it vulnerable to mistakes such as one recently involving the provision of Austudy which was blamed on loss of "the Department's corporate memory and expertise" so that mistakes were made in the forms "making them nonsense" (Richardson, 1997, p.5). While numeracy is a key issue there appears to be nobody in DEETYA who actually has a solid background in this area, let alone anyone who can link literacy and numeracy or even understand that they are related.

DEETYA has responsibility for administering research funding so its operation impacts on allocation of funding for mathematics education research. While a peer review system largely isolates this from political interference, lack of resources to support this can lead to problems. Further, as Australia does not have a national science policy, funding for research operates in something of a policy vacuum.

Post the budget, mid 1996: The level playing field

The level playing field mentality meant that all federal government departments had budget cuts. In science the most damaging in the long term are likely to be cuts to research and development (R & D) concessions as this has caused some industries to re-think investment in these areas. This both decreases development in Australia and reduces the number of careers for graduates. Compared with other areas, science did well. In the long term however, science depends on the health of the education sector and the budget effects there have been much more dramatic and damaging.

Schools remained relatively unaffected by the federal budget as the bulk of their funding comes from grants to the States. Some states, including Victoria, had already dramatically cut funding to schools in recent years. The higher education sector is much more dependent on direct federal funds. Prior to the budget, and over a period of years, the university sector had taken in many more students but levels of teaching staff had remained more or less the same. One set of figures from the University of Melbourne quoted by the Dean of Science at the end of 1995 showed 20% more students for no increase in staff. This would be the norm rather than the exception.

Further, the number of students taking mathematics and science at advanced levels at year 12 has been decreasing (for example see DeLaeter & Dekkers, 1996) and it was becoming difficult to fill places in science based courses, including engineering. In turn,

this was further reducing the number of people considering careers in mathematics and science teaching.

The budget cut operating grants to universities, failed to fund a pay increase for staff and introduced a three tiered band for the Higher Education Contribution Scheme (HECS) which placed science based courses in the middle band. Taken together these measures are likely to have a profound effect on the mathematical sciences.

On November 20, 1996 a forum was held in Canberra to address the collective concerns of many in the sciences about education at the school and tertiary level. The Forum was hosted by the Federation of Australian Scientific and Technological Societies (FASTS) with assistance from the Australian Academy of Science (AAS). Participants, speakers and support came from FASTS, the Academies, Deans of education, science and engineering, the Institution of Engineers, mathematics and science education researchers, teachers and business groups. The Forum identified a number matters of urgency (Thomas, 1996):

- The supply of teachers
- Professional development and the professional status of teachers
- Career awareness and advice for students
- HECS changes
- University funding cuts
- Networking

From this list it is apparent that many serious issues were developing. It was widely predicted for example that applications for science and engineering in 1997 would drop which they appeared to do. What is more problematic is determining to what extent this was due to falling year 12 enrolments and to what extent it is due to changes such as those in the HECS. Further, competition between tertiary institutions can lead to aberrant interpretations of data that do not really reflect reality. For example, in the interests of perceived prestige, a university can report higher cut-off entrance scores but not report that this has been achieved by accepting fewer students into the particular course being discussed. What the Forum did demonstrate was the willingness of quite diverse groups to work together on a number of issues.

Mathematical sciences in Australia: Early 1997

Advanced mathematical sciences

The review of advanced mathematical sciences had found a fragile base for advanced mathematical sciences in Australia. In the current climate of constraint, data collected by the National Committee for Mathematics (Healy, 1996) and the situation at Deakin University at the start of 1997 exemplifies how fragile. Mathematics at Deakin has been reduced to service teaching with new students having to be admitted to another subject major and studying mathematics as a second choice. The staff has been decimated and the Head of Mathematics, a mathematician of considerable professional reputation, secured a research position in Berlin and left taking two PhD students with him.

Mathematics education research

Little has changed in the situation in regard to mathematics education research funding. However, continued difficulties relating to attracting students to become mathematics teachers means fewer opportunities for positions in universities. For example, the number of people in full-time positions in mathematics education in Victoria has dropped and mathematics methods courses are being combined either within or between universities. In New South Wales a campus of the University of New South Wales which was concerned largely with education is to be closed and a number of positions lost. Mathematics education courses have been eliminated at other universities.

The impact of the HECS changes to higher degrees is expected to greatly affect the participation of teachers in this valuable form of professional development unless state Ministries of Education offer incentives that in some way compensate for the additional costs teachers will have to bear. The alternative is that either the universities or the federal government will find a way to ensure that these do not all become full-fee paying. As this could open a pandora's box of special pleading it is unlikely to happen.

Mathematics education in the universities

Continued improvement in course delivery will be difficult with fewer staff and every indication that even more students, especially in service courses, will be ill-prepared. There is concern that mathematics departments at some universities may cease to exist with the few remaining staff being deployed to other departments and faculties to provide service teaching. This is more or less what has already happened at Deakin.

Many second and third year courses are small. This makes them very vulnerable in a situation where universities are given no encouragement to maintain areas of strategic importance and university administrators are setting minimum sizes for classes.

The provision of appropriate support for teaching that incorporates use of computers remains a dilemma. The Strategic Review recommended low cost loans to students so they could purchase a computer and appropriate software. The increases in the HECS means that this is no longer an option as it would further deter some students from mathematics. The use of computers in the teaching of mathematics is a major equity issue as increasingly tertiary students have access to their own personal computer. The ones least likely to are those from lower socio-economic groups who are already under-represented in mathematics and are also more likely to be deterred by the increased HECS. Further, young women are well aware of how long it is taking women to repay HECS at the old rates and additional costs associated with the use of computers in their studies in addition to increased HECS, may affect their participation in mathematics.

Mathematics education in the schools

The malaise in school mathematics becomes more apparent each year as fewer students enrol in advanced level mathematics courses at year 11 and 12, especially in some states. The reasons almost certainly include lack of appropriate course and career advice in schools, issues relating to teaching and the curriculum.

The Third International Mathematics and Science Study (TIMSS) results for Australian 13 and 14 year olds were reported on very favourable in the Australian press but the results from the different states had considerable variations and the gap between Australian students and those in high achieving countries was considerable (Lokan, Ford & Greenwood, 1996). The spread of results was disturbing. A major difference between Australia and Singapore, the highest achieving country, is that in Australia it would appear that the community and schools are prepared to accept the concept of failure in mathematics and in Singapore all students are expected to be numerate. Achieving significant improvements in Australia is going to be very difficult.

A hidden teacher shortage has existed for years and has manifest itself as reduction in time for mathematics and teachers with little or no mathematics teaching at primary and secondary levels. Unpublished data collected by the Mathematical Association of Victoria had demonstrated that, in secondary schools, using under-qualified teachers is not confined to the junior secondary years but can exist at year 12 and suggests that 34% of teachers teaching mathematics are not fully qualified to do so. It is clear from recent data (Preston, 1997) that this is about to get much worse. Further, another disturbing finding of the TIMSS study was that over 50% of Australian (and New Zealand) teachers would choose another career if they had the opportunity (Lokan, Ford & Greenwood, 1996). It is a worrying finding in need of urgent research. It is possible that this figure could be reduced if teachers teaching without appropriate discipline knowledge could be reduced as being asked to teach mathematics without appropriate content knowledge can be a demoralising experience. In junior secondary years it is likely to be confounded by not having studied mathematics method either.

Finally, the outcomes based curriculum which has been imposed on schools must be questioned both in how well it is serving students and the effect it is having on teacher morale. The Eltis Committee Report (Eltis, 1995) remains the one comprehensive report where teachers were genuinely consulted and it rejected the National Profiles. This report showed both the lack of research base for outcomes based education and teachers' concern for how it could be implemented. The slowness of the implementation of the recommendations of the Eltis Committee have recently been the subject of stories in the

NSW press. This is not surprising as there is little indication that many of the key education bureaucrats in NSW either wanted the report or were keen to implement it.

The re-thinking of the outcomes based approach to mathematics education through a more collaborative approach has yet to eventuate. As suggested by Clements and Thomas (1996), the Profiles have now become 'benchmarks' and a recent 2 day meeting was hosted by the Curriculum Corporation (CC) in Melbourne to establish this project. The Australian Mathematical Sciences Council (AMSC) was asked to nominate a representative who has since submitted a report to the Council (available from http://www.dms.CSIRO.AU/~noel/AMSC/amschome.htm).

The 'benchmarks' need much greater involvement of MERGA members and others in the mathematical sciences. It makes sense in any project to build on existing documents. However this project has been given to a group who have been ardent supporters of particular approaches and groups like the CC and the Australian Council for Educational Research whose finances are largely dependent on implementation of existing documents. A recent Mathematical Association of Victoria Newsletter reported that the AAMT has serious reservations about the 'benchmarks' so in their current form there would appear to be little support for them within any of the major groups in the mathematical sciences. They are, however, likely to dominate approaches to mathematics curriculum in schools in the foreseeable future.

Where to now? 1997 and beyond

It is clear that the mathematical sciences are in for a very difficult time in Australia. The problems are not unique to mathematics and neither are the causes and the solutions. To exemplify I will digress and recount a personal experience. Recently, on the Wednesday of one week, I heard one of Australia's top scientists note that some people were going to have to die in a capital city before the dangers of deregulating water supplies was recognised. The next day, Thursday, I heard another top scientist discuss his concerns that an antibiotic might be released for veterinary use against the recommendations of the expert committee and that this might further decrease the effectiveness of the last line of defence antibiotics now available. On the Friday the President of a peak body of medical scientists told me a sorry tale of interference in the allocation of research funds for cancer in one of the States. On the Saturday it was announced that two people had die of salmonella poisoning and over the weekend there was another major outbreak of food poisoning in Victoria, both blamed on self-regulation in the food industry.

The bottom line is that political expediency driven by economic rationalism means that even safe food and water can get sacrificed. If expert opinion runs counter to economic arguments, economics wins. In this climate, the mathematical sciences are just as vulnerable as every other important or valuable activity. If they are not publicly defended by those who understand their importance, including their economic importance, then they will be neglected and ignored. There has probably never been a greater need for political action in the mathematical sciences. MERGA members need to constantly remind government that much of the expertise in mathematics education is not being used effectively in resolving some serious problems with both achievement and access in mathematics. We also need to listen to, and speak for, teachers.

Teachers to a great extent have been silenced and can face threats of dismissal and lost career opportunities if they make public comment critical of government actions. A national forum for teacher professional associations was established in 1996 from which discipline based societies and academics were largely excluded. This exclusion included both MERGA and the Mathematics Education Lecturers' Association. The Executive Director of the Australian Science Teachers' Association (ASTA) believes the formation of this group has led to the Ministers of Education consulting with this 'peak' body rather than groups such as ASTA (personal communication) and presumably the Australian Association of Mathematics Teachers (AAMT). I have argued on other occasions that mathematics and science teachers need to work with the discipline people for an effective voice (see for example the MERGA Newsletter for December 1996) and I believe this is becoming even more important. They must be members of peak educational bodies but

mathematics and science teachers are always in a minority in these forums and they should also use their natural allies in the disciplines and science and mathematics education research.

Teachers have become so used to minimal support that $260 per teacher for professional development in Victoria evoked a headline "Why the generosity?" (Richards, 1997, p.8). What industry, if it was serious about staying economically competitive, would allocate such a paltry sum to keeping its professional staff up to date? The National Professional Development Program (NPDP) funded by the previous government was so well promoted to teachers that they saw it as generous but it represented a fraction of what has been allocated in previous years.

Media issues

The problems and challenges of the mathematical sciences are being debated publicly in the press and other forums in spite of very active government media and public relations. This is largely the work of bodies like FASTS and AMSC and some individual academics. Increasingly the focus of some of this activity is on alerting business and industry to the dangers of allowing educational infrastructure to further deteriorate (see for example, Thomas and Guttmann, 1997). The Strategic Review of advanced mathematical sciences comprehensively documented just how important the mathematical sciences are to Australia, culturally and economically. The pervasiveness of the mathematical sciences is both a strength and a weakness as it has the effect of sometimes rendering the mathematical sciences invisible. Their role in underpinning science, business and industry can be taken for granted. This aspect has however been picked up by the media with the result that the 'bad' news stories of teacher shortages, for example, are often balanced by 'good' news stories about uses of mathematics to solve important problems.

An aspect of the invisibility of mathematics is the current concern about literacy and numeracy. While literacy and numeracy are sometimes linked, increasingly literacy is being separated from numeracy at the school level and the numeracy focus is being lost. This is an unfortunate development as to understand text that has mathematical or scientific concepts embedded in it is very difficult, if not impossible, unless those concepts are understood. Science awareness programs also sometimes fail to emphasise mathematics, an issue that is constantly being raised by the AMSC.

The changing face of lobbying

The value of the internet in lobbying has yet to be realised. The use of e-mail to generate support for petitions was an early use adopted by the mathematics community (Ellerton and Clements, 1994). However what government has been slow to realise, and what professional associations have still to fully utilise, is the way in which e-mail and the Web can be used to tap into communities, gather new information, share information and so on. There is no need for secrecy or for decision making by small groups acting on behalf of much bigger groups. A leader in this has been the FASTS Web page which contains regularly up-dated material including media releases, drafts of documents for comment, policy and so on . The recently established AMSC site is following this lead. (The sites can be located at http://bimbo.pharmacol.su.oz.au/fasts/fastshome.html and http://www.dms.CSIRO.AU/~noel/AMSC/amschome.htm).

Conclusion

There is room for optimism. There are many reviews under way at both the state and federal level and eventually the talking will have to stop and some concrete actions taken. The AMSC, in conjunction with the National Committee for Mathematics and other bodies such as FASTS and the Institution of Engineers continues to build a network of people who are concerned about mathematics education and mathematics education research. These bodies have learnt lobbying and media skills and they know how to use them. The teachers, largely silenced by government, have powerful allies in these groups. In two years as President of the AMSC I never heard mathematics teachers criticised but I heard plenty of criticism of what they were being expected to do and the lack of support for them. In the next few weeks there will be two meetings with Dr Kemp involving

representatives of the mathematical sciences. It would be nice to think the juggernaut may be about to be tamed and a wider professional view of the mathematical sciences start to influence government. Only then can all the parts of the mathematical sciences thrive and equity, economic and cultural objectives be achieved.

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Secondary school mathematics teachers' views of assessment: some insights from the caribbean

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There appears to be a lack of information about what teachers at large believe and understand about assessment and how this may impact on their instruction in mathematics and subsequently on students learning of mathematics. In an effort to address this issue, this paper explores views about assessment held by a small sample of secondary teachers in St. Vincent. The results indicate that the teachers perceived limitations in traditional assessment methods.

Introduction

When I think back to 4th September 1981, I vividly recall my first mathematics session at Teachers' College. Sixty of us were given a written test, on the basis of which we were put into ability groups for instruction.

There has been increasing concern that written tests assess only a sample of the mathematics felt to be important (Baird and Mitchell, 1986; Caribbean Examination Council, 1990; NCTM, 1989; 1993). This has resulted in the move towards other forms of assessment such as projects, portfolios and journals. While these alternatives are positive developments there is need for more research in the area of assessment of mathematics. For example, little is yet known about teachers' views of assessment, and how these impact on students' learning (Ernest, 1989; Southwell and Khamis, 1992).

The value of knowing about teachers' views of assessment

Teaching and learning at the school level are often bound up with the initiation and socialisation of teachers into subject subcultural settings (Goodson, 1985). Each subculture operates within its own set of norms, beliefs and values. As mathematics teachers, we bring to the classroom and develop within it our own set of beliefs, including beliefs about assessment. While there seems to be much research conducted on teachers' views of the nature of mathematics and the teaching and learning of mathematics (Webb, 1992; Grouws, 1992), an ERIC search for literature related to teachers' views of the assessment of mathematics failed to reveal any.

The focus on teachers' views of assessment in mathematics is significant because what we as teachers do grows out of our understanding of how learning takes place (Baroody, 1987; Siemon, 1989; Webb, 1992). An understanding of teachers' assessment ideas also provides a basis for gauging their likely acceptance of alternative assessment strategies. As Ernest (1989) noted, reforms in teaching would only take place when those deeply held beliefs about learning and teaching are confronted and changed. Webb (1992) endorsed this view, suggesting that because teachers are engaged in assessment activities a large percentage of their working day, they will have a strong impact on students' learning through these practices.

This paper aims to throw some small rays of light on secondary teachers' views of assessment.

Methodology

The data reported in this paper is part of a larger study that focuses on exploring the use of alternative assessment approaches in mathematics. This research was conducted in St. Vincent during the latter part of 1996.

Instrument

A teachers' views of assessment in mathematics questionnaire (see Appendix 1) was designed to gather (i) background information, (ii) information relating to reasons for assessing, (iii) usage of different assessment approaches, and (iv) ideas about whether they considered there was a need to reconsider assessment in mathematics.

Sample

The subjects for this aspect of the study were twenty (n=20) secondary school mathematics teachers from 15 schools in St. Vincent. Three of the teachers were female, and teaching experience ranged from one term to more than fifteen years with a mean of 10.3 years.

Results and Discussion

Teachers' responses to the questionnaire are categorised and discussed in terms of their (i) acceptance of current assessment methods, (ii) reasons for assessing, (iii) views about whether assessment should be reconsidered, (iv) use of various assessment approaches, (v) description of assessment, and (vi) thoughts about the need for improvements to the present system.

Teachers' acceptance of current assessment methods.

One of the teachers was non-committal about the methods used, thirteen were reasonably satisfied or quite satisfied, while six were unhappy with them. These six felt that present methods were too narrow, did not tap into students' mathematical thinking, and restricted teaching accordingly. They wanted to see a greater range of assessment methods used. The comments of one teacher illustrates some of these concerns:

> "the traditional methods of teaching and assessing the subject seem not to be as effective as they were before. We cannot truly assess students' performance until these obstacles of learning be removed...Students should be challenged and shown the inter-relatedness of mathematics. Assessment techniques should not only include written tests but observations, discussions, SBA and interviews".

The majority of teachers who were satisfied with current assessment practices seemed to base this on a belief that written test and examination results provide a true indication of student achievement and understanding. For instance, one teacher wrote,

> "they [present methods] normally give a relatively accurate measurement of students' performance and of the effectiveness of the teachers' strategies".

One of the thirteen satisfied teachers wanted even more of the same. He said,

> "Since I manage the assessment, I am satisfied. I would like to see final term exams in all terms. Students need as much reinforcement and repetition in a subject such as mathematics where a solid foundation is absolutely necessary".

Three of the thirteen satisfied teachers seemed to contradict themselves. One, for example, while claiming to be happy with present methods, nevertheless wished to see a change in the balance of regional and school assessment. This teacher responded,

> "Yes, a series of in-course tests is done each term. Thirty percent of this goes to the final mark at the end of the year. I hope CXC [Caribbean Examination Council] will introduce SBA [school-based assessments] in mathematics for the examination".

Reasons for Assessing

On the basis of a content analysis of the responses, the sixty responses to Question 2 were categorised as shown in Table 1.

Table 1 Teachers' claimed reasons for assessing

Category	No of responses
• Diagnosis of	
(a) what a student knows or monitoring students' progress	18
(b) what difficulties students encounter	2
• Providing feedback for students; self-assessment	7
• Obtaining instructional feedback for	
(a) planning and informing instruction	11
(b) monitoring or testing effectiveness of teaching	7
• Grading	3
• Measuring students' attainment	5
• Providing opportunities for creative and critical thinking	3
• Reporting	1
• Motivating students	3

Of interest, Table 1 reveals that the teachers in St. Vincent perceived the main reasons for assessing students' mathematical learning as, diagnosing both what they know and difficulties they are having, monitoring students' progress and obtaining feedback to inform their instruction practices as being. Although issues of selection and qualification - aspects of summative assessment - have dominated testing in St.Vincent, these teachers did not suggest these as reasons for assessing. The issues of grading, motivating students and providing opportunities for creative and critical thinking were perceived by three teachers in each case as reasons for assessing. While the tradition of grading will remain with us in St. Vincent, many secondary school mathematics teachers appear to be looking beyond this emphasis, to identify students' actual understanding.

Reconsideration of assessment

Although thirteen of the twenty teachers questioned suggested they were satisfied with the present assessment methods, all the respondents felt that assessment should be reconsidered. They felt that the present focus on right and wrong answers was too restricting and did not take account of students' procedures, logical thinking and reasoning. Further, it did not allow consideration of the usefulness of mathematics or allow it to arise out of practical or real life situations. One teacher claimed that mathematics

> "seems to be a 'cut-and-dry' matter, where pupils follow rules and formulae or some sleight-of-hand method without understanding why. Additionally, routined time assessment measures how quickly students can respond but not necessarily how well they can think and apply their knowledge...assessment should not be limited to paper-and-pencil tests".

The teachers felt that the present system should be broadened to include assessment approaches that provide opportunities for:

- practical work as well as observations on the various approaches students use to arrive at their solutions,
- encouraging discussion between teacher and pupils and between pupils themselves,
- continuous assessment throughout a child's school career rather than performance on a single external examination, and
- the inclusion of higher order thinking and the development of creativity.

The arguments presented here point to the need for serious reconsideration of the assessment practices by mathematics teachers and educators in St. Vincent.

Frequency of use of various assessment approaches

Teachers' reported use of various assessment methods are summarised in Table 2.

Analysis of the data presented in Table 2 shows the dominance of written tests as the primary means of assessment; all teachers used it. Grimison (1992) noted that this is a clear indication of the classroom subculture at work. Clarke, Clarke and Lovitt (1990) argued that this dependence on a single form of assessment has not only shaped teachers attitudes but also their teaching practices. The old adage *if the only tool you have is a hammer, everything you see will look like a nail* seems to aptly describe assessment based on a single approach. Thus, to get around the hammer-nail problem a full range of approaches needs to be considered.

Table 2 Frequency in using assessment approaches

Assessment Approach	Daily	Once per week	Monthly	Not applicable
* Interviews	8	2		10
Debates		1		19
* Quizzes	3	7	6	4
* Oral presentation	8	4	2	6
Projects			5	15
Investigations	2	4	2	12
* Students self-assessment	5	2	4	9
Peer assessment	3	2	4	11
* Written tests	1	9	10	
Portfolios		1		19
Journals		1	1	18
* Observation	14			6

* approaches more readily used

Looking at the data presented in Table 2, one may be struck by the variety of approaches used by the teachers. While all teachers use tests, sixteen also use quizzes, fourteen use oral presentations and observations, and over half use students' self-assessment. These other approaches represent a few ways in which teachers can examine the depth of ideas that students hold about certain concepts and thereby gain a clearer picture of students' understanding. Thus, this finding may be an explanation as to why so many teachers felt that the traditional written form of assessment needs to be reconsidered.

Although the observation of students is carried out daily by teachers, it was surprising to note that six of the twenty teachers indicated that this approach was not applicable for assessment purposes. Further, the results showed that these teachers did not see debates, portfolios or journals as potential assessment techniques.

One assessment approach not included in Table 2 but mentioned several times by some of these teachers as a significant alternative is that of continuous assessment in the form of School Based Assessment (SBA). SBAs could be considered as investigations and or projects, however, twelve and fifteen of the twenty respondents did not consider investigations and projects respectively as applicable to the assessment of mathematics.

Words associated with Assessment

Teachers were asked to use descriptors to indicate their perceptions of assessment in mathematics. An analysis of their responses showed that the teachers used both the final external examinations and their school examinations as referents when describing their feelings about the assessment of mathematics.

Some teachers associated assessment with stream-lining students and preparing them for the job market. Others focussed upon assessment as skill-testing, that is, as determining students' ability to calculate, memorise tables and formulae, and to work accurately within a limited time.

Another important finding was the linkage of assessment with feelings of anxiety. Teachers feared that not only were tests too difficult or in some cases too easy but that they sometimes tended to be biased or one-sided. Additionally, some teachers felt that tests were brain teasers and definitely outside many students' ability to pass.

There were some teachers who were concerned that their assessment practices were restricted by syllabuses, and the focus on written tests and long examinations. They felt that one of the ideas students were getting about assessment of mathematical learning was that it was a competition.

Views about the need to improve the present system of assessment.

Another major dimension of teachers' views of a subject is their response to change. The respondents views about whether there is a need to improve the current assessment system are summarised in Table 3.

Table 3 Teachers' responses to need for changes

Responses	No. of respondents
No change required	2
No response	5
Not eager to change but willing to adopt other approaches	2
Changes needed	11

Although just six of the twenty teachers had previously indicated their dissatisfaction with the current assessment practices, eleven teachers thought changes were needed.

Table 3 reveals that two of the teachers surveyed indicated emphatically the need for things to remain constant while two other teachers were not eager to change their present practice but were willing to trial other approaches. This kind of response, according to Fullan (1982), is natural because many teachers fear changes, sometimes because innovations do not consider the subjective realities of teachers, that is, their views and perceptions. In the words of one respondent:

> "when learning is released from the bondage of a system which excludes us from being partners in decision-making activities, to one that includes our views and input, then and only then, will we more readily be open to changes."

Of the eleven teachers who indicated a desire for changes in the way assessment is presently being done in St. Vincent, five highlighted the need for broadening the range of assessment approaches to include SBAs, practical work/individual projects, group work, oral presentation, interviews and quizzes. Three of the other teachers indicated a need for better assessment in terms of having a higher pass mark and "increasing the percentage of the in-course test marks to 50% of the final mark".

Interestingly, of the five teachers who offered no response about the need for change, four wrote previously that assessment should be reconsidered.

Conclusions

On the basis of the findings presented here, when one thinks about secondary school mathematics teachers in St. Vincent and innovations in mathematics education the picture is far from being gloomy. Rather than being resistant to any change in the traditional manner of assessment many teachers already seem to be exploring alternative forms. However, as suggested by Fullan (1982), once teachers have examined their feelings and reflected upon their practices, many trials of new approaches will need to occur gradually to convince them to adopt these approaches. Additionally, teachers will require strong support systems to assist them in the transition.

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APPENDIX: The Teacher Assessment Questionnaire

Dear Friends,

I am in the process of conducting research in the area of assessment in mathematics. Your participation is greatly appreciated. Please note that your views and opinions will be valuable contributions to this study and that all responses are confidential.

Teachers' Views of the Assessment of Mathematics in Secondary Schools

Please answer either by ticking the appropriate column or by writing in the spaces provided.

- Are you happy with the assessment method(s) currently being used in your school? Give reason(s) for your response.
- In your opinion, what are the three most important reasons for assessing mathematical learning? Give reasons to support your responses.
- Should we, as mathematics teachers, reconsider assessment? Why/why not?
- How often do you use the following approaches when assessing students? Indicate your response with a tick.

Assessment Approach	Daily	Once per week	Monthly	Not applicable
Interviews				
Debates				
Quizzes				
Oral presentation				
Projects				
Investigations				
Students self-assessment				
Peer assessment				
Written tests				
Portfolios				
Journals				
Observation				

- When you think of assessment what words come to mind?
- What changes, if any, would you make to the present system of assessment? Give reasons for your response.

Personal Background

Gender: Male/Female

Number of years teaching experience

0 - 4 ☐ 5 - 9 ☐ 15+ ☐

Name of School

Thank you for your cooperation.

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The Introduction of Probability into Australian Schools Interpreted Within a Broad-Spectrum Ecological Framework

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This paper argues that the two models of curriculum development currently used to interpret Australian mathematics education history—the Colonial Echo model and the Muddling Through model—are both deficient, and proposes a more complex model—the Broad Spectrum Ecological model. This considers the physical, social and intellectual forces operating within a specific environment. One small aspect of mathematics education history, the introduction of probability teaching into Australian schools, is used to illustrate the superiority of this model.

The Importance of a Sound Historical Framework

Australian educational practice has changed much in recent years, but without significant consensus and with substantial criticism and dissatisfaction. It seems to lack widely agreed principles for assessing sound educational practice, and the practice of any particular time is strongly influenced by groups influential at that time. One function of an historian is to provide frameworks which can assist all members of society to see their society more clearly so that it is easier for society to develop constructively and efficiently.

At the moment, Australian mathematics educators are more interested in social, affective and philosophical issues than pedagogic ones (Sullivan, Owens & Atweh, 1996). So the political aspects of education are highlighted, raising questions about the mechanisms which influence the nature of our society.

But such mechanisms have been poorly examined. Standard historical writing tends to describe environments and how actions conform with or fight against such environments, but tends not to examine the underlying forces which influence these actions. Blainey (1966), however, has examined the effect on the Australian economy of forces arising from transport difficulties and shown how shared original cultures can move apart when forces act differentially on the component parts. While Blainey's method does not necessarily have predictive power, it is a much more convincing explanation of events than one asserting that *X* happened because of a rise of interest in *Y*.

Educational research is concerned with both understanding and initiating change. For such research to be fully effective, an understanding of forces acting within and upon education should underpin all research. This paper is presented as a contribution towards raising the understanding of the importance of developing a sound interpretative historical framework among mathematics education researchers.

The limited debate which has so far occurred in Australian mathematics education has been mainly between the proponents of two distinct frameworks: the "Colonial Echo" model, proposed at least as early as Clements (1978) and codified in Clements, Grimison & Ellerton (1989), and the "Muddling Through" model, whose development may be traced in Truran (1991, 1994a, 1994b & 1995). The discussion below examines both models briefly before proposing and explicating an alternative model: the "Broad-Spectrum Ecological Model". It concludes by examining how well these models fit data pertaining to one small segment of mathematics education history.

The Colonial Echo Framework

Clements et al. (1989) argue that a "Colonial Echo" theory, sometimes conscious, sometimes unconscious, is sufficient to account for major events in mathematics education and for the overall uniformity of Australian schools until the mid-1970s and that it still had some validity in the late 1980s, although by then, they claim, a more mature approach had

developed. They attribute such influence to the assumption that mathematics is culture-free and to many structural and social links with Great Britain, and go on to argue that this led to Australian school mathematics becoming both sexist and elitist. Furthermore, they assert that the School Mathematics Project (SMP) and the Nuffield Project, both from England, were the major inspirations for the introduction of the new mathematics into Australia, that even the criticisms of the new mathematics which arose in Australia were copies of criticisms being made overseas, and that

> it was the naive willingness of Australian educators to accept untested English ideas in the area of school mathematics which persuaded education authorities ... to commit schools to large-scale, but ill-fated, reconstructions of their primary and secondary mathematics curricula, through the Cuisenaire and "New Maths" movements. (Clements et al., 1989, p. 68)

For Clements (the name is used in this paper to include his collaborators where appropriate) the changes of the 1970s and the 1980s arose from factors like the high level of co-operation between mathematics educators in the different states, the high quality and distinctive character of recent Australian mathematics education projects, and the development of new assessment procedures (Clements et al., 1989, p. 71).

The Muddling Through Framework

Truran (1991) examined the South Australian situation in the late nineteenth century, and argued that Clements' claims were too wide, and sometimes not applicable to all States of Australia. In particular he argued that some significant changes in South Australia antedated similar changes in England, New South Wales and Victoria, and that others were successfully introduced when they had failed elsewhere. His analysis of the reasons for this significant state of affairs was inconclusive.

Later, Truran started to analyse the forces influencing the teaching of probability in South Australia, still attempting to set his findings against the Colonial Echo model. He observed (Truran, 1994a) that there were too many different overseas developments for all of them to be copied so some sort of selection had to be made and justified, and also (Truran, 1994b) that the forces for change were "idiosyncratic, poorly integrated, and poorly justified". As a result he proposed Muddling Through as a better model.

However, Truran (1995) then proposed a Broad-Spectrum Ecological Model based on Crombie (1994), which, he claimed, encompassed the Colonial Echo model and allowed for a much greater degree of purposefulness by administrators than did the Muddling Through model. The model took more account of the forces operating within education, and explained why echoing was sometimes an efficient response to a specific situation. It is a more detailed working out of this model which is discussed below.

Debate about the Two Frameworks

Some time after Truran had moved past his Muddling Through model, Clements agreed that it did accurately describe some aspects of Australian educational practice, but he also asserted that

> ... those controlling the direction of the "muddling" still look to the United Kingdom, the United States of America and, increasingly, to the discipline of economics for inspiration. In particular, policies affecting practices in school mathematics have been, and continue to be, based on decisions made by governments in the United Kingdom and the United States of America. (Clements & Ellerton, 1996, p. 141, lightly paraphrased in Clements & Thomas, 1996, p. 141)

To some extent this position puts paid to the view they expressed earlier that the 70s and 80s ushered in a new era, and the introduction of a new force—economics—suggests

that the original model was not comprehensive. It is surprising that they have retained the Colonial Echo model. It is true that many recent changes in Australia have been similar to those elsewhere, and that many overseas visitors address Australians on educational matters. Such facts are not in dispute. What is in dispute is whether they are best explained by assuming that Australians "consciously" copy what is happening overseas (Clements & Thomas, 1996, p. 91, italics removed). For Clements and Ellerton (1996, p. 139) the similarities between the contents strands in nationally developed curriculum statements in Australia and the UK [sic: England & Wales is what is meant] are best explained by the Colonial Echo model, even though an explanation based on the intrinsic logic of how we perceive mathematics might be far more parsimonious.

This paper presents a model which, it will be claimed, fits more of the facts than does the Colonial Echo model. In one short paper it is only possible to support this claim by considering a very small part of Australian mathematics education history, but at this stage in the debate it seems to be more important to explicate the theory clearly.

The Broad-Spectrum Ecological Framework

There are two principles which underlie the culture of twentieth century biological investigation. One is that organisms behave in a way which optimises the balance between energy expended and satisfaction obtained. The other is that organisms operate within a competitive environment which ensures that only the most efficient survive. These principles form the basis of the model being proposed here. Educational systems may be looked at as though they are ecosystems containing individuals and groups of individuals. These organisms and groups interact, sometimes co-operating, sometimes competing. What actually happens may be interpreted in terms of the interacting forces, the mechanisms for minimising energy expenditure, and the decisions which individuals and groups make about whether to co-operate or compete.

This approach is not new. Popkewitz (1988, pp. 242 - 245) has described how some curriculum developers have proposed an evolutionary model of curriculum development, which looks for ways of introducing new items into the school practice with minimal disruption. However, he claims, they have tended to over-emphasise harmonious reconciliations, without adequately considering the tensions inherent in most situations. As a result Popkewitz proposed a more mature model which tried to reconcile these tensions and to juxtapose stability and change.

But this approach has been taken much further, and beyond traditional biology by Crombie (1994, v. I, pp. 63 - 66), an historian of science, who has proposed that historical investigation needs to consider concurrently several different aspects.

Historical ecology: which is at the level of nature, "the reconstruction of the physical and biomedical environment and of what people made of it";

Cultural dispositions: concerned with habits, motives, opportunities and responses

Scientific thinking: concerned with conceptions of the discoverable in nature and of scientific enquiry and explanation in relation to intellectual commitments, scientific context and experience, and available technical possibilities, with their historical sources.

For Crombie an historical narrative requires an examination of where people lived, how they lived and how they thought. I shall refer to these aspects respectively as *physical, social* and *intellectual* ecology. The term "physical" is preferred to "historical" because it emphasises that the forces are natural; the term "social" is preferred to "cultural" because it is more likely to be seen as encompassing a wider range of non-physical forces existing within a society. The three approaches taken together will be called the Broad-Spectrum Ecological (BSE) model. Together they provide a wide-ranging metaphor for developing historical interpretation. The Colonial Echo model is set mainly within social ecology, the Muddling Through model within physical ecology.

Ecology has developed within biology, being concerned with organisms' relations with each other and with their physical environment. It is particularly concerned with the way

in which individuals respond to *forces* operating within their environment. The BSE model looks at humans in their social and intellectual environment as well. There are important ecological principles which are well understood within biology, but which seem not to be well understood by some who apply the term to education. In particular, they fail to consider the forces which are operating. For example, Reed (1993, p. 46) sees individuals as acting autonomously *within* the environment rather than as *under* its influence, and Jardine (1994, p. 111) treats human ecology as a description of interpsychic and dialectical processes mediated by joint intentions. Some of these principles need to be considered here.

Ultimate and Proximate Factors

Ecologists distinguish "ultimate" and "proximate" factors causing biological behaviour (Baker, 1938; Immelmann, 1972). For example, breeding behaviour may be seen as being brought on by a proximate factor such as increasing hours of daylight or by an ultimate factor such as the desirability for the species to produce young when food and shelter are most likely to be abundant. Proximate factors may be linked with their response across a brief time span. Ultimate factors have a much larger time span. They are the environmental and evolutionary factors which have led to the mechanism which selects those proximate factors which are advantageous to the individual.

This approach may be applied to education. Our current rapid changes result in part from the development of the computer and its related technology over the last 50 years. The development has not been an overnight phenomenon; there have been quantum leaps, like the design of the Macintosh, but they have taken time to consolidate. It is very hard for individuals or small groups like schools to place themselves accurately within such a rapidly changing world. Their decisions are subject to much more proximate forces like the funds available to purchase technological equipment and expertise. Everyone is moving in the same direction, but the rate of progression is subject in a significant way to proximate factors.

Optimisation criteria

A second concept of importance is the criteria used for establishing optimal behaviour. In zoological terms the term "best" is often applied to behaviour which requires the least effort to obtain adequate food or shelter, and this approach seems to be a helpful one for education where resources are always scarce. So in this thesis, society is seen as an organism whose individual members respond to a variety of pressures in ways which are practicable, in the best interests of those individuals, and in many cases in the best interests of society as perceived by those individuals. The BSE approach extends the meaning of "ecology" from one concerned only with the physical environment to one which examines how natural forces (often the results of limited resources) operate within a social *milieu* (such as respect for education) and an intellectual *milieu* (where content and pedagogical knowledge of teachers and administrators are important) in such a way that individuals and society are sustained and improved. This will often not be easy to assess, because what may be best for the group may not be best for the individual and because the range of forces operating will not always be clear.

The Principle of Convergence

In zoology, it is quite common to find unrelated families of animals which are very similar in appearance. Evolutionary theory argues that similar environments have led to a convergence of appearance quite independent of the families' different genetic background. In education genetic transmission does not occur, so the parallels with zoology are not straight-forward. But it is reasonable to assert that if similar educational practices are observed in different countries, then the possibility that they arise from convergence, rather than cultural transmission, should, at the least, be examined.

An Example from the Introduction of Probability into Australian Schools

General Principles

So, when discussing the teaching of probability it is necessary to examine the physical environment within which schools are set, the social culture operating within them, and what was known about probability at the time under discussion. Issues such as time allocation, class size and available equipment represent the physical forces. Issues about the nature and purpose of teaching and assessment, which will reflect the differing perceptions of politicians, teachers, administrators, children and parents, represent the social forces. And understandings of probability and its pedagogy held within different sections of the academic community constitute the intellectual forces.

There are, of course, strong links between the three sets of forces. Teachers can only teach what they know. For probability, most teachers initially knew little, so the links between classroom practice and the academic community were especially important for the ways in which the topic was treated. Physically, the large classes and very limited resources in schools in the 1960s also influenced what could be satisfactorily done in a classroom. The argument that understanding probability would be a social asset to children was a new social force, one not often well understood. To understand change it is necessary to examine more than ability and organisation: "wider factors such as resources, status and legitimacy" are also important (Openshaw, Lee & Lee, 1993, p. 145).

A Specific Example

The General Environment: As an example, let us consider the period in the late 1960s when probability was first taught as a major topic in Australian schools. For some years there had been very strong advocates for the teaching of probability, because stochastic thinking was becoming increasingly important in science and industry, and because a clear understanding of chance was seen as an important part of a general education. These views were strongly presented at the 1959 Royaumont Seminar (OEEC, 1961) and at the Cambridge Conference on School Mathematics (1963). Intellectually, probability had become so well understood that it was able to be used constructively in many fields. Its consequent advocacy as a subject suitable for schools marked a significant change in the social forces operating on the mathematics curriculum.

Applicable knowledge spreads rapidly across the world within the academic community, so the intellectual changes were well understood by Australian academics, but they bypassed the majority of Australian secondary teachers, who were working under many difficulties and interested primarily in what went down well in the classroom (UNESCO, 1965). From their point of view, responding to physical forces was their most urgent task. Few had studied probability, so their knowledge was limited. Nor were there strong social forces operating within teaching which saw probabilistic understanding as a social asset. Such forces were present for statistics, which was seen both as an applied tool and as easy to teach in a classroom, but the intellectual links between probability and statistics were poorly understood.

Furthermore, there was no established pedagogy for teaching the topic. The most significant work had been done by the American College Entrance Examination Board (1959), but there had not been time for this work to be refined in practice and modified for wider use. So when the intellectual forces were successful in arguing that probability should be part of the Australian secondary mathematics curriculum, the topic was placed into a physically difficult school environment and an impoverished social one.

Solving a Specific Problem: Did Australians solve this problem by meekly copying overseas practice? The answer must be "No". A wide variety of approaches specifically designed for Australian schools was developed, some of which are discussed here.

The Canberra Mathematical Association released a small work on probability for teachers (Neumann, 1966) which was seen as the best available at the time (Newman & Wall, 1974), even though its approach was far too academic for ordinary students and teachers. In Western

Australia Hume (1966) developed a course for Year 12 based on CEEB (1959). The book was practical, and attractively presented, but the modifications required were found to be far more than anticipated (Hume, 1970). Hume later felt that the level she wanted to reach could not be achieved until here was a higher level of teacher knowledge (Hume, 1970). South Australia had started to change its academic syllabus by including statistics as early as 1964, well before the main movement for change in Australia (University of Adelaide. Public Examinations Board (1966). Soon after, a committee of teachers, administrators and academics decided to use the SMP textbooks, which included probability, as a basis for classroom work because these were believed to be the best available (Baxter, 1972). The books were found to be too hard for the students, and were soon replaced by locally produced material. At least two quite different approaches to teaching probability were taken in Victoria, accompanied by considerable debate, and some acrimony. The official approach was to emphasise formal structures, and Preston & Watterson (1972), two academics, took this to extremes. Lucas & James, two independent school classroom teachers, started to produce more straight-forward texts from 1967.

Discussion

There were several reasons why Australian mathematics education had to change in the 1960s. These included changing school clientele (physical forces arising from social changes) and changing views about mathematics (intellectual forces). The introduction of decimal currency in 1967 freed up the mathematics curriculum, and meant that change of some sort had to occur, and provided an opportunity for major restructuring.

When we examine the wide variety of responses to these changes with respect to probability across Australia it is hard to see how a Colonial Echo theory is tenable. The sheer diversity of responses alone means that it cannot be argued that any copying which did take place was automatic. At the least, each State had to choose from many competing messages. And we must presume that the administrators who made these choices were responsible people trying to do a good job. While there is some evidence that they had limited perspectives, and perhaps an eye to profits from textbooks, there is no evidence that they deliberately tried to foist poor material onto the schools.

It is true that some of the states made a conscious decision to use overseas materials, particularly South Australia. This State had been cautious about change, but when decimalisation demanded change a decision was made which was informed by years of deliberation, which could be defended as an efficient use of scarce resources and which involved the use of what was probably the best and most attractive course available. It was not "Muddling Through". Judged on the physical and intellectual forces operating at the time the decision was essentially a good one though it paid inadequate attention to the very academic emphasis of the project. Neither the teachers nor the students were able to adjust to the academic open-ended style. Social forces saw it replaced by an approach which conflicted less with the existing classroom environment.

The same may be said of many of the other innovative approaches described above. Hume's project and Neumann's text were too hard for teachers. Preston & Watterson's text was too formal for the classroom environment of the time. The environment could have been changed: after all, European classrooms are much more formal than ours, but this would have required far more energy than was available.

In the long term, the approach to probability in most parts of Australia ended up being remarkably uniform. A definitional, pure mathematics approach. After a short practical introduction, the main emphasis of the courses centred on simple calculus of probabilities in a concrete setting. This uniformity is a classic case of educational convergence. A deterministic approach to probability fitted best with the social and physical forces in classrooms at the time without breaching intellectual rigour.

May I be forgiven for quoting an example from personal experience of how the optimisation principle operates in practice. In 1969 I was teaching at an Australian school which used its own internally written textbook. The approach to probability was formal and

abstract, and was not popular with staff or students. I had had experience with using a more concrete approach to the same material as a result of my time with SMP and it was easy to persuade the department to revise this aspect of the course. This was not echoing, it was making use of pedagogical knowledge (within intellectual ecology) to propose a more efficient way of communicating an idea. Given approaches all of which are academically acceptable, it is natural to choose the one which goes down best in the classroom.
This discussion has concentrated on proximate forces. The ultimate force operating is the increasingly data-driven nature of our society (Hacking, 1990). In spite of this the long-term position of probability in schools is not yet secure. So a study of probability in schools is especially able to illuminate the uneasy relationship between proximate and ultimate forces on the mathematics curriculum. This work remains to be done.

Conclusion

Crombie (1994, v. 1, p. 69) calls the process he uses "historiography"—"an art resembling natural science both in its commitment to searching for a particular and general truth through the control of evidence by critical scholarship, and in the design and presentation of its arguments in order to convince and persuade". I would argue that, for the issue discussed here, the Broad-Spectrum Ecological framework is more able to convince and persuade than the other frameworks discussed here, and so deserves more detailed examination.

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Understanding of Association and Regression by First Year Economics Students from Two Different Countries as Revealed in Responses to the Same Examination Questions

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This paper addresses the broad issue of relating research findings with pedagogical practices by analysing the responses to questions set in an undergraduate statistics examination using Eisner's connoisseurship and criticism approach, supported by general pedagogical and psychological principles. Comparisons are made between responses to the same course given in two different countries to assess similarities, differences, and weaknesses in order to indicate possible ways in which future courses might be modified to improve student learning.

The Purpose of this Investigation

There is currently an agreement between the University of Adelaide, Australia, and the Sepang Institute of Technology, Malaysia, which permits Malaysian students to prepare and sit for examinations at the University of Adelaide by attending lectures and tutorials at Sepang. One such course currently available is "Business Data Analysis" (BDA) which is a compulsory first-level course in basic statistics undertaken mainly by students studying for a Bachelor of Economics or Bachelor of Commerce degree. Currently it is possible for Malaysian students to take the first two years of their degree at Sepang, after which they come to Adelaide to complete their studies and receive an Adelaide degree.

As far as possible, the course offered in Sepang matches that in Adelaide. For the semester considered in this paper the Adelaide lecturer in charge of the whole course kept a firm hand on all aspects of the course, her notes and instructions to both students and staff were detailed and precise. The Malaysian lecturer received all of these notes as a basis for his lectures, his students had the same amount of contact time, and did the same exercises, assignments and examinations. Tutorial exercises, which counted for the final assessment, were marked in Malaysia, but special assignments and examinations were marked by the teaching and tutorial staff in Australia.

In second semester, 1996, a total of 272 Adelaide students and 88 Malaysian students sat the final examination. Observation of the author's three tutorial groups suggests that the Adelaide class is fairly diverse, with representatives from most major cultural sub-groups in South Australian society and with about 10% of its members being student visitors to Australia, mainly from a number of different Asian countries. Little is known about the Sepang group, but all the students have Chinese names, so it may be presumed that they are predominantly Malaysians of Chinese origin.

Such an arrangement provides about an excellent opportunity to compare the approaches of two quite different groups to an almost identical course in order to see if there are significant differences in understanding or approach which need to be considered in designing the presentation or examination of the subject. In order to match findings with other work it was decided to investigate similarities and differences in students' responses to those questions in the final examination which were concerned with the ideas of association and linear regression.

Methodological Principles

Analysis of examination scripts is rarely reported by researchers. It has limitations, including an inability to follow up answers in more detail, and, as Lipson (1994) has clearly shown, a tendency to assess product rather than process learning. On the other hand, students have a strong motivation to present their best work, so it may reasonably be claimed that the answers provided represent a good estimate of what they consider to be relevant to the questions as presented. Furthermore, questions tend to be precisely

formulated and clearly stated, students know that the marking scheme is additive, so there is no penalty for incorrect statements, and usually they quickly discover that they have adequate time to answer the questions thoroughly. For the researcher the method is time-efficient, and is relevant to classroom practice because it has the potential to improve both the teaching and examining processes.

On the other hand, it is not possible for a researcher to obtain further information to clarify what has been written, or to assess by alternative approaches whether similar responses from different students really do represent similar understandings. There may well be inconsistencies within an answer which may represent either a slip or an important misunderstanding. And it is often easy for phrases learned merely by rote to give the appearance of sound understanding. This is of special importance in a subject like BDA where much of the work comprises routine processes and routine interpretations.
But the principles of textual analysis have been developed to cope with just these situations. They have, however, rarely been applied in mathematics or to examination scripts. Clearly such analysis needs to be done cautiously and with a clear sense of the provisional nature of any conclusions which are drawn. The approach employed here follows Eisner's (1985, ch. 6) proposal for using *connoisseurship* and *criticism* in order to make helpful evaluations. Connoisseurship he sees as an appreciative and very personal art; criticism as a way of describing what is present so that others may enter into the work, and make their own judgements. In this paper a mixture of systematic elementary data analysis and broad mathematics education principles are used to examine the data in order to determine which aspects of the teaching of one main topic have been successful or unsuccessful, and to provide a basis for subsequent modification of the course. In other words, both qualitative and quantitative analysis are seen as relevant.

Such an approach fits within the evolutionary model of curriculum change proposed by Popkewitz (1988, pp. 242 - 244) which has limitations, but can be very efficient for implementing minor changes quickly, and for strengthening good practice. It is of special value when limited resources particularly encourage the use of time-efficient forms of assessment like written examinations and is able to estimate quickly the discriminatory power of the questions and to suggest appropriate improvements.

The Course

The course concerned, "Business Data Analysis", is a compulsory semester-length course, taught by one lecturer for 2 hours per week. Each student should attend a weekly one hour tutorial class usually with about 10 participants, taken by a number of tutors, of whom this author has been one. In practise about one-third of the Adelaide students do not attend tutorials. The figures for Sepang are not known. Tutors mark assignment questions, and discuss work which the students are expected to prepare for each tutorial. Basic principles and expectations are reiterated many times in lectures and tutorials. Students are encouraged, both verbally and by the mark schemes employed, to memorise appropriate language for reporting statistical interpretations.

Most of the teaching staff concerned with the course do not have any formal training in education, and their pedagogical principles tend to be traditional. This does not imply that they do not do their job well, but rather that they see their job as the transmission and retention of information and skills. The constructivist position that there is no mathematics "out there", but only in the mind of the learner, would not be understood by most of them. So the analysis in this paper cannot assume technical understanding of educational theories of any form. This makes the connoisseurship approach particularly appropriate, because if the analysis is to be of any value it must convince intelligent people who are "outsiders" to modern educational thinking, while remaining consistent with sound principles of mathematics education.

One aspect of the conflict between constructivism and traditional approaches is of special importance here. Within the constructivist position there is no such thing as a "misconception", but within the transmissionist position misconceptions are one of the most important features that a teacher needs to identify and try to eliminate. The position taken in this paper is that misconceptions do exist and it is useful to identify the existence and prevalence of significant understandings and misunderstandings by students.

Methodology

Many misconceptions are not widely held, but are still important, so aberrant responses are of special interest. Therefore it was decided to examine a relatively large number of scripts. In Adelaide 272 students (90% of the enrolled students) sat the examination and half of these scripts were analysed, selected in chunks from the full alphabetical ordering more with regard to physical convenience than to any other criterion. This was considered adequate for the precision required by the research. All 88 Malaysian scripts were analysed in the same way as the Adelaide ones.

All the data were entered onto an electronic spreadsheet. For each part of each question the different responses from students were listed on the horizontal axis. The electronic spreadsheet approach allows the headings to be built up as the scripts are read. Clearly, the greater the number of scripts the greater the number of different responses there is likely to be. Even so, building up the responses is not easy, because it may not be obvious at the beginning that certain aspects of the responses are of special importance. It will be seen below that on a couple of occasions inadequate data were recorded. Given the very limited amount of previous research done into these topics, this is not surprising. On the vertical axis a code for each student was entered. Each cell in the table was filled in with either "1" or "0" or "blank", indicating a correct, incorrect, or absent response, respectively. This approach is consistent with the reality that new classifications are being added to the table as the data set is being built up.

The relevant parts of the one question examined are presented here. While no choice of questions was permitted, it was still clear that this question was popular, because it was frequently attempted early in the examination, and few students seemed to be under time pressure when attempting it.

Question 4 presented a summary of information illustrating a relationship between the age of a motor cycle and its resale value, and asked students to calculate measures of association and the simple linear regression. Formulae were provided; the calculations were usually done correctly. The following parts are analysed here (sections in italics were not examined but are included here to clarify the context of the parts which were considered):

(a) *Calculate and* interpret both the correlation coefficient and the coefficient of determination.

(c) Interpret both the intercept and slope of [the regression equation you have calculated].

(d) Predict the selling price of a motor cycle that is
(i) 3 years old
(ii) 6 months old.

(e) What reservations, if any, do you have about prediction (i) and prediction (ii) in part (d).

Two major techniques of analysis were employed. The first involved simple analysis of the relative frequency of each response encountered. This was useful for identifying the popular responses, and weak areas in the group's responses as a whole. This, together with experience gained while teaching the course, was helpful in indicating where responses tended to have been rote learned, and in identifying subtle differences in students' understanding of similar ideas. The second major technique used electronic copies to re-order the columns so that similar ideas in the same question or in different questions were adjacent. The data could then be sorted electronically into numerical order by row entries (1, 0, blank) to identify, for example, which students had a "1" entry for, say, all the similar ideas, or for various subsets of the ideas.

Previous Research into Understanding of Association and Linear Regression

The concepts of statistical association and linear regression are fundamental to statistical thought. Calculating regressions is straight-forward, but my experience is that the underlying ideas sometimes prove difficult, a view supported by Franklin (1988) and

Tamura (1994). A systematic search of *Teaching Statistics* and *Journal of Statistics Education* found a number of articles presenting interesting ways of teaching both regression and correlation (Goode & Gold, 1987; Laviolette, 1994; Wilkie, 1980). These suggested, either implicitly or explicitly, that using such ways would improve students' understanding of the concepts, but they did not specifically identify students' difficulties or misconceptions. The emphasis was on *teaching,* rather than on *learning,* statistics. Indeed, there has been little done in this field (Batanero et al., 1994, p. 538; Shaughnessy, 1992; J. & K. Truran, 1996). It has been observed that students often have difficulties in interpreting a single measure of correlation, especially in elementary courses (Hawkins et al., 1992, pp. 51 - 56). Students' understanding of correlation as represented in contingency tables has been analysed in detail (Batanero et al., 1996) and tertiary students' understanding of the significance of *r* has been analysed briefly and somewhat imprecisely (Truran, 1995) using a method similar to that described here.

Results

The purpose of this survey is partly to inform the teaching of the BDA course in subsequent years, so the results presented here consider the most striking differences between the two groups and/or those situations which most clearly indicate important misconceptions which need to be addressed in teaching the topic. Where percentages are given as ordered pairs, it is the Adelaide figures which are given first.

Question 4 (a) - Interpretation of r

The correct value of *r* was -0·57. Students were expected to indicate that this indicated a moderate, inverse, linear relationship between the variables. Only correct responses for *r* were analysed further (84%, 89%). Of these, almost all students appreciated the significance of the negative sign (93%, 97%). A significant minority of both groups failed to observe that the relationship was linear (27%, 24%) In both cases the differences between the groups were small.

The importance of stating that the relationship was both inverse and linear was emphasised throughout the course, and in general both were treated together. The striking difference between the students' willingness to state both of these needs to be considered. While we have no evidence about how these students thought, it is possible to use principles from mathematics education to explain the discrepancy. Skemp (1971, ch. 2) has argued that in the development of a concept it is important to be clear about its boundaries. Since the students have met examples of both positive, negative and zero association between variables, they have been able to see the importance of this classification. But the students have met only linear associations in any detail, and have not encountered the importance of deciding whether such a model is appropriate at the beginning.
The difference in success rates for two items treated together indicate that emphasis in class and rote learning do not necessarily produce correct responses in examinations, and this effect seems to be similar across both groups.

Assessing the strength of the relationship requires and expression of an opinion, and there were definite differences in the groups' perceptions as may be seen in Table 1. (Responses like "moderate to weak" were classified by the first term stated.)

Table 1
Verbal Meaning for *r* = -0Æ57

Response	Adelaide	Sepang
Strong	0%	8%
Fairly strong	4%	6%
Moderate	61%	47%
Weak	25%	22%
No term used	10%	17%

In so far as any language has meaning across cultures, the Sepang students tend to see the relationship as stronger than the Australians do. This result shows that vernacular terms are unreliable instruments for describing the strength of a relationship. The cross-cultural differences and the fact that no student indicated considering the number in the

sample means that there is a need to provide students with a less individualist way of making judgements about correlation. A significance test is one obvious approach. Finally, we may note one set of responses which came only from Sepang. It seems that these students were alerted to the provisional nature of their judgements and advised to use the word "possibly". A number of them did. However, rather than writing something like "the relationship is possibly a moderate, negative, linear relationship", in several cases the students transferred "possibly" to another position which led to statements like "the relationship is a moderate, possibly negative, linear relationship", even though the inverse nature of the relationship is in no doubt. Character based languages like Chinese do not provide as many clues within their construction about the relationship between the ideas embedded in each character. It is not possible to tell whether the students concerned had a problem with English or a genuine misunderstanding. However, it is possible to note this issue, and to pay special attention to it in future with any students for whom English is a second language.

Question 4 (a) - Interpretation of r^2

Since the *interpretation* of r^2 is not affected by the initial value of r all consequentially correct answers were accepted here. Using this criterion, most students did the calculation correctly (88%, 99%). Of this large difference in what is essentially a button pushing exercise, 4% is explained by Australian students who did not obtain any value for r in the first part of the question and a further 4% who obtained a value for r but did not attempt to calculate r^2.

Inadequate attention was paid to some aspects of this question when collating the Adelaide data so a direct comparison is not possible here. Only 58% of the correct respondents stated that r^2 showed that 32% of the variation of Y was explained by either X, the variation of X, or the regression (all three forms were accepted as correct for the purposes of the examination). The Sepang responses were much more uniform. Of the correct respondents 76% believed that r^2 showed that 32% of the variation of Y was explained by one of these three features—X (15%), variation in X (59%), the regression (2%). The Sepang responses suggested significant rote learning.

The Australians provided a wide variety of other responses: 4% believed that r^2 measured the percentage of explained *change*, 23% that it measured the proportion of the *time* that concurrent change occurred, 6% that it measured the probability that the variation could be explained. In contrast, there were very few other responses from Sepang—the most common was concurrent change (3%). However, 10% gave no response at all. The concept of the coefficient of determination is not easy, especially when, as in this course, it is not presented as the ratio of the explained sum of squares to the total sum of squares. It is not clear why there were quite different profiles of "other" responses. But even though the Sepang responses suggests a stronger element of rote learning than the Adelaide ones, the significant number who gave no response suggests just as strongly as the very varied Australian responses that this concept is generally not well understood.

Questions 4 (c) and (e) - Interpretation (and Reservations) of Predictions of Intercept and Slope

While part of this question is concerned merely with calculations, it also provides some indication of the extent to which the students perceive regression to be a deterministic, rather than a stochastic, concept. Research with undergraduate New Zealand students has found them to have "a tendency towards overly deterministic thinking", which "reflects their lack of awareness or understanding of variation" (Pfannkuch & Brown, 1996) and some Spanish students believe that association is a deterministic concept (Batanero et al., 1996, p. 166).

The standard form of the regression equation used in the course is $\hat{Y}_i = b_1 + b_2X_i$ where $\hat{Y}_i$ indicates the estimated value of Y_i. For this question the correct results were $\hat{Y}_i = 2096 - 129\ X_i$ where the units are in dollars. The calculations were usually done correctly (85%, 84%). There were several places where students could indicate their understanding of the stochastic nature of the regression, and these will now be discussed.

One measure of whether students appreciated that this equation is a stochastic equation can be gained by seeing whether they indicated, by using either the "cap" or words, that this equation provides an estimate of Y_i. There were weaknesses with this approach because some of those who did not include the estimate idea in their original equation did use it at other stages in their interpretation, and this was not recorded during the analysis. However, the percentages which did include the estimate idea at this stage (60%, 70%) suggest, at the least, that the significance of the symbols used in the equation has been overlooked by a significant minority of both classes.

In interpreting the coefficients of the regression equation some students did use "estimate language", and not all of this was recorded. However, one form was recorded which is particularly interesting. Some students used the term "on average", as in "the selling price of a motor bike will decrease by $129 per annum on average". This is of course reasonable. But of the 21 Australians who used the term for one parameter only, 16 applied it to the slope, 5 to the y-intercept. Of the 11 Malaysian students who used the term for one parameter only, 5 applied it to the slope, 6 to the y-intercept. These results are strikingly different. One might hypothesis that the slope, which extended over a long time period was more likely to be seen as an average than the y-intercept, so the Australian figures are not surprising, but it is not clear why the Sepang results are so close. The strongest differences between the groups arose in discussing their reservations about the regression. Of those who correctly calculated the regression, most (63%, 85%) were well aware of the dangers of extrapolation, and indicated in some way that the estimate for six months (question 4 (d) (ii)) was outside the range of the data. (It is true that an expression of reservation about the y-intercept was not asked for, but no-one who was concerned about extrapolation to 6 months expressed any concern about extrapolation to 0 months.) The difference in percentage is quite strong. The Sepang data suggests that this important point was strongly emphasised and heard by the students. But it was strongly emphasised in the Australian course as well. It is not clear why so many Australian students did not hear it.

But the Australian students were far more willing to express other reservations. Relatively many more expressed concern about the low coefficient of determination (45%, 28%) or the small value of the sample (17%, 7%). The examination paper stated that there were three marks for this part of the question, which strongly suggested that there were at least three reasonable grounds for having reservations. So all of these results are disappointing, especially the Malaysian ones. It is possible that the traditional way of dividing questions into clear parts, which makes marking very much easier, discourages students from seeing the context as a whole. It would be interesting to see what differences arose if different members of the same class were presented with two different versions of the same question—one balkanised, the other holistic.

But on another matter the results were even more disappointing still. Very few students (9%, 5%) questioned the assumption of linearity. It is difficult to believe that so many young adults are unaware of the distinctly non-linear nature of prices for used motor vehicles. Nor did students comment on the unrealistically low value of the motor bikes. Freudenthal (1991) and Schoenfeld (1988) have argued that students merely play the teacher's game, conforming as closely as they can to what they believe the rules to be, regardless of "common sense". They hold to this position even when they are strongly encouraged to question the reality of the problems they had been set.

It is possible that gender is relevant to this issue. Of the 11 students who questioned the reasonableness of this question, 10 were male, and the one female was a mature age student with substantial self-confidence, and a particularly determined approach to her studies. Ability does not seem to be relevant: the final percentages of those who had reservations were fairly evenly spread between 29 and 94.

For the Malaysians there were cultural issues as well. What interpretation could Malaysians bring to the question when the prices were expressed in a foreign, unspecified currency? The significant minority of Malaysians who did offer other reasons suggests strongly that the other reasons and the importance of contextualisation were discussed in classes, but the students seemed unwilling to offer such answers. Unfortunately, because gender cannot be deduced from Chinese names, it is not possible to see if there was a gender bias in the responses.

Conclusion

The connoisseurship approach presented here which links exploratory data analysis, pedagogical and psychological theory, and critical analysis is able to provide a deeper understanding of the students' understanding of the concepts of association and regression. What follows is a personal appreciation of the question and answers which tries to combine systematically a value judgement with a description which might help others to view the students' understandings in a different light.

The analysis suggests that the question was straïght-forward and reasonable. While some marks could be obtained by accurate calculations, and others by rote learning, others required deeper thinking, and, perhaps more importantly, an appreciation of the whole problem. It is possible that the fragmented structure of the question inhibited this overall view, and it is also possible that deeper investigation of some ideas, such as "variation" and "linear" would have exposed more misunderstandings than were apparent. Most interesting of all is the strong finding that even where there is substantial evidence of general rote learning among the class, a significant minority still do not succeed.
The analysis revealed both similarities and differences between the two groups. Some differences probably arose from linguistic difficulties experienced by the Sepang students, but others probably represent cultural differences. In general, the Sepang responses were more uniform, and less critical than the Australian ones. And it was only a small minority of Australian males who were inclined to be critical.

In terms of marks the Sepang students did better than the Adelaide ones overall. To some extent this was caused by their high computational accuracy. But this analysis shows that the situation is more complex than that indicated by marks, and identifies areas where both groups could benefit from further help. Such an approach is labour-intensive, but is efficient in isolating areas of importance. In particular, has the potential to focus teaching emphasis where it is most needed. Also, it is possible to make more deductions about process learning for the group as a whole than Lipson (1994) has suggested, though not for individuals. Given that more and more teaching and examining is likely to be done in groups for economic reasons, this is not a serious disadvantage. Finally, the responses make one question whether the traditional approach of dealing only with linear association in a first-year course is educationally sound.

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Beliefs About Teaching Stochastics Held By Primary Pre-service Teaching Students

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This paper explores some observed confusions held by pre-service teachers about concepts of probability and statistics. The writer uses information about confusion and misconceptions held by pre-service teachers gained by examination of teaching assignments written by her tertiary students. It considers some other research in this field and makes some suggestions about what steps may be taken to provide pre-service teachers with a better understanding of stochastics.

Introduction

The primary pre-service teaching course at the University of South Australia uses the *National Statement on Teaching Mathematics for Australian Schools* (AEC, 1991) and the related documents *Mathematics A Curriculum Profile for Australian Schools* (AEC, 1994a) and *Mathematics Work Samples* (AEC, 1994b) as major references. They are used because they provide common planning documents for students and the teachers who will supervise the practicum.

The mathematics curriculum subject offered to these students is of the 'pressure cooker' variety—only 100 contact hours over a four year period with practicum experience spread throughout the course. The emphasis of the subject is on childrens' cognitive development, and the mathematics of the primary curriculum as set out in the National Statement and associated documents. Students are encouraged to use a wide range of teacher reference books for additional lesson ideas.

Methodology

This paper examines the methodology used with the above group of students and discusses some of the difficulties that these students, who were in the second year of the course in Semester 2, 1996 experienced. They spent nine weeks in lectures broken half-way by four weeks on practicum. Students were required to prepare lesson plans for a mathematics topic which they would then teach during the practicum. This planning was generally done with the support of their supervising teachers. The task required students to plan, teach and then to reflect on a series of four lessons. It is this task that will be discussed in this paper, because such responses indicate important beliefs by students about the teaching and learning of Chance & data.

Of the group of 180 students, 23 chose Chance & data as their topic and it is from these 23 that the examples in this paper are taken. Five sets of lesson plans referred solely to the *National Statement* and *Curriculum Profiles* as source material while the remainder also referred to other linked documents or related references.

The students, however, found these documents very difficult to work with overall and the Chance & data strand caused particular difficulties. Russell & Mokros (1990), Shaughnessy (1992) and Greer & Ritson (1993) have all reported similar difficulties by students in other systems using similar curriculum documents. The majority of students used the outcome statements from the *National Statement* and the *Profiles* for their lesson planning. Both are seen as authoritative documents to be followed to the letter. In an analysis of the documents we proposed that '[O]utcome statements emphasise one aspect of the learning process. They are absolute and isolated. They do not present each outcome in the wider context of a child's mathematical thinking.' (K. & J. Truran, 1994). The drawing together of individual aspects of mathematics into a cohesive whole is a long (cognitive) process which many children will not comprehend before they finish primary school. Students' comments, my observations and other research, for example (Green, 1982) suggests that this difficulty does exist and I shall argue here that it extends to pre-service teaching students.

Student Background

The majority of the students discussed here have a limited mathematics background. Some claim to have completed Year 12 mathematics; however, the majority of these have studied Business Mathematics which should be called 'Business Arithmetic', based as it is on simple calculations involving number, percentages and decimal fractions. Very few students have studied the traditional, academic Maths 1 and 2 courses. Therefore, pre-service students come to this subject unsure of their own mathematics knowledge and with strong misconceptions about topics like Fractions, Algebra and Number. A few openly profess to hate mathematics and related subjects.

Furthermore, the university's policy of exposing all teaching students to both junior and upper primary classes was viewed with alarm by some, and by real fear by most of those students who had intended to specialise in junior primary teaching and had made this decision because they viewed maths for young children as being 'easier'. Southwell (1995) claims that both cognition and affect are factors in mathematics learning, and points out that if a student feels to be lacking in the necessary ability to achieve good results in mathematics or does not consider mathematics to be very important then this student will not feel confident enough to try to achieve good results. One student who confessed to having taught only one mathematics lesson during her 4 week practicum was delighted by this and told me about it very proudly.

Edwards (1996) is carrying out a survey among pre-service primary teachers in New Zealand in respect to teacher knowledge and confidence in teaching statistics. He observes some anomalies in his study between teachers' perceptions of their own statistical ability (or lack of same) and their high teaching confidence. In fact, in his study, few saw their lack of ability as a problem for their students. I have also observed this among the students discussed in this paper, who claimed, among other things that because the outcome of random events is non-deterministic then the approach to teaching probability doesn't matter and the children will 'pick it up sooner or later anyway', others believe that asking children to*predict* an outcome is sufficient criteria for teaching probability. The non-deterministic aspect of probability caused students other difficulties. Many stated that of all the topics Chance is hardest to teach because 'there usually wasn't a right answer'. Throughout the workshops before their teaching practicum some students maintained their view that there are no right or wrong answers in probability, and everyone's contributions are valid. Therefore, (by implication) the kinds of questions used don't matter either. Much of the literature on teachers and statistics education highlights concerns about the lack of 'preparedness' for teaching statistics. Greer & Ritson (1993) believe has its origins in teacher training. The move from lack of training to difficulty in teaching statistics is readily made and assumed by many researchers in statistics education. Hawkins (1990, p. 24) goes so far as to compare the lack of knowledge with a 'disease' which teachers pass on to their students. I now look at details in an attempt to effect a remedy.

Discussion of Results

The following examples demonstrate some interpretations of teaching stochastics by these students. My results examine not the general but the specific in the context where change is still possible by those involved in both pre-service and in-service training. Some suggestions as how that might be done are made in the conclusion of this paper.

Limitations of Formal Documents

There was evidence that the students had not considered the implications of the mathematics of the lessons that they had planned in relation to the age of children in their class and the children's existing experience or pre-knowledge. Many challenging follow-up activities which could have encouraged the children to question their beliefs about chance were not taken, generally because they were not considered part of the lesson planning or because the student had not realised how long preceding tasks would take.

Unfortunately most students saw the activities that they had planned as the basis for a single lesson (even thought they had to plan four), not as part of an ongoing sequence with potential for development of further mathematical ideas.

For example, EG (name coded) had children consider dice being used in a game of Snakes & Ladders and used questions such as:

> Is there one number that is hardest/easiest to get?
> How long does it take to get a six?

Both of these questions could have provided some interesting starting points for discussion and further work; neither occurred, and there was no evidence of follow up lessons on this theme. SN also used Snakes & Ladders and asked some particularly imaginative questions:

> Are there more snakes than ladders, if so is this fair?
> Do you have a lucky number?
> Is there a particular number that is easier to roll?
> Is it possible to finish the game without going up/down a snake or ladder?

Some of these questions could have provided a basis for very useful investigations. However the answers were summarily dealt with and the opportunity for a challenging investigation lost.

Some students did not consider the mathematics of probability. LI played a game by drawing multi-coloured marbles from an urn with no replacement, it was intended that this game would last for a 20 minute lesson. Her class consisted of 6-8 year old children and LI had not analysed the difficulty of the mathematics involved in non-replacement. After each draw the probability of the remaining colour changes, presenting the participants with a different probability situation. She had not experimented before the lesson, and, because she had found the suggestion in a resource book, accepted it without question. This lack of preparation is of course not limited to teaching probability, but LI's lesson was disastrous, with the game being over in a few minutes and she was left with no idea what to do next.

It is not just the curriculum documents that create such problems. A common reference for primary teachers Griffiths & Clyne (1988) uses children's story books as a way of integrating mathematics and the language curriculum for primary teachers. From one story, 'Phoebe And The Hot Water Bottles', a suggested activity is that the children should draw graphs of the presents they get when they have been very good. The graph illustrated for this activity shows a class graph of their pets made by some children in one of the trial schools. There is no information other than the heading. The scale used is unclear and appears to be inaccurate. This type of reference only serves to confuse students more.

Probable/improbable events

In *Mathematics - a Curriculum Profile*° (p. 62) is the suggestion that children 'order a few easily understood situations from least likely to most likely, e.g. 'my teacher will come to school with green hair'. Ortiz de Haro (1996, pp. 60 - 62) points out that such activities cannot be repeated (as can the tossing of a coin or die) and are therefore not caused by randomness. This is a matter for concern because such activities are suggested in a majority of teacher references.

Using the *Profiles* led some students to compile lists for discussion with the class of probable or improbable events, and frequent examples of the 'I will come to school tomorrow with blue/ green/ purple hair' *genre* were observed. In using this type of activity students suggested as many patently impossible situations as possible:

> Tomorrow will be Christmas day;
> Martians will land on the school roof;
> If I roll a dice the number will be forty-seven;

it is equally important for students to consider situations that are likely, possible, or certain. These are equally realistic outcomes of a random event and must also be presented for consideration and understanding by children. It is important for example, that children are clear about the difference between, likely, possible and certain. Questions like, 'if we

rank events on a line with 'impossible' at one end and 'certain' at the other, where would events that are likely, possible or certain fit'? Questions like this need to be presented, the need to categorise random events in this way was not considered.

SN who integrated her planned mathematics lesson with a topic on 'weather' examined weather reports from the newspapers and television. Weather reports from the previous day were examined by the children in the class who were asked questions like, 'How did they [weather forecasters] know it would not be hot today?' and were then asked to predict the chance that 'it will be hot tomorrow and we will go swimming'. This lesson was taken in the middle of an Australian winter, but the illogicality of the questions used was lost on this student. She considered that asking children to *predict* was what probability was about and therefore considered the lesson and the questions to be good.

Dice

Dice as random generators are preferred by a majority of teachers and students: dice are familiar and comfortable and students feel that here, at least, is an object with which they can feel in control. It may be seen that here that there are elements of language and prediction involved, factors they have already identified as being important in teaching probability.

Many students when justifying lessons with dice as the basis, quoted the Overview statement of Chance & data in Band A, (AEC p. 165) which begins with the statement:

> During the early primary years children should carry out experiments which involve chance processes (e.g., toss a die) and examine the outcomes; that is, they should try an experiment and note the 'results'.

While some of the settings for lessons chosen were imaginative, the questions and experiences offered to the children were limited and often lifted straight from references.

Activities that involved tossing dice and recording the results were frequently used, without any follow up activity or very much questioning. A show of hands to indicate who got which number was used by two different students. In both cases there was no indication as to how the resulting data would be used, and the next lesson moved away from using dice altogether and introduced another, unrelated lesson.

Games like Snakes and Ladders and Lotto were frequently used. In a study of young children's perception of random generators, Truran (1994) observed that 'the chance aspect of probability is seen as being related to games, and the use of 6 sided die, and this is the [teaching] approach taken in many classrooms'.

LJ wished members of her Year 1 & 2 class to 'investigate the meaning of chance and outcomes of one-chance experiments'. They were to do this by 'playing a game and investigating the outcomes of rolling a dice students will investigate the meaning of chance'[sic]. How this was to be done was not made clear, nor was LJ sure how to tell when the children had understood the meaning of chance. KF asked her class members aged 5 - 8 years to 'predict the outcomes' of tosses of an individual die. There was no indication of a follow up lesson or even discussions about the results of the tosses.

Sometimes dice were used as a basis for a data activity. The task that DH set asked that 'the students individually will roll a dice ten times and communicate the results in a graph shown by me'. SN asked children in her class to roll 2 six-sided dice and to record the results.

My suggestion that there might be other random generators or that others might be constructed were met with amazement. Students' own personal experiences are often limited to the use of a single, six faced die there comments often indicated that the use of a die would make the concept of chance clear to the children. OB had his class of year 6 students tossing pairs of dice; he was confident that 'by the end of this activity students should begin to have a healthy scepticism for situations involving probability'. It would seem that students consider that manipulating the objects themselves transmits concepts of chance.OB also considered that this activity would enable students to identify fair and unfair situations.

Data

Students were encouraged to concentrate on either chance or data and 7 chose to concentrate on data. It seems from this study that data is viewed with as much confusion and reliance on references as Chance. The most common confusion facing these students was what data collection is, how the information should be recorded and why you do it.

EL had considered The *Profiles* as a source for planning a series of lessons on Traffic for 11 year olds. Lesson 1 presented children with two similar, but not identical pictures of a freeway, and the children were asked to 'find the eight differences that exist between two pictures' which they were then to go on and graph. While this lesson may well have provided an example of collection of data, EL had not thought about how this data could be interpreted. EL's response to the children's confusion at this part of the lesson was his first indication that the question was inappropriate. He finished the lesson at this point, telling the children that 'they would do the graph tomorrow'. On the following day a new task was presented to the children and the issue as far as DK was concerned was over.

Another of EL's lessons involved children using a street directory to 'find the quickest route based on the information on a map like distances, speed zones, heavy traffic areas and stop lights'. EL explained:

> Irrelevant data will also be included so that students must analyse the data and evaluate what is required. Children will also use S=D/T formula for time calculations. Examples will be given.

It is difficult to link these activities to the concept of data. They eventually spanned 5 lessons and all proved too difficult for the children, EL was devastated as his attempt to 'teach data' fell into disarray. The worrying thing about EL's plans was that they were written in close collaboration with his supervising teacher who had obviously approved them for use with the class. This teacher did not provide EL with any real support at this time and DK was very angry about this when we talked about these lessons later.

Equally worrying were the lessons devised by WM. The aim of one of her lessons with 8 year olds was that the children 'represent, interpret and report on data, and answer questions posed by themselves and others'. (AEC, 1991 p. 172). In order to do this she set a task for children working in groups of four, with each child working from a different supermarket price sheet. They were given a list of ten products which they were to find on their individual sheets and to then graph their results using a column graph. It was intended that at the end of this lesson:

> Children would discuss brand names that they bought, whether they exceeded their budget and how easy or difficult they found the exercise. Children discover which supermarket overall is the cheapest.

It was not made clear how WM thought children would graph the information that they were asked to collect, or the purpose of such an exercise. WM could not remember how this lesson had progressed or what her teacher had said about it but believed that it had been a good lesson.

LI was working with 7-8 year olds and set them a task which required them to 'toss this die ten times and communicate the results on a graph which was shown by me'. LI modelled the graph, there was no opportunity for the children to construct a graph of their own design. For LI creating the graph was the objective, with no reference to the relationship between the shape of the graph and the outcome of the ten tosses.

Conclusion

Azcárate Goded (1996) has carried out a study into the understanding of chance and probability concepts held by prospective primary school teachers in Spain. Her findings, listed below, identified four relevant concepts held by the group. Evidence of a similar devaluation, described in this paper is listed in italics.

Very few students displayed a clear understanding of the characteristics of random events.
Very students had the mathematical background or skill to clearly understand what is involved in the teaching of stochatics. It does not help them very much if they consult some of the curriculum references.
Most reason from causal pre-suppositions which they use to justify the chance nature of phenomena, or even their deterministic character.
Most approached their teaching from a position of half-understood truths and indicated that the presuppositions from which regard stochastics as a topic has no real definition;
Personal experience determined their ideas and their beliefs rather than the phenomenological outcomes.
There was little evidence from students notes that personal experience determined their ideas and their beliefs, unfortunately I did not see any of these lessons given and did not have an opportunity to discuss this point with the students.
Levels of comprehension and use of normative models are minimal and, usually, restricted to the context of games. This presents great difficulty when interpreting probabilistic information arising naturally , particularly frequencial data, and many of their arguments seem to be based on their own deterministic thinking.
Models that were presented were generally dice based games which were used extensively by the students in this study 10 students chose this format for at least one lesson.

Further Comments

A significant concern for me was that although students were asked to plan four lessons there was little evidence of consideration of a need for development of children's ideas. The majority of students planned four individual, unrelated lessons.

Chance & data both involve complex ideas that need to be seen as a related, integrated whole, both with mathematics and the rest of the school curriculum. The common element of the material discussed in this paper appears to be the total confusion the students had about defining teaching outcomes of, and mathematics involved in this topic. They were enthusiastic about beginning their practicum, had met and liked the children that they would teach and wanted to do their best to provide them with stimulating, interesting, useful lessons. Presenting stochastics ideas to children is complex; and even the most enthusiastic teaching studies student appeared to find the task immensely difficult.

'Probability is a wet Friday activity. It's a nice thing to do when its been raining all day and the children are restless. Bring out the dice and coins and let them play with them, learning will surely follow. Data can be covered in any project or activity'. These comments, and others like them have been made frequently by practicing teachers when discussing with me 'what is Chance & data?' at in-service courses in primary schools in S.A.

The fact that the majority of these lessons were planned with supervising teachers who believed that they were appropriate, both in terms of suitability for the class and statistical accuracy, indicates that many classroom teachers do not understand concepts of Chance & data themselves. Lecturers who are not maths specialists supervised some of these lessons found the majority of students had not come to terms with the concepts of Chance & data and spoke of general confusion about the topic, lessons that had no clear definition or conclusion, and children unsure of what the lessons were about. While these same observations, particularly the last two, were made to a lesser extent about lessons based on other mathematics topics there were not the same reports of general confusion.

We need to re-think the time and effort put into both pre-service, and in-service courses for practising teachers whose own experiences in this area appear to be lacking. If we are to overcome teaching students' confidence and lack of knowledge we must have classroom teachers who are confident and in control of this important topic. We need to consider examples of good questions to begin a discussion; appropriate and challenging question sequences and analyses of children's responses to questions. How this can be done given the short time available and the other demands of mathematics topics within the course is a mystery. In my own institution I have re-structured each of the subjects so

that in each year the students consider in depth, two Strands of the *National Statement.* This may limit the course, but the advantage is that it allows for a more in depth approach and gives more time to consider the difficulties and confusion related to learning and teaching some topics, particularly Chance & data. I intend to try this approach for 2 years to see if there are more positive teaching outcomes for students. The observation and quotes from students throughout this paper are not made to devalue them or their work. They were used to demonbstrate the confusion and difficulty that pre-service students experience in planning and teaching stochastics lessons.

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Geeks and Nerds: Computers gender and the mathematics classroom

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This paper presents some of the findings of a pilot study concerning gender differences in behaviour in a secondary mathematics classroom that used computers. Analysis of interactions between students and the teacher and engagement with the task, the computer and the mathematics suggest some evidence of gender difference. A number of factors were hypothesised as relating to behaviour and this paper discusses the preliminary findings in relation to students' experience with computers and their style of interaction with computers.

For a long time now we have been struggling to overcome the disadvantage of women and girls in mathematics. It would seem that the research and strategies that have been developed to encourage continued participation and success in mathematics for girls are starting to pay off (Fennema, 1996). In Victoria, recent data concerning ratio of females to males taking the "hardest" VCE mathematics subject shows a slight deterioration (38: 62 in 1992 to 34:66 in 1995). However, females are performing on average as well as, if not better than, males in VCE mathematics (Vale, 1993). On the other hand participation in VCE Information Systems has a marked gender difference in participation (12% female and 88 % male). This clearly demonstrates the masculine culture that surrounds and mystifies advanced technology. As teachers make greater use of computers for the teaching and learning of mathematics it is unclear how this will contribute to the classroom culture and to the gender differences which may characterise the complex nature of that culture.

The research literature is generally concerned with technical aspects of computer software and cognition in relation to computer enhanced mathematics programs. Little attention has been paid to affect or gender differences in changing mathematics learning environments. A few studies have reported on gender difference in achievement. Vale (1993) found that female VCE mathematics students who used computers to do the problem solving assessment task performed better than other groups of students. Rowe (1993) and Yelland (1992) found that primary aged boys performed better than girls on particular programming tasks using LOGO. In her study of the use of personal computers of primary children for learning across the curriculum, Rowe (1993) found that gender differences developed over time with familiarity and use. Males became relatively more confident and enthusiastic and used computers for a wider range of activities. Boys sought out the class expert or the teacher when in need of assistance whereas girls sought out their friends. Studies of classroom culture have found evidence to support the view that computer enhanced classrooms are male domains (Beynon, 1993; Elkajaer, 1992). A whole range of slang has developed to label people who are very interested and skilled in computers. Terms such as 'geeks' and 'nerds' are not gender specific but are generally assumed to apply to males. Some young women who are computer enthusiasts have adopted the name 'geek girls'. A few studies have explored individual differences in the way students interact with computers: Rowe(1993) reported on categories of effective users of computers, Turkle(1984) on types of interactors with computers and Turkle & Papert (1990) on types of programmers. These studies did not however report on gender differences.

This paper presents some of the findings of a pilot study concerning the culture of mathematics classrooms that used computers. The qualitative study was designed to investigate how boys and girls behave when they use computers for the learning of mathematics. Particular factors were hypothesised to be related to behaviour: the students around them, the attitudes and actions of the teacher, the nature of the task, their performance in mathematics, their attitude to mathematics, their experience with computers, their preferred learning style and their style of interaction with computers.

This paper presents results concerning gender difference in behaviour as it relates to experience with computers and style of interaction with computers.

The Study

The study investigated the behaviour of a class of year 10 students in a large Victorian co-educational secondary school, who were required to use computers as part of their mathematics program. There were 12 girls and 11 boys in the class. The teacher had 17 years of experience teaching mathematics and had begun using computers in the previous year. The topic was parabolas and quadratic functions and the teacher chose to use Graphmatica, the only graphing software available for use in the school. The teacher selected this topic and planned the unit of work, assessment tasks and the specific lessons that were used for the study. The students were required to complete two tasks for assessment: 'Graphing Project' and 'Problem Solving'. The teacher determined the timing of the study to coincide with the teaching of this topic. Special room allocations were made in order for the class to have access to a computer laboratory for four of the lessons that were observed. The computer laboratory contained 28 personal computers that were part of an entire school network of computers. Each student and staff member had a password for accessing the network.

Method

Six consecutive lessons of one year 10 class were observed. Data was collected in a number of ways: four lessons were videotaped and two were audio-taped, field notes were taken during each lesson and on informal discussions with the teacher and the teacher and four students were interviewed. These interviews were audio taped. The four students selected by the teacher had different levels of ability in mathematics: a girl and a boy who were above average (GK and BA) and a girl and a boy who were average or below average (GM and BD). Each of these students attended all of the lessons with the exception of one boy, BA, who was absent for the fourth lesson. The interviews of the students collected demographic data and data concerning attitudes and experiences of mathematics and of computers. During the interview the students also used the computer to repeat one of the problems from the 'Problem Solving' worksheet. The student interviews therefore collected a sample of work of each student using the computer for mathematics.

A large amount of qualitative data was generated and data reduction (Miles & Huberman, 1994, p. 50-89) was used as the process of analysis. The video tapes were transcribed and coded using codes adapted from the literature. These codes were descriptive and to some extent interpretative. The codes were grouped into five categories. The first category of codes concerned the participants of interactions, for example, B-G was used for an interaction between a boy and a girl initiated by a boy. The second category described the cognitive nature of interactions between members of the class including the teacher: informative, organisational, argumentational, exploratory or metacognative (Geiger & Goos, 1996, p. 231). The third category described the content of these interactions: mathematical, systems, software, task and off task. The fourth category interpreted the attitudes and feelings about people conveyed in their behaviour: positive socio-emotional and negative socio-emotional (Lee, 1993, p. 554). The fifth category interpreted the attitudes and feelings about the computer conveyed in behaviour: emotional verbal, emotional non-verbal, personalised and depersonalised (Turkle, 1984, p. 108). An additional category of codes was later included concerning the degree of collaboration: parallel activity, peer tutoring, collaboration (Goos et al., 1996, p. 238). Four additional codes for this category were created to make some finer distinctions in behaviour: peer demonstration, teaming, conspiring and peer takeover. A working definition of each of these additional codes is included in Table 1.

Table 1. Additional codes to describe the degree or nature of collaboration

Code	Working definition
peer demonstration	A version of peer tutoring, but in this case the student demonstrates by taking control of the mouse or keyboard to show how something is done.
teaming	Two or more students working together to complete the task sharing roles and assisting each other. Differs from collaboration because they don't necessarily have a problem with understanding the work, maths or software but are more efficient doing it together.
conspiring	Two people working together to give the appearance that each has done the work independently. One student benefits from thework of the other.
peer takeover	When a student rather than tutoring or demmonstrating does the work for another.

The student interviews were transcribed. A summary of each interview was compiled using categories which followed the set of questions used for the interview: attitude to mathematics, belief about mathematics ability, attitude to computers, experience with computers including personal ownership, use, activities, knowledge of software and finally opinions about using computers in mathematics. A summary of the student's work on a problem from the 'Problem Solving' worksheet included a description of the process followed by the student and a descriptive assessment of their knowledge of the mathematics content and the computer software tool.

Results

Whilst four students in particular were the focus of data collection during the lessons and through interviews, data concerning many other students was also collected. This section presents the results of a gender analysis of behaviour as it related to students' experience with computers and style of interaction with computers.

Behaviour and experience with computers

None of the students who were interviewed reported having previous experience of using computers in mathematics. The class had taken an information technology subject during the previous two terms, so they had been issued a password for accessing the network and had been introduced to a range of software. Three out of the four students who were interviewed (the two girls and one of the boys, BA) had a computer at home. The number of students in the class who had a computer at home is unknown, though the teacher believed that most of them did. None of the students who were interviewed reported using the computers at school out of class time.

All of the students who were interviewed commented positively on using computers for mathematics. For them, computers were familiar, easy to use or easier or better than writing. They also enjoyed using computers to play games, though they each played different types of games. One of the boys (BA) enjoyed adventure games and the other (BD) occasionally played sporting games at his friend's place. One of the girls (GM) said she played "Carmen San Diego" the other (GK) played a variety of games. The three students who had access to computers at home used word processing software for homework and one of the girls (GK) made use of reference material on CD-Rom. Of these three students, none spent long periods of time using their computer at home. The most was GK who spent about 7 hours per week and the least was GM at 3 hours per week. Each student remembered something different about what they did in their computing subject during the first two terms of the year. The more mathematically able students (GK and BA) recalled more examples of software used in the subject than the other two students who were interviewed.

Gender differences in relation to knowledge or expertise with software were not obvious. Three girls and one boy were observed to access the word processor to document their answers to the assignment questions (transcript and field notes,

Thursday). Whilst both boys and girls acted as tutors during the lessons with the computer it was observed that two girls (GE and GR) moved around the room helping other students.

One of the boys (BA) spoke of his particular interest in computers, his recent work experience with computers and his desire to do a particular tertiary course in computer communication:

> went to work with computers for work experience.... I had to fix a few computers up... I had to install CD-Roms and configure the computers... (transcript, student interview BA).

This boy demonstrated his knowledge of specialist computer terms or jargon during the interview by interacting with the icons on the screen to show his understanding of the term 'configure'. During the lessons he was observed demonstrating features of the software to a neighbouring girl and answering questions asked by boys.

Other boys in the class also demonstrated knowledge about hardware and systems during the lessons. Boys talked more often than the girls about issues concerning access to the computers; they also used more jargon associated with computers (eg. virus, configure). Two episodes, recorded in Table 2, demonstrate boys' knowledge and their interest in accessing the network. In the first one boy (BJ) persisted with investigating the network to discover the reason for the inability of one the boys (BD) to log onto the network. The second episode involved the teacher. The teacher had decided to solve BD's access problem by using her own password and five boys were around her watching what she was doing. This episode illustrates the teacher's fear that she did not know enough about the system and did not like to be dependent on other people (teacher interview, p.1-2). She was pressured and challenged by the boys and had to defend her knowledge and skills.

Table 2. Boys knowledge and interest in hardware and accessing the network.

Problem with logging on	The teachers' password
BJ: R are you logged in? BD: Na...?.. BJ: Someone's logged in with your name. BD: Why? BJ: Someone's in with your name. BT: No wonder you couldn't get in. BD: How do you know? B?: Most likely in the other room or the library. BJ: Two people have your name. B?: Yeah it'll be in the other room. BD: How do you know? BJ: It says it here. BD: What?..?.... BJ: That's you yeah? BD: Yeah J. (video transcript Thursday, p.7)	BT: Miss is that ? your own one. T does not respond. BT: ?..why don't you just try to log in first maybe that might help you. T: ?..why...? BT (shaking his head in his hands): No but you shouldn't. You do have to. T: ? BD: Now log in....?.. BD: Now log in again. BT: Don't you have to write star first, cos we've got to...? T: No, no I don't. BD: You can't log in you can only log in like that. BR arrives to watch what the teacher is doing to log in. BB is also looking on. BD: .?... Now write it now. BT: You've done it wrong all the other times because you forgot the y. T: ? BT: Na, you did. T: Yeah but I did it intentionally to ....?... (video transcript, Thursday, p. 6)

The boys also showed a willingness to back up their mates who had less experience with computers. BD, who had the least experience with computers of the four student interviewed, finally completed the first task by 'conspiring' with another student to use his work and change the name on the graphs to indicate that it was his own work:

BD is leaning over BB's keyboard.
BB: We'll change the name. You got to go to title.
BD backspaces to remove BB's name and then types in his own.
(BB tries to shield this from the camera.)
BT: Who done it?
BB: He's getting my work.

(video transcript Thursday, p. 6)

Girls also encountered technical problems. In one episode two girls were unable to get a computer working and they resolved the problem by choosing to work together as a team and share one computer. If girls had knowledge of the computer system and hardware it was not evident in the discussion recorded during the lessons.

Behaviour and style of interaction with the computer

The students were task orientated, that is there were many interactions between students or between students and the teacher that were concerned with the organisational details of getting the work done. Furthermore, most students appeared to view the computer as a tool that they were required to use to do the work. The students made comments such as "just make it easier" in much the same way as they might see calculators as making arithmetic easy or word processing as making writing (the physical task) easier. The teacher commented that it is difficult to get the students to use pen and paper, whereas they had been motivated to use the computers to do the work. A view of computers as a tool was probably enhanced by the teacher's view of computers. The first task required the students to use the computer as a tool for graphing rather than as a medium with which to engage and explore mathematical concepts.

There was some evidence to suggest that girls in particular, viewed computers as a tool. Three girls (and one boy) accessed the word processor to document their answers to the assignment questions. One girl (GK) who had a broad experience with software and was above average mathematically, worked independently through the tasks set by the teacher. She did not interact with other students but successfully used Graphmatica to find turning points and intercepts for other questions on the assignment sheet (rather than use symbolic reasoning as intended by the teacher). (One boy (BT) had wanted to do this but had been redirected by the teacher. His response was off task behaviour.) GK also successfully interpreted the screen and used the features of the software to solve the real problem. By contrast another girl (GM) who was very task orientated adopted a methodical approach to completing the tasks, using the software without interpreting the mathematics. She requested information from other students around her and she worked in parallel with the girl beside her. She personalised the computer, talking to it, making emotional respondes (especially when things went wrong) and talking herself through the processes.

GM: Oh shit I didn't label em.
BT: Didn't you label them?
GM: Na.
BT: Have fun.
GM: I'll have to start all over again. Oh you turd.
BT: (inaudible)
GM: It is hard actually. Very.

(video transcript Wed., p.2)

She used the same methodical approach to solving the real problem during the interview. Rather than interpreting relationships between the maximum value of the turning point and the values she was trialing from the screen, she made small adjustments to the value to reach the solution.

The interaction above also supports the notion that girls are more likely to personalise their interactions with the computer. One other girl (GV) used emotive language to describe her successes and difficulties. She made sounds and signs of shock or delight as the computer behaved unexpectedly or as she had hoped. They were some episodes when boys showed emotional responses, these were normally physical (for example, BT hitting himself on the head) and were responses to their own mistakes rather

than to the computer. Boys also were public with their announcements of success (BB, BD, BT and BJ) whereas girls did this privately (GV, GF and GM).

The behaviour of two of the boys, BJ and BA, perhaps points to a particular style of interaction with computers. BJ in his attempts to discover the reason why one of the students couldn't log on demonstrated engagement with the computer and its information system in way which may be described as 'hacking' behaviour (Turkle, 1984). BA appeared to engage with the software and system by moving around in it. He used the mouse and the keyboard to demonstrate to others rather than articulate instructions from a knowledge of the features of the software and the design of the interface system, that is the tool bar. This student successfully interpreted the screen to develop an understanding of parabolas and solve the real problem efficiently (video transcript Friday and BA student interview). The following episode depicts him in the role of peer demonstrator.

> GV (to BA): You know how (inaudible) makes the thing bigger or ..
> (BA takes the mouse and starts to change the range.)
> GV: Mate, oh (she holds her hand to her head in surprise, GE arrives to watch what's happening on the screen)
> (BA continues to operate the mouse. Then GV takes over operating the mouse and says something inaudible.)
> GE (BA watches on): OK go to view (she points to the tool bar). No that's (pause she watches as GV operates the mouse) Range! Now you change this one (points to the screen) (inaudible)... five and that (inaudible).
> GV (typing): Negative one.
> GE: Now we change the four to six. (pause and then inaudible, points to the screen and then moves off).
>
> (video transcript Wed., p. 11)

This episode show that the female student (GV) was more comfortable with the instruction given by the girl (GE) rather than demonstration given by BA. This boy's use of demonstration may reflect his style of interaction with the computer and perhaps be derived from or related to his interest in adventure games and computers themselves. It may also be a display of control or power.

Discussion and conclusion

The students appeared comfortable in the computer surroundings; it was familiar territory for them and they were motivated to complete the tasks. In general a lot of the behaviour that was observed was common to both girls and boys in the class. However there were some gender differences observed in the way the students engaged with the task, interacted with each other or the teacher, interacted or engaged with and used the computer and software and engaged with the mathematics. The boys were dominant in the computer classroom: they were larger, louder, took up more space, more negative, more demonstrative and public, but this may not be any different to normal behaviour in mathematics lessons. The content of boys' interactions in general revealed their interest and pre-occupation with hardware and the networking system and their knowledge, experience and expertise with hardware and software systems. There was some evidence to suggest that girls were more likely to view computers as a 'tool' and more likely to personalise their interactions with the computer (Turkle,1984). Whilst some of the characteristics of computer interaction, knowledge and interest displayed by at least two of the boys in the class might be considered 'nerdish', the behaviour of girls who were competent users of software is probably not 'geekish'. It remains unclear however, how differences in behaviour related to experience and style of interaction with computers relates to a student's interest or performance in mathematics.

The episode where the boys challenged the teacher's knowledge of accessing the school network suggests that some female mathematics teachers will need encouragement and support to 'take on' boys with superior knowledge of computer systems. It also suggests that the shift in power in computer enhanced mathematics classrooms may not only concern democratisation of learning (Rowe, 1993; Thomas et al, 1996) but also concern a realignment of power in terms of gender.

These conclusions are tentative and will need to be tested in a more thorough ethnographic study. Students need to be observed for a longer period of time in order to confirm patterns of gender difference in behaviour. A more thorough study would involve the observation of a class where students used computers for a variety of mathematical tasks and/or used a variety of mathematics software. An appropriate set of codes for analysing styles of interaction with the computer would need to be developed. It will need to be born in mind that the nature of the hardware may be a factor influencing behaviour in the classroom and that the nature of the mathematics tasks will determine the parameters for such a study.

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The problem of the real in curriculum implementation

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This paper takes seriously the claim that postmodernism has undermined our 'modern' understandings of mathematics educational research. It therefore seeks to reinterpret this understanding with particular reference to curriculum implementation in a manner that confronts the challenge that postmodernism has posed. In order to do this the paper clarifies how postmodernism discredits the assumptions on which curriculum implementation has been erected. It argues for a reconstruction of the relationship between curriculum inquiry and the truths of curriculum reform.

Locating the real in curriculum implementation research

In these days of cultural and epistemological crisis much is made of allusions to the end of a 'modern' era, epitomised in the grand stories that developed and provided the legitimation for modern society and scientific knowledge. From the time when the seductive theoretical positions of the postmodern took hold in the 1980s and began to question those very ideas and claims to truth of the 'modern' era, the scientific certainty of previous years came to be replaced with a skepticism of belief in universal truth or in the possibility of a totalising narrative. It is with some nostalgia then that one looks back to that time before those games rules for science changed when the research talk was of notions of truth, progress and objectivity.

In this paper I want to look at mathematics educational inquiry as implicated in these cultural currents constitutive of the postmodern condition. Central to all theory of the postmodern lies the troubling preoccupation of the 'real'. I want to look at the place of the 'real' in curriculum implementation as it is for a small group of secondary school teachers. I want to argue that curriculum implementation research has traditionally been positioned within modernist ideas, subsumed within the grand metanarrative of science. Such inquiry unfolds around the script of the new curriculum document in which the lead roles are assumed by those in schools relegated the responsibility of 'getting the document up and running' and the supporting roles are taken by authorial intent , planned outcomes and so forth.

In this story of curriculum implementation there has never been any difficulty with the 'real'. Since its very inception curriculum implementation has been a commonly understood notion, and those debates that have surfaced have been more about efforts to achieve efficiency and effectiveness than about critique of the foundations, the function and the status of curriculum inquiry itself. Committed to the discovery of ultimate causes and to the identification of unitary themes it presupposes and is dependent upon the belief that it is possible to control the meaning of the text. Implicit is the idea that the text reflects an unmistakably transparent stand that will descend down into the schools. The real is self-evident.

In such work the text and its implementation are perceived as discrete entities in which the former is privileged. The curriculum policy text is generally interpreted as an expression of political purpose, a statement of the course of action that policy makers and administrators intend practitioners to follow and the meaning of implementation is restricted to narrowly-defined strategic interests in how to put a given curriculum plan into practice. Curriculum is said to be implemented when the authorial intent presumed to lie behind the policy text becomes embedded in schooling; more specifically when the syllabuses meet the required specifications,

the textbooks and other teaching materials have been made available and the planned outcomes are achieved.

There are problems with this theorising and the explanations that have been given for curriculum implementation cannot be considered to be 'whiter than white'. While generally professing to be on the side of rationality and progress curriculum research analyses still fabricate a reality. In accordance with the codes and conventions of positivist research that has domesticated such inquiry it posits itself within the boundaries of prediction and control, and in so doing commits modernity's mistake in ignoring the particular relations of power and oppression inscribed within the pratices it considers to be universal.

Where curriculum research is guided by technicist concerns it remains indebted to and enmeshed in the work of the human sciences. Put differently, truths arrived at through orderly method, scientific inquiry and prescriptive theorising predispose and deeply constrain the way one understands curriculum implementation. By ignoring the social location of agents and the social construction of knowledge curriculum implementation has, common to many fields of social inquiry, failed to recognise the embeddedness of its own assumptions within specific historical context, immune to the influences of politics or values.

Such an over-reliance on the restrictive and self-confirming methodology of scientific analyses seems to have little persuasive purchase in the postmodern era. I am not taking issue here with research that responded to curriculum questions in the modern era. These researchers deployed the theoretical tools which were available to them at that time and they responded as best they could given those bounds. What I am suggesting is that given the postmodern challenge to the Cartesian hold on what is considered real and legitimate ways of knowing and being, curriculum inquiry needs to look closely at the 'real' of inquiry as the relation between knowledge production and systems of power.

What I want to stress is that curriculum implementation research has been premised upon the construction of objects of study, the 'text' and the implementing teacher and those very objects are not real nor timeless, but produced for particular purposes within specific historical, social and political conditions. How do we as mathematics educational researchers rethink curriculum implementation in a world where the realities of the text and of the implementing teacher are no longer self-evident? Because the 'real' is linked with power systems it cannot be emancipated from the forms of hegemony, social, economic and cultural, within which it operates at any given time. The argument is not about whether curriculum change happens within classrooms but how that change is understood and the effectivity of its discursive constitution.

It seems then that implementation inquiry's claims to truths must be looked at, both in terms of their specificity, their emergence, they ways in which they are shaped by, and are themselves shaping, reality. In turning away from traditional commentaries on the curriculum text and curriculum practices that define it, I want to foreground the relation between the text and the reader and probe the complexity of actions and individual consciousness within socially and historically defined circumstances. In so doing teachers' reciprocal and often changing connections to the social and cultural forces in which they negotiate curiculum meaning might be appreciated.

I am arguing that the very concept of curriculum implementation is not universal but extremely specific and in its specificity occludes marginalised stories. What I am at pains to avoid then is a totalising configuration of what curriculum implementation means. I propose to rethink such inquiry not by drawing on the theoretical tools of deconstruction, though taking apart the fictions of curriculum inquiry has an important place in this work. Rather the proposal points to the emergence of a new (dis)order, no longer framed within or legitimated by the systematic set of relations organised by the metanarratives of modernity.

I want to capture some of those other stories - the stories of teachers who are simultaneously defined and resisting definition, endlessly asked to be autonomous while endlessly asked to conform in curriculum work. I offer a bricolage of perspectives, that seems to jostle for position, from teachers who produce their unique creations as they attempt to capture and tame the issue of implementation itself. In this approach I draw attention to the variety of readings, the partiality of any one view and its implications in historical social relations. In encouraging a multiplicity of readings, I seek to demonstrate that the 'real' of the text can never be exhausted, and how a text is able to participate in multiple meanings without being reduced to them.

In placing a high value on teachers' knowledge as constructed and provisional, I attempt to unravel the ways in which expectations, fears, conflicts and questions shape 'readings'. As they locate themselves relative to their roles as agents of curriculum change in four institutional sites, issues of power, authority, and truth surface. Through their curriculum work I want to demonstrate the contingency of curriculum truth; that the 'real' can be defined only for specific concrete and local practices. In its representation to you, the reader, the meaning of the 'real' which I need to communicate slides away. And we, both you and I, are left not with the reality, but with an approximation which, however much I try to make more 'real', is always already deferred and irrecoverable.

Reading the curriculum document

How can one determine the 'real' of the text? How can one decide what the text 'really' means? Texts are systems or economies of truth. None of us has access to an out-side text, to what is 'really real'. The real, in postmodern theorising, is subject to multiple interpretations, multiple readings and multiple uses. In reading, interpreting and assigning meanings one can but appeal to the schemes and texts that one has available through history, culture, language, politics, economics, ethnicity, gender and so forth. 'Reading' becomes overdetermined by contingencies of language, rhetoric, power and history. These forces are seen to work through texts in ways neither their authors nor their interpretors can fully control.

A reading is always relational. The reader makes sense of what is conveyed by attending to social convention and circumstance as well as to words and utterances. What appears 'real' to the reader in the text is always analysable as a restrictive and expressive set of social codes and conventions. Postmodern theorising locate interpretors in many sorts of reciprocal contexts, negotiating a power-laden and conflicting 'reality' in the communicative process, telling many divergent stories. The text will reach many types of readers - teachers, students, administrators, parents and the wider community - all of whom bring different past experiences, beliefs, attitudes, and commitments with them. They start from diffferent places. Some will find comfort with the text; others will not.

> The new curriculum probably suits my teaching and learning style so I'm fortunate in that I feel comfortable with it reasonably and that probably isn't true of everybody. So that it's something I have developed over the years and feel comfortable with anyway and that's not true of everybody within the department.
>
> (AL)

No one interpretor is 'master' of the 'real' than any other. There is no privileged access to the 'real' of the curriculum text. Upon its delivery to teachers, the singular document that is the text arrives with expectant transference. In a concern for grasping the 'sense' of the text, readers will search for its identity, constituted by its reference to the world of school mathematics. The text cannot be the expression of an already constituted experience. Interpretation then becomes necessary and requires bringing the reference of the teacher's world into the curriculum without collapsing into its specificity. The social knowledge which she brings to a 'reading' will serve to decentre its meanings. It might be said that the teacher is always reading oneself when reading a text, captured by a number of domain assumptions which allow her to see only certain aspects of the situation. She will reject, choose or ignore part of the text as she focuses on personal advantage.

Every act of reading the policy document and every teaching occasion produces meanings, the social and political implications of which are determined both by the position within the discursive field from which the teacher reads and acts, and the knowledge inscribed in the discursive field from which she reads and acts. The text could be thought of then as a permuational incessantly mobile space and within it a multiplicity of discursive events interweave. Thus the author of the text cannot be the source of the authority of meaning any more than the individual reader can be its origin. The postmodern emphasis shifts then away from the inaccessible authorial intentions onto the text itself.

> I think it is a different way from sitting down, getting out your books, ruling up and doing work from textbooks
>
> (RI)

> It's not a series of techniques. It's a philosophy really behind the teaching. Not actually the techniques itself. Obviously some things will become more important. Group management and class discussion management as well. Trying to get away from the old chalk and talk sort of style into a more hands-on experience type of teaching and I guess that it will require added training in that respect. But I think it's more the philosophy than the techniques that are important.
>
> (TI)

> Activities, doing something rather than sitting down doing sums out of a text book. Actually getting involved, problem solving, investigation. Something where it can be seen the same as other subjects like art where they find it interesting. There's some creativity in there. There's some activity. There's actually something where the kids can have their own input into it rather than just something that's been imposed upon them. Rather than just learning a whole lot of rules and a whole lot of steps to do something, they're actually developing steps themselves as well as coming in with their own creative ideas. I like that.
>
> (BV)

Since the reader of the text is intimately linked to a set of institutional contexts, any understanding of teachers' curriculum work must be grounded in the social experience of daily school life. Traditional curriculum inquiry has paid little attention to institutional and wider societal factors in which pedagogical practice is embedded. The implementation of a curriculum requires, then, an understanding of schools as organisations in society, since possibilities are defined not only within the culture and political climate of the classroom, but also within the school itself and the wider school community.

Because mathematics teachers participate in networks beyond their own classrooms, the values and interests constituting the norms of mathematics pedagogy have to be understood in the context of widely held beliefs about mathematics. These cross a range of discursive fields from the school, the parents, the community, and employers, to the representation of mathematics in the media. To speak of the institution of school mathematics is to draw together a number of institutional sites and practices, which although they are differentially positioned with respect to access of power, together produce the discursive field of school mathematics. The attempt in New Zealand to change such meanings through the introduction of *Mathematics in the New Zealand Curriculum* constitutes a part of the wider Western academic endeavour which seeks to transform the dominant discourse of school mathematics.

It is important to recognise and not ignore the ways in which the school itself claims control over curriculum. This claim exists not only in the interpersonal conflicts between senior management teams and teachers but is more importantly couched in the culture of efficiency and effectiveness of the school's bureaucracy. Parameters of prediction, control and accountability in the school that have woven educational ideas and practices into a technicist web of 'performative privatisation', profoundly constrain the kind and extent of changes towards *Mathematics in the New Zealand Curriculum* teachers can effect.

Professional responsibility for the mathematics teacher no longer begins and ends with mathematics nor even with teaching and learning. The major school restructuring of 1989 is not simply a structural change, it also represents a profound change in organisational culture. Oriented towards and constructed within a discourse of cost, income, efficiency, financial planning, image presentation and enterprsie, the 'new management' competes for control over curriculum in schools. It sets itself against sustained personal and departmental educational and professional development. In doing so, it generates tensions, ambiguities and ambivalences in schools and acts to divert energies and efforts away from the vision of *Mathematics in the New Zealand Curriculum.*

> Apart from the normal duties of running a department I am a member of the Senior Management team. This involves meetings with the Principal and the other PR3s. I am also involved in the writing of the school timetable as timetabler, exam timetable, and the preparation of a booklet on report writing and the computerisation of reports, This team is also investigating the implementation of the Framework. Curriculum committees and other administration areas such as Teacher-in-charge of soccer and duty team also take up some of my time...I know I am not being an effective HOD and teacher. If I had more support and the time to give to my staff and the development of resources and programmes I wouldn't be so concerned.
>
> (MM)

> I'm in charge of the Sixth Form Certificate in general and that requires making sure the teachers record their marks properly on the computer, work with the grades we're given and allocate them and send everything off...That does take quite a bit of time. I am also part of the Assessment and Reporting Committee for the school.
>
> We are expected to do a lot outside the classroom...We are also expected to work a lot of weekends doing a lot of PR, talking to the parents, ringing up parents when there might be a problem
>
> (CL)

Since the school is a public institution and integral part of the polity, deciding what to do about curriculum problems in schools is also inherently a political act. Any activity that is inherently moral and poltical is likely to be contested and, from time to time, embroiled in controversy. Curriculum knowledge production resonates with all the oppositions and tensions of education - the private and the public; the unknown and the known; the individual and the collective, with conservation and with transformation. Curriculum knowledge production is a process of interpretation whose rules are determined by the discourse of the classroom, by community politics, by bureaucratic regulations all serving to undermine its open inquiry and democratic participation.

Sensitivity to human needs is stifled by a growing pressure on the social, political and economic infrastructure of the school to respond to the 'education means business' approaches. For these teachers opportunities for changing pedagogical practice becomes limited. The purpose of educational endeavour has now become antithetical to critical notions of relevant professional practice. Paradoxically the school climate that makes the curriculum change seem desirable or even necessary, is also one in which any thorough change in school mathematics may seem to these teachers themselves impossible to achieve.

These teachers were concerned not only with the world of theory but were also engaged in the important arena of practice. They created curricula for students whose mathematical experiences were already established. They attempted to mediate the world of abstract theory and the lifeworld of the school. On the one hand *Mathematics in the New Zealand Curriculum* urges that teachers take seriously the imperatives of living in a multicultural democracy. On the other, the organisation of the school is increasingly dominated by educational policy constitutued by a set of values and practices that take as their model the laws and ideas of the marketplace.

The happy illusion of their engagement with seductive new school mathematical theory is shattered and redefined through corporate ideas in which competition is ethically prior to cooperation, excellence to equity and standardisation to difference. In their curriculum efforts they inevidently run up against established perspectives and methods which when considered alongside institutional constraints, programme demands and public expectations, militate against any significant reformulation of mathematics pedagogy. They come to understand what is meant by the 'real' in curriculum work as contested, multiple and partial.

Conclusion

This paper has not attempted to offer an alternative and more secure foundation for curriculum implementation, but rather sought to illmuminate the complexity, contingency and fragility of forms and events. It considers implementation of curriculum as a cycle of continous policy making, which evolves amongst all the parties in the process. It recognises the multiple and contradictory aspects of teachers' curriculum work within the complex and often conflicting historical, interpersonal and ideological contexts of individual and organisational involvement. The point was to break through the sameness, the flattened landscape, to disrupt official readings of curriculum implementation and to question its assumed progressive accomplishments.

Implementation then becomes not a site for working through more effective transmission strategies, but for helping teachers learn to analyse the discourses available to them, which ones they are invested in and how they are partially and contradictorily inscribed by the dominant. As such it is a search not only for the evidence of those forces that 'diminish' teachers, but also provides a medium for them to reduce the dichotomies between social and cultural forces. It moves beyond proposals that merely speak on behalf of the researched teachers, to a position that makes available an exploratory tool for them to move beyond their present positions.

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Generalising from and transferring between algebraic representation systems: Characteristics that support these processes.

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In response to misconceptions students hold with understanding the concept of a variable new teaching approaches have been introduced into many Australian Schools. These approaches entail generalising from visual patterns and tables of data. While recent research has reported many of the difficulties students experience with these new approaches, it has failed to delineate why these difficulties are occurring. It seems that these approaches call on an array of reasoning processes that, in the past, have not been considered as important to the algebraic domain. This paper delineates some of these processes and pinpoints particular characteristics of successful students.

Extensive studies such as The Second International Mathematics Study - The Fourth Assessment of Mathematics in the U.S.A. (1982) reported the misconceptions many students hold not only with understanding the concept of a variable but also in solving algebraic equations, and translating word problems into algebraic symbols. In Australia, one response to these concerns was to introduce the algebraic ideas using new teaching approaches (Pegg & Redden, 1990; Quinlan, Low, Sawyer, & White, 1987). For these approaches students are required to view the variable as a means for expressing the generalisations discovered by looking at tables of data and patterns represented by concrete or pictorial representations. Research with regard to these approaches is only beginning to appear in the literature.

Yerushalmy and Shterenberg (1994) suggested that the links between patterns and algebra are not trivial. Recent research has delineated some of the difficulties students are experiencing with these approaches (MacGregor & Stacey, 1993; Orton & Orton, 1994; Redden, 1994). Most students seemed to find it easy to describe the connection between two adjacent terms in a sequence but had difficulty describing the rule that created the sequence (MacGregor & Stacey, 1993; Yerushalmy & Shterenberg, 1994; Orton & Orton, 1994). Most did not naturally choose to express their generalisations in algebraic symbols (Kaput, 1992; Yershalmy & Shterenberg, 1994). The reflex of checking the formula against the given data was not present (Lee & Wheeler, 1989). Arithmetical incompetence prevented progress for some (Orton & Orton, 1994). The role of language in reaching generalisation has also been investigated. Stacey and MacGregor (1995) reported that the correct verbal descriptions were more likely to lead to the correct algebraic rules. Kaput (1992) suggested that over-learned natural language can in fact inhibit appropriate algebraic symbolism. Thus the focus of recent research seems to be identifying difficulties and not so much on delineating why these difficulties are occurring or identifying specific student characteristics that support these approaches.

One of the key processes required for success with these approaches is the ability to generalise. Dreyfus (1991) distinguished two different generalising processes. The first requires one to generalise from the specific to the general (e.g., generalising from tables) and the second requires transfer from one representation to the other (e.g., table to visual patterns to variable), a process requiring both synthesising and elaborating. If the aim of utilising a variety of approaches for introducing the concept of a variable is to enrich understanding of the variable then students not only need to establish links between each representation and the concept of a variable but also to be able to successfully transfer from one representation to another (Lesh, Post, & Behr, 1987). Not only do students need to successfully translate between the representations, they also need to envisage the dynamic

nature of both representations and abstract the linear function represented, that is, envisage $ax + c$ as an expression in its own right. Sfard (1991, 1994) referred to these dual roles as structural and operational.

The structural concepts are 'static, instantaneous and integrative," whereas the operations is 'dynamic, sequential and detailed." (Sfard, 1991). Sfard (1991, p.18) believed the student passes from an operational conception to a structural conception by progressing through three hierarchical phases namely, *interiorisation*, *condensation*, and *reification*. Firstly, the learner becomes acquainted with the processes which will eventually give rise to the new concept. Secondly, lengthy sequences of operations are compacted into more manageable units. The student becomes more capable of thinking about a given process as a whole. There emerges a growing facility with alternating between different representations. Thirdly, the ability to see something familiar in a totally new light evolves; one has abstracted the concept.

While Sfard's (1991) theory provides considerable insights into the processes of algebraic abstraction of a concept it does not explicate the means by which the learner might construct generalities and abstract the underlying concept (English & Sharry, 1996). English and Sharry felt that analogical reasoning plays an important role in this process. Reasoning by analogy is defined as a mapping from a base to a target (Gentner, 1983). For example, in mathematics the analogy "4 is to 6 as 10 is to 13" the base is "4 is to 6" and the target is "10 is to 13." Elements in the base are mapped into elements in the target. 4 is mapped into 10 and 6 into 13. The relation of "less than" in the base then corresponds to the same relation in the target. When using concrete representation systems in mathematics the base is the concrete representation and the target is the concept to be learnt (English & Halford, 1995). Before making use of analogical transfer one must notice the correspondence between the target problem and base problem and retrieve the base as a generalised structure. The role of analogical reasoning in the algebraic domain is yet to be delineated.

When transferring between tables of data and visual patterns, students need to map the top row of the table to the step number in the pattern and the bottom row to the number of students, that is mapping the base (the table) to the target (the visual pattern). Many students experience difficulties with this process. This failure to notice the critical sameness between situations is particularly apparent when transfer of learning fails to occur. English and Halford (1995) provide some insights into why these difficulties might occur. Firstly, the students could be mapping the problem into an inappropriate scheme. This could depend on either the materials used to represents the concepts or on the difficulties they experience with reasoning analogically. Secondly, the processing load imposed by the proposed mapping may be such that some students find it difficult to reason by analogy.

The literature identifies other specific reasoning processes that appear to enhance associating representations. One of these is the ability to think flexibly (Warren & English, 1995). There seems to be two different interpretations of the concept of flexibility. The first refers to how students perceive geometric shapes and visual patterns. If their perception of the shape or pattern hinders their ability to visualise the shape or pattern in differing orientations or within more complex patterns then they are judged as inflexible (Tartre, 1990). The second refers to how students approach a problem solving situation, that is, their willingness or unwillingness to change their approach to the problem (Chancellor, 1991; Krutetskii, 1976). Lipman (1985) believed that some of the key processes crucial to flexibility are spatial thinking including a facility to mental rotation, logical and analogical

reasoning, classifying and hypothesising, and an ability to complete patterns and generalise. The role these reasoning processes play in early algebraic experiences need to be examined.

The aim of this study was to begin to delineate why difficulties are occurring with these approaches and to attempt to identify specific student characteristics that seem to support these approaches.

Methodology

The design for this study consisted of two components, a correlational design followed by a clinical interview. Research that uses more than one method to collect relevant data is often referred to as a multiple measurement approach. Bewer and Hunter (1989, p. 17) suggested that the fundamental strategy of a multiple measurement approach is to "to attack a research problem with an arsenal of methods that have non-overlapping weaknesses in addition to their complementary strengths." One aim of the clinical interview was to probe the thinking students engaged in when reaching generalisations. It also served to explore important areas that are inherent in using the patterning and table of data approaches for introducing the concept of a variable, for example, do students identify the common structures inherent in these approaches.

Written test

The written test consisted of three components, namely, generalising from visual patterns, generalising from tables of data, and understanding the concept of a variable. Both the components for measuring the ability to generalise from visual patterns and tables of data consisted of four questions. Each question focused on ascertaining the students' ability to look at visual patterns or tables of data and make appropriate generalisations. Each represented the four possible linear generalisations that can be made from visual patterns, that is, $x + c$, ax, $ax + c$, and $ax - c$ where 'a' and 'c' are constants and positive integers, and x is the variable. The functions were chosen such that each patterning question matched a corresponding table of data question. The questions chosen for the written test were as follows:

Patterning component

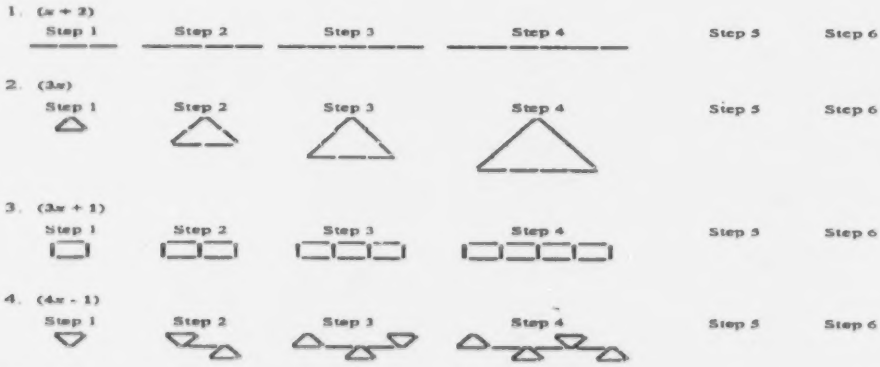

Table component

1. (x+3)

Input	1	2	3	4	5	6	7
Output	4	5	6	7	8		

2. (2x)

Input	1	2	3	4	5	6
Output	2	4	6	8	10	

3. (3x+2)

input	1	2	3	4	5	6	7
Output	5	8	11	14	17	20	

4. (2x-1)

Input	1	2	3	4	5	6
Output	1	3	5	7	9	11

Each question for each component began with the following set of instructions:

(1) Continue the pattern for 2 more steps
(2) How many sticks would you need to step 30? *Or* What number would the computer turn 30 into
(3) Write your rule in your own words
(4) Write your rule in symbols.

Semi-structured interview

The function chosen for patterning question was the same as question 3 from the written test, that is $3x + 1$. Students were asked to make a square using sticks. They were then asked to make two squares, three squares, and so on. The patterns students were required to generate were:

The patterning approach is used not only used to introduce the concept of a variable but also to explore the notion of equivalence. With this approach, students are required to physically construct and reconstruct the pattern in a variety or ways, each leading to new expressions of generalisation. For example, the above pattern can be described as both $3n + 1$ and $4 + 3(n - 1)$. Thus these expressions are equivalent as they represent the same pattern. In the interview, students were asked to: describe the pattern in their own words; represent their descriptions with concrete materials; ascertain how many sticks were needed for 30 squares; describe the general pattern using their own words; express the general pattern using algebraic symbols; reconstruct the pattern using another method; describe this second method in both words and symbols; and relate the two expressions generated by the two different methods. As one of the aims of the interview was to delineate students ability to transfer between the two representations, the linear function chosen for the table component was also $3x + 1$.

Sample

The sample comprised of 379 students, aged between 12 years and 2 months and 15 years and 10 months. Both schools chosen for the study consisted of students from lower-middle socio-economic status, with a variety of ethnic backgrounds represented. Both schools were large metropolitan schools comprising up to two thousand students. The classes included in the main study were randomly selected from the total student population. The results from the written test were used to select students to be interviewed. The results for each component were ranked and three students were selected at the 1^{st}, 25^{th}, 75^{th} and 100^{th} percentile for each of the written components. These particular percentiles were chosen to highlight the difference between 'high achieving' and 'low achieving' students. In all 36 students were interviewed, 12 from each component of the written algebra test. Each interview was videotaped.

Results

Written test

The percentage of student with correct responses to each part of each question was calculated. From this data it seemed that students on the whole experienced greater success in articulating the rule (part (3)) than in writing the rule in symbolic form (part (4)). In order to compare the results of the patterning component with the results of the table component, a McNemar (1969) test was utilised. For this study, students' responses were matched according to the question number and whether the response was verbal or symbolic, for example, their verbal responses to question 1 of the patterning component was matched with their verbal response to question 1 of the table component and so on. A fourfold frequency table was established for each pair of data and a McNemar test was

utilised to test for statistically significant changes between the cells. Table 1 summarises the results for the verbal responses.

Table 1

Comparing the Verbal Responses for the Table and Patterning Questions (N=379)

Question	Chi-square (Verbal)
1	16.15*
2	106.25*
3	1.92
4	10.25*

* $p < .001$

For Questions 1, 2, and 4 , for the verbal responses, the number of students who correctly answered the table question and incorrectly answered the patterning question was statistically significantly higher than the number of students who incorrectly answered the table question and correctly answered the patterning question. This seems to indicate that students found it easier to verbalise the generalisations in the table questions than the generalisations in the visual patterns questions. The trends for the symbolic responses mirrored those for the verbal responses. Thus, on the whole, students experienced a greater degree of success with the table component as compared with the patterning component.

Semi-structured interview.

Students' responses to the patterning and table of data questions were transcribed and examined in order to identify trends in the responses. On examination of the students' responses it seems that, as expected, students at 1st, 25th, 75th, and 100th percentiles exhibited a growth in differing characteristics. These characteristics related to students' ability to: deal with number (arithmetic competence); initially see the pattern; articulate their thinking; identify the generalisation, with a particular emphasis on the strategies used; express the generalisation as an algebraic expression; and think flexibly, that is, their willingness to change their approach to a problem.

Low achieving "patterning" students, that is, students at the 1st and 25th percentile, tended to make arithmetic errors. For example, when ascertaining the number of matches needed for 4 squares, Alison said, *There are 4 for one square and you just keep adding 3 that makes 15, no 16.* They displayed difficulty in initially seeing the pattern and articulating their thinking. They tended to see the pattern in terms of the ratio concept or the additive strategy, that is, *If 10 sticks (were) needed for 3 (squares) so just times 10 for each 3 (squares)* (the ratio concept) or *Start with 4 for the first square and add 3, add 3, add 3, and so on* (the additive strategy). Low achieving "patterning" students were unable to express their generalisations algebraically, and exhibited an inability to change their approach to the problem, that is, to think flexibly. For example, Matthew wrote '10 in 3' when asked to express his expression in symbolic form, and once Kaylin identified the additive strategy she could not go beyond it and thus was incapable of finding the correct functional relationship. Low achieving patterning students also exhibited an inability to reconstruct the pattern by another means.

Low achieving "table" students shared many of the same characteristics as their patterning cohorts. They did show a greater capability for articulating their thinking as compared with their counterparts on the patterning question. For example, for the table question, Simon said, *the top line goes up 1 each time and the bottom line goes up three each time.* By contrast, for the patterning question Alison said, *Divide 4 by 30. Because each time the first square is 4 and you keep adding 4 on so I got to 28 and that gave 105*

and half it and so it's 2. This supports the finding from the written test that students seemed to find it easier to verbalise the generalisation in the table questions.

High achieving "patterning" students, that is, students at the 100th percentile, exhibited a high level of arithmetic competence, recognised the generalisation at an early stage and articulated their thinking in a manner that aided expressing the generalisations in symbols. For example, Bede said, *Each time a stick is added to the top and the bottom and then one on the side and it's repeated. (To work out the number of sticks (sic)) Amount of squares by 3 add 1.* They exhibited a high degree of flexibility in their thinking (i.e., were willing to change their approach to the problem). For example, when Bede was asked if he could reconstruct the pattern by another method, he said, *Take all the centres out (leaving the 1st and last vertical) so you have a rectangle and then fill the sides in to make squares.* They tended to break the pattern into differing repetitive parts and reconstruct the pattern using these parts. Not only could they all recreate the pattern in a number of ways but also could express each method algebraically and recognise that all of expressions they generated were equivalent.

High achieving "table" students, that is, students at the 100th percentile, also exhibited a high level of arithmetic competence, and recognised the generalisation at an early stage. They all seemed to select a pair of data and attempted to find the relationship by trailing addition and multiplication. They exhibited a reluctance to trial subtraction. Once they found a relationship that successfully linked one pair they quickly checked it by considering another pair of data. If their initial 'guess' was incorrect they quickly found another relationship. This approach contrasted with the low achieving "table" students. Once these students identified an initial function they exhibited a reluctance to change either the coefficient of x or the constant. For the high achieving students, an inability to recognise the correct functional generalisation usually resulted in their not being able to attempt the question at all. They did not use either the additive strategy or the ratio strategy.

All students were asked to complete both the patterning question and the corresponding table question. None of the low achieving students in each group successfully completed either question. While the high achieving "patterning" students were all able to complete the table question, only one of the high achieving "table" students could identify the correct generalisation for the patterning question. She could also reform the pattern in differing ways, that is, flexibly manipulate the pattern. She said, *Times the sides by 2 (to obtain the number of horizontal sticks needed for the pattern) and add the number plus one (to obtain the number of vertical sticks needed for the pattern).* She then said, *You could multiply one less than the number of squares by 3 and add 4 for the start (the first square).* The other two high achieving table students failed to exhibit the same degree of flexibility.

In summary, it seemed that students, on the whole, experienced a greater degree of success on the table component than on the patterning component. The low achieving table students shared many common characteristics with their patterning cohorts. Students in both groups made arithmetic errors, had difficulty in expressing their generalisations in symbols, and were unable to think flexibly. But the low achieving students in the table component experienced a greater degree of success in articulating their thinking.

To be successful in the patterning component it seems that students required some added characteristics above and beyond those exhibited by their table cohorts. They needed to be able to continually manipulate, both physically and/or visually, the materials to form new generalisations. This flexibility in thinking not only allowed them to understand the

notion of equivalence but also to see the common structures between the patterning question and the table question. They were the only group who could successfully map one problem onto the other, that is, reason analogically. A typical response was, *the input number in the table is the same as the number of sticks*, thus successfully mapping the relational components of each question. Their table cohorts were unable to do this. Another added attribute of the successful patterning group was that they continually related their generalisation to the problem under consideration. This was reflected in the permissible values they assigned to the variable in their generalisations. They continually checked if the values resulted in 'imaginary' situations, such as, a negative number of boxes.

Discussion of findings

Many of the findings in this study confirm and extend the results found in this recent research. As indicated by Stacey and McGregor (1995), language seems to play an important role in the generalising process. The present findings confirm the view that students not only find it easier to articulate their generalisations than to describe the generalisation symbolically (Kaput, 1992; Stacey & MacGregor, 1995; Yerushalmy & Shterenberg; 1994) but also find it easier to describe generalisations in tables of data than in visual patterns. The expression of patterns as numbers seemed to assist students in their search for numeric relationships. This could reflect a change in the processing load imposed by these tasks (English & Halford, 1995).

The literature does not appear to delineate characteristics of successful and unsuccessful students in this area of research. Orton and Orton (1994) acknowledged the importance of arithmetic competence and Stacey and MacGregor (1995) acknowledged the importance of correct verbal descriptions. The results indicated spatial visualisation, that is, the skill of mentally manipulating, rotating, twisting and inverting pictorially represented stimuli (Tartre, 1990), seemed important for generalising from visual patterns. For example, successful students manipulated the materials to form patterns and recognised the pattern being generated. They also broke the pattern into repetitive parts and reconstructed the visual pattern in a variety of ways. All of these processes seemed to involve spatial reasoning.

Analogicial reasoning also seems to play a significant role in successfully transferring between and linking representations. In the semi-structured interview, students who linked the two questions exhibited an ability to map the relational structure of one problem onto the other (i.e., reason analogically). The number of students who failed to link the patterning question with the table question confirms Lesh, Post and Behr's (1987) belief that students rarely seem to get things into a single coherent representation.

The results of the semi-structured interview intimated that both dimensions of flexible thinking were important for generalising from visual patterns and tables of data. Successful students seemed to be able to perceive the visual patterns from differing perspectives (Tartre, 1990). They also quickly changed their approach to reaching a generalisation once they realised that their generalisation was not applicable for all the cases presented. For example, in the semi-structured interview, high achieving students seemed to search for a generalisation by considering one pair of data at a time and then tested if the relationship fitted all other pairs. If the relationship was incorrect, high achieving students quickly refocussed on the original pair and discovered another correct relationship.

This research gives initial insights into why students are experiencing difficulties. Both approaches seem to require an array of reasoning processes not generally believed to be needed for early algebraic understanding. For example, the role of spatial reasoning in the algebraic domain is not generally acknowledged in the teaching of algebra. The intention behind these approaches is to introduce the concept of a variable in a more meaningful way. As this research

begins to indicate, such approaches may be so complex that they hinder rather than help understanding.

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Which jar gives the better chance? Children's decision making strategies.

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In task-based interviews 48 Kindergarten to Year 6 children were asked to choose between two jars containing different mixes of red and yellow toy bears, with the aim of giving themselves a better chance at drawing out a red bear. The children applied a variety of strategies, ranging from idiosyncratic reasons to proportional reasoning. These strategies are examined in relation to the ratio pairs presented in each jar and are compared to other strategies reported in the literature.

Probability can be defined as being "..composed of two subconcepts: chance and proportion." (Falk, Falk & Levin, 1980; p183). It is necessary to be aware of uncertainty in a situation so that proportional reasoning can be appropriately applied. Some studies have established that young children, with under-developed proportional thinking, are able to apply non-numerical or estimation strategies to make appropriate probability judgements (Spinillo, 1995; Falk, Falk & Levin, 1980). Fischbein, Pampu & Manzat (1970) advocate that, with appropriate instruction, young children can learn to develop their intuitive strategies into proportional reasoning. Their research revealed that 9 and 10 year olds were particularly responsive to the instruction that involved the use of discrete objects to model ratios and to form subgroups to assist in the comparison of ratios. However, other researchers argue that proportional reasoning is slow to develop and is not consistently used until formal operational thinking is well established (Piaget, 1975; Green 1983, Lovell & Butterworth, 1966), although there is disagreement about the age at which this occurs. Perhaps the type of research most useful to mathematics educators is the type that provides some insight into the strategies that children use in probabilistic decision making as they gradually develop proportional reasoning.

The classic 'two-urn-choice task', introduced in Piaget's studies has often formed the basis of research tasks for exploring proportional thinking. In such tasks, the child is shown two containers holding different mixes of two colours of small objects. The target, or 'favourable colour' is specified, and the child is asked to choose the container that would provide the better chance of drawing out the target colour without looking. According to Piaget (1975), applying proportional reasoning to such a task would require one of two approaches: constructing a ratio for each container that compares the number of each colour; or constructing a probability of the specified outcome occurring with each container. The first approach involves examining the part-part relationship within each sample space. The second approach involves part-whole thinking. Also of importance is the conservation of ratio and proportion, which enables us to work with equivalent fractions and recognise the equality of two ratios (Falk, Falk & Levin, 1980; Lovell & Butterworth, 1966).

There are, however, other appropriate strategies that require the understanding of proportions, but do not require precise calculations. Behr, Harel, Post and Lesh (1992) refer to research investigating children's qualitative reasoning when dealing with order relations between fractions and between ratios. The problems they presented to children were based on the equation $a/b = c$. Information about the relationship between two of these variables, without allocating specific numerical values, was provided to the children, who were required to determine the nature of the effect this relationship would have on the third variable. For example, if a remains the same and b decreases, then the value of c would increase. The problems also required comparison between two sets of relationships; $a_1 / b_1 = c_1$ and $a_2 / b_2 = c_2$, not unlike the two-urn-choice' task referred to previously. For example, if a_1 is less than a_2 and $b_1 = b_2$, which is greater c_1 or c_2. The authors refer to the orange juice mixture tasks used in Noelting's (1980) research, and to Siegler's (1976) masses on a balance beam

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tasks, to illustrate the differences between two main categories of task type; invariance of ratio and invariance of product. These categories are each further split into two subcategories; find-product-order and find-factor-order, and find-rate-order and find-rate-quantity.

These categories and subcategories are relevant to two-urn-choice studies in several ways. The children are asked to make a comparison between two sets where a is the number of target-colour objects and b is the number of other-colour objects. Some children choose to focus on the relationship between a_1 and a_2 , or on the relationship between b_1 and b_2 or to determine c_1 and c_2 . Even though the children are presented with specific numbers for each variable, it is possible to produce a 'correct choice' and valid explanation by making qualitative judgements to determine the direction of a relationship rather than calculate precise relationships. In some tasks, depending on the colour-colour ratios, additive or subtractive strategies may be all that is required for a successful choice, while other tasks may demand multiplicative thinking to support an appropriate choice.

Method

The particular piece of research reported in this paper is part of a larger study using task-based interviews to investigate children's understanding of basic probability concepts prior to formal instruction. The main purpose of the task reported here was to examine the levels, and types, of proportional reasoning being applied by the children. The total sample was drawn from two different NSW schools, one a country school (sample J) and the other from the western fringe of the Sydney metropolitan area (sample S). K-6 teachers were asked to provide four 'average' ability children from each grade who would be reasonably comfortable in a one-to-one interview situation within the school. Some of these children may have experienced some lessons dealing with basic probability concepts, but because this is not yet part of the NSW Primary Mathematics Syllabus, they will not have received any systematic instruction. Table 1 shows some details about the sample.

Age Group	Male	Female	Total
5/6 yrs	7	7	14
7/8 yrs	4	8	12
9/10 yrs	9	3	12
11/12 yrs	5	5	10
Total	25	23	48

Table 1: Games 1 to 4 Samples J and S

Age Group	Male	Female	Total
5/6 yrs	5	4	9
7/8 yrs	3	3	6
9/10 yrs	4	2	6
11/12 yrs	2	2	4
Total	14	11	25

Table 2: Games 5 to 7 Sample S

All the children played a 'warm-up' game to familiarise them with materials and procedures. The task took the form of a game, where the goal was to choose the jar of small plastic red and yellow bears that would give a better chance of picking out red bears without looking. The contents for each of the two jars were first lined up in front of the jars, in a line of red and a line of yellow. The question asked was: "Will one of these jars give a better chance for picking out red bears, or do they both give the same chance?" An explanation of the reason for the choice was sought. The bears were then placed into the jar/s chosen and five draws, with replacement, were made by the child. A reward of 'smarties' (tiny chocolate & candy) was given to the child according to the number of red bears drawn.

The main interview set, common to the children from both schools, contained four such games. The children from the second school (25 children) were given an additional three games which involved different types of colour mixes. The actual ratios of red and yellow used are presented in the column headings of Table 9. Each interview was tape recorded and transcribed into a data base.

Results

The choices made by each child in each game were determined to be either correct or incorrect based on the calculated probability for each jar. A reason was deemed to be 'correct' if it used the information provided by the contents of the jar (sample space) appropriately. Tables 3 and 4 contain these results in the form of percentages for each age group for the four main games, and for the additional three games. Tables 5 and 6 provide similar details for each separate game. In the overall results there is an obvious progression with age and a clear difference between the number of correct choices and 'correct' reasons. However, Games 1 and 3 produced some surprising results in the 5/6 year old age group, where these children performed better than older children. A possible explanation for this emerges when the details of the actual tasks and the choice strategies used by children are examined.

Age Group	Correct Choices	Correct Reasons
5/6 yrs	62.5	21.4
7/8 yrs	67.4	20.0
9/10 yrs	83.3	33.3
11/12 yrs	85.0	58.3

Table 3: Percentages correct. Games 1 to 4

Age Group	Correct Choices	Correct Reasons
5/6 yrs	70.4	14.8
7/8 yrs	66.7	22.2
9/10 yrs	88.9	16.7
11/12 yrs	100	16.7

Table 4: Percentages correct. Games 5 to 7

	Game 1		Game 2		Game 3		Game 4	
	Choice	Reason	Choice	Reason	Choice	Reason	Choice	Reason
5/6 yrs	92.9	78.6	21.4	0	92.9	0	35.7	0
7/8 yrs	83.3	58.3	41.7	16.7	63.6	0	81.8	0
9/10yrs	83.3	83.3	58.3	33.3	75.0	16.7	91.7	16.7
11/12yr	100	100	70.0	66.7	70.0	44.4	100	22.2
TOTAL	87.5	76.6	45.8	25.5	76.6	12.8	72.9	8.5

Table 5: Percentage Correct each Game. Games 1 to 4.

	Game 5		Game 6		Game 7	
	Choice	Reason	Choice	Reason	Choice	Reason
5/6 yrs	88.9	0	44.4	0	77.8	44.4
7/8 yrs	66.7	0	66.4	0	66.7	66.7
9/10yrs	83.3	0	83.3	0	100	75.0
11/12yr	100	0	100	0	100	75.0
TOTAL	84.0	0	76.0	0	84.0	56.0

Table 6: Percentage Correct each Game. Games 5 to 7.

Closer examination of the reasons given by the children for their choices revealed five distinct categories, with a further six subcategories. There was some indication that a larger sample of children may have provided opportunity for creation of a few more subcategories. These categories are explained below.

Reasoning Categories

1. Idiosyncratic

a) *No reference to the number* of bears in either jar: For example: Samantha (5 yrs 2 mths), "It's the jar that won before."

b) *Reference to number* is made but does not refer to any comparison of colours, contradicts the choice or makes no sense to the researcher. Sally (5yrs 6mths), " The same. There's 1R and 2R there."

2. Comparison of Favourable Events.

The children refer to the sample spaces but are only really concerned with comparing the number of favourable colours (Red) in each jar. The jar with more Reds is chosen. For example, John (9yrs 1mth) "This has got 3 red, that's got 1 red. It's a better chance having the red."

3. Comparison of Unfavourable Events

The children refer to the sample spaces but are only really concerned with comparing the number of unfavourable colours (Yellow) in each jar. The jar with fewer Yellows is chosen. For example, Sinclair (8yrs 11mths), "There's less yellow than the other one."

4. Subtractive Comparison

The difference between the number of Reds (favourable) and Yellows (unfavourable) in each jar is determined and compared. The jar with the smaller difference is chosen. Sometimes the child clearly states the numerical differences in their explanations, and sometimes the comparison is less specific. For example, Jason (9yrs 11mths), "They're the same because that one's got 3 more yellow (than red) and that one's got 3 more yellow (than red)."; and Sven (9yrs 4mths), "You have more red than the yellow (in the 2nd jar). In this one (1st jar) you have nearly exactly the same (amount of red and yellow)."

5. Proportional Comparison

To fit into this category a response had to indicate that an attempt had been made to define the relationship between the number of Red and Yellow objects in each jar, and that this relationship formed the basis for comparison of the two sample spaces. Within this category the range of responses can be further sorted into subcategories. Although the order in which the subcategories are listed suggest a gradual increase in the level of sophistication of reasoning, this may not necessarily be the case for all responses. This is because there is some evidence that the nature of certain ratios used in the interview prompted certain strategies.

a) Multiplication (including doubling by addition): In some cases fractions were not constructed but multiplication was used in the reasoning. For example; Justin (9yrs 6mths), "2 red against 4 yellow (in 1st jar), and this one (2nd jar) it's got double the red (of the 1st jar) and double the yellow."

b) Approximation: Some children used approximations to express the comparison they had made. Once either the balance or the direction of the imbalance had been determined, sufficient information was available for making an appropriate decision without determining a specific numerical relationship. For example, Jane (11yrs 5mths), "There's 1 red and 3 yellow (in 1st jar). There's more yellow on this side (in 2nd jar) - because there's only 2 red and there's 8 yellow it's not even a half, it's probably only a quarter." (Note: This strategy illustrates the qualitative reasoning reported by Behr, Harel, Post and Lesh (1992)).

c) Component Ratios: A small number of children used component ratios, or subgrouping, to define the relationship between the numbers of colours in each sample space. In the following example a child is trying to explain that one jar containing 2 red and 4 yellow objects presents the same probability for drawing a red as the second jar containing 4 red and 8 yellow, because the objects in the second jar can be physically divided into two groups of 2 red and 4 yellow; Julie (12yrs 3mths), "2 fours go into to 8 and that's 4 ..um..it's the same amount, 2 there, 2 there. (In the 1st jar) those 2 (red) match (with 2 yellow) and there's 2 again (total of 4 yellow). (In the 2nd jar) 4 (red which she separates into pairs) match 4 (yellow) and there's 4 (yellow) again." (Note: This is the strategy used in Fischbein, Pampu & Manzat's (1970) instructional study).

d) Fractions, actually part-part ratios: Some children constructed precise fractions from the number or red and yellow objects in each container, then compared the fractions and chose the 'larger' fraction. No child constructed a part-whole probability fraction. For example, Jack (11yrs 8mths), "'Cause that's half and that's only quarter."

Strategy Category	5/6 yrs	7/8 yrs	9/10 yrs	11/12 yr
1. Idiosyncratic	24.1	24.2	7.6	4.2
2. Comparing favourable	53.0	32.3	18.2	14.6
3. Comparing unfavourable	6.0	27.4	40.9	8.3
4. Subtractive	16.8	25.8	42.4	41.7
5. Proportional	0	1.6	18.2	25.0

Table 7: Percentage of children in each age group using each strategy. Games 1 to 7.

Of interest is the tendency for 5 to 8 year olds to apply strategies 1 and 2, and the strong similarity of the figures for the 9/10 yrs and 11/12 yrs groups, except for Strategy 3. There is quite a clear increase in the use of proportional reasoning with age. It is also worth noting that, with the exception of 5/6 year olds and Proportional Reasoning, children of all age groups used the full range of strategies. However, very few children applied the same strategy to all games.

Strategy Category	1	2	3	4	5	6	7
1. Idiosyncratic	8.7	23.9	10.9	19.6	12	24	12
2. Comparing favourable	84.8	17.4	50	13	32	8	12
3. Comparing unfavourable	0	10.9	6.5	34.8	0	4	4
4. Subtractive	6.5	17.4	17.4	23.9	56	64	72
5. Proportional	0	30.4	15.2	8.7	0	0	0

Table 8: Percentage of children using each strategy in each game.

The top of Table 9 shows the ratio of red and yellow bears in each jar, with the first vertical pair referring to the first jar and the second vertical pair referring to the second jar. The remainder of the table indicates whether application of a particular strategy would result in the correct jar selection. It can be seen that although Proportional thinking is appropriate for all games, there are less demanding strategies that lead to correct choices in all games except Game 2.

The results in Table 9 suggest that certain tasks are more likely to stimulate particular strategies than others. For example, Game 2 produced more proportional reasoning than any other, while Games 1, 5, 6 & 7 produced none. The relationship between the type of ratios used in each game and the type of strategies used warrants further examination.

	Game 1 1R 3R 4Y 4Y	Game 2 2R 4R 4Y 8Y	Game 3 1R 3R 4Y 6Y	Game 4 1R 2R 3Y 8Y	Game 5 3R 5R 2Y 3Y	Game 6 4R 6R 2Y 5Y	Game 7 3R 3R 5Y 7Y
1. Idiosyncratic a) No number	Not approp.	Not approp.	Not approp.	Not approp.	Not approp.	Not approp.	Not approp.
b) Number	Not approp.	Not approp.	Not approp.	Not approp.	Not approp.	Not approp.	Not approp.
2. Comparing favourable	Correct choice	Wrong choice	Correct choice	Wrong choice	Correct choice	Wrong choice	Not approp.
3. Comparing unfavourable	Wrong choice	Wrong choice	Wrong choice	Correct choice	Wrong choice	Correct choice	Correct choice
4. Subtractive	Wrong choice	Wrong choice	Wrong choice	Correct choice	Wrong choice	Correct choice	Correct choice
5. Proportional a) Multiplication	Correct choice	Correct choice	Correct choice	Correct choice	Not approp.	Not approp.	Correct choice
b) Estimation	Correct choice	Correct choice	Correct choice	Correct choice	Correct choice	Correct choice	Correct choice
c) Component ratios	Correct choice	Correct choice	Correct choice	Correct choice	Correct but hard	Correct but hard	Correct but hard
d) Ratio construction	Correct choice	Correct choice	Correct choice	Correct choice	Correct but hard	Correct but hard	Correct but hard

Table 9: The potential of each strategy in producing a correct choice in each game.

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Discussion

The main focus of this paper is to report on the use of various strategies employed by the children in problems involving pairs of ratios. There have been a number of studies conducted using simular problems that have also revealed a variety of response categories These include Chapman, 1975; Fischbein, Pampu & Manzat, 1970; Green, 1983; Lamon, 1993; Siegler, 1986; Truran, 1992. The categories reported by these researchers range from being too narrow to include all the categories found in this study (Siegler, 1986; Fischbein et al.,1970), to being too finely divided to be of great practical use (Truran, 1992). The strategies listed by Lamon (1993) perhaps supply the best compromise between these extremes. There are, however, sufficient overlaps across all the reported categories to lend support to the idea that there is a set of consistently identifiable proportional reasoning categories. Table 10 provides a summary of the overlaps between some of these studies, particularly in reference to the categories found in this study. An expansion of this table to include the examination of several other significant studies should provide sufficient information to propose a stable set of categories that could become a very useful resource for formulating specific research tasks and informing educators on beneficial instructional approaches.

Even though not all types of ratio pairs were included in this study it is obvious from the results that particular ratio pairs are likely to prompt particular reasoning strategies. Comparing the information in Tables 8 and 9 provides support for this idea. For example, in Game 1 it was only necessary to compare the favourable colours (red) in each jar to make a correct choice, and 84.8% of the children used this strategy. In contrast, Game 2 required some form of Proportional reasoning for a correct choice and 30.4% of the children used this strategy; the highest percentage of use across all games. Games 5, 6 and 7 involved ratios that were difficult to compare proportionally because they relied on one's ability to create equivalent fractions. Consequently, even children who had used Proportional reasoning in Game 2, relied on the less appropriate and less reliable Subtractive strategy.

In general, the outcomes of this study suggest that children as young as 5 years possess a repertoire of strategies to select from in reaction to the type of ratio pairs presented to them. These reactions may disguise an underlying developmental sequence to these strategies. Closer examination of the interview transcriptions may reveal response patterns in individual children, that may give further insight into the nature of these reactions. Several children possessed powerful intuitive or qualitative strategies that also warrant closer investigation. There were also hints of some interesting intuitive strategies in the Idiosyncratic category, particularly amongst the younger children who had some difficulty in verbalising their decision making processes. Again, closer examination of the raw data may reveal response patterns. These investigations, together with a completed synthesis of existing research on the emergence of proportional strategies (such as begun in Table 10), could provide the foundation for designing a revealing research project on children's reasoning in two-choice probability tasks.

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Table 10: Summary of commonalities between reported strategies.

Way 97	Lamon 93	Truran 92 *	Siegler 86	Green 83	Chapm
	Avoiding	Strategy 1		Non-response	
Idiosyncratic a) No number	Visual or			Other, and "They are the same"	0 Rating No quan relations
b) Number	additive	Strategies 2 &5.		Comparison within bag	1 Rating Referenc number
Comparison of favourable	Preproportional reasoning	Strategies 7 & 8 Partly Strategy 17	Rule 1	Comparison between bags of one colour	2 Rating
Comparison of unfavourable		Strategies 9 & 10	Includes Rule 2		Compari colour
Subtractive comparison		Strategies 14 & 15.	Rule 3	Difference comparison	
Proportional a) Multiplication	Qualitative proportional reasoning. (Pattern Building?)	Partly Strategy 17			3 Rating Both col and both containe
b) Estimation		Strategies 12 & 13 & 16	Rule 4	Ratio comparison	compare
c) Component ratios	Quantitative proportional reasoning	Strategy 11.			
d) Construction of ratios		Strategies 18 & 19			

*Several of Truran's categories have not been included because they did not correspond sufficiently with other identified categories.

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Recognition of abstract angles in familiar physical situations

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A sample of 480 students from grades 2, 4, 6 and 8 were interviewed concerning their recognition of abstract angles in nine familiar physical situations. The major factor influencing students' interpretation of an angle situation was not the number of arms of the abstract angle visible in the physical situation. Even by the end of grade 8, many students had difficulty recognising angles in situations where both arms of the angle were not visible. Implications are drawn for the teaching of angles.

The development of an abstract concept can be considered to be the result of a classification process which starts with the identification of similarities between different experiences (Mitchelmore & White, 1995; White & Mitchelmore, 1996). A definition of any given concept is a succinct description of the predominant similarities in all the different situations from which the concept has been abstracted. In the case of angle, the definition given by the New South Wales K - 6 syllabus is "the amount of turning between two lines about a common point" (New South Wales Department of Education, 1989, p.79). This definition derives from the argument that the development of the angle concept consists of establishing a particular relationship between the two arms of the angle; and a turn is a familiar, physical action which relates the two arms (Wilson & Adams, 1992). For example, the angle of slope in a hill needs to be seen as the amount of turning between the horizontal and a line which gives the slope of the hill. Given that a hill does not involve physical turning and the horizontal line is not visible, the question arises as to how well students can actually visualise abstract angles in familiar physical situations. How well do they see the turn, the two lines and the common point?

There is little published research on students' interpretations of angles. Krainer (1989) reports that 12 year olds tended to give examples of angles which involved some sort of turning motion. Davey & Pegg (1991) found that, among students from Grades 1 to 10, descriptions of an angle went through a sequence of four stages: (1) a corner which is pointy or sharp; (2) a place where two lines meet; (3) the distance or area between two lines; and (4) the difference between the slopes of the lines. Clements & Battista (1989) report that grade 3 students commonly described an angle as a sloping line, a place where two lines meet, the two lines themselves, or a turn (the last only by students who had studied Logo). Such studies suggest that students do conceptualise their surroundings in terms of angles, and that their concept of angle develops in sophistication as they get older. However, these studies do not show how the developing angle concept is linked to children's physical experiences.

Recent studies (Mitchelmore, 1996; White & Mitchelmore, 1995) suggest, for example, that young children may not interpret many physical angle situations in terms of turning. In these studies, children in each of grades 2 and 4 were presented with models of a number of physical situations (a doll turning about a fixed axis, a variable hill, a pair of scissors, a map of some bending roads, a ball game and floor tiles). It was found that children often did not see the desired angle-related features in the situations, but focused on features arising from other specific details within the situation. In particular, angles

occurring in the turning doll and rebounds in the ball game were generally seen as quite different to angles in the corners of tiles, slopes of a hill, bends in a road and opening/closing objects like scissors. Although the grade 4 children had all studied angles as an amount of turning in school, very few stated this definition of angle or gave turning as a example of an angle. Furthermore, about a quarter of the students explicitly stated that the turning doll situation did not involve angles.

Mitchelmore and White (1996b) specifically investigated how children from each of grades 2, 4 and 6 conceptualise and classify various turning situations. It was found that between grades 2 and 6 there was a general increase in children's tendency to recognise angle-related similarities between turning situations. However, relatively few children referred to the common movement in these situations as turning. In fact, children more often reported angle-related similarities based on the static appearance of the angles than ones based the dynamic way in which the angles were formed.

We have recently completed a study of how students in grades 2, 4, 6 and 8 interpret familiar physical situations in terms of angles. The grade 2 results were presented in Mitchelmore and White (1996a). The present paper reports on grades 4, 6 and 8. Some grade 2 responses have since been rescored for consistency with the analysis of the data from grades 4-8.

Method

Sample

The sample was gender-balanced and consisted of 144 children in each of grades 2, 4, 6 chosen from six schools in Sydney. A further 48 grade 8 students from two high schools were also interviewed.

Materials

Nine physical angle situations were used: wheel, door, scissors, hand fan, signpost, hill, road junction, tile and wall. The first four of these were movable while the last five were fixed. Each movable situation was represented by a single adjustable model. Each of the fixed situations was represented by a set of three models representing a "neutral" configuration (angle 0° or 90°), an angle of 45°, and a "middle" angle (22.5° or 67.5°). Adjustable models of the fixed situations were deliberately not used, in order to avoid suggesting a turning interpretation. Examples of the models used are shown in Figure 1.

A drinking straw which could be bent at various angles was used as an abstract angle model. A second straw fixed at 45° was also used.

Procedure

A trained research assistant administered interviews which investigated how well children modelled angles in the standard manner. The term "standard manner" refers to generally accepted angular modelling in which the vertex and the two arms of the abstract model match a point and two lines on the physical model. For the moveable situations, the vertex is the point of rotation and the opening of the arms represents the amount of rotation. For the fixed situations, the two arms represent two specific lines and the vertex is the point where these meet.

During the interview, the students were asked to complete three tasks.

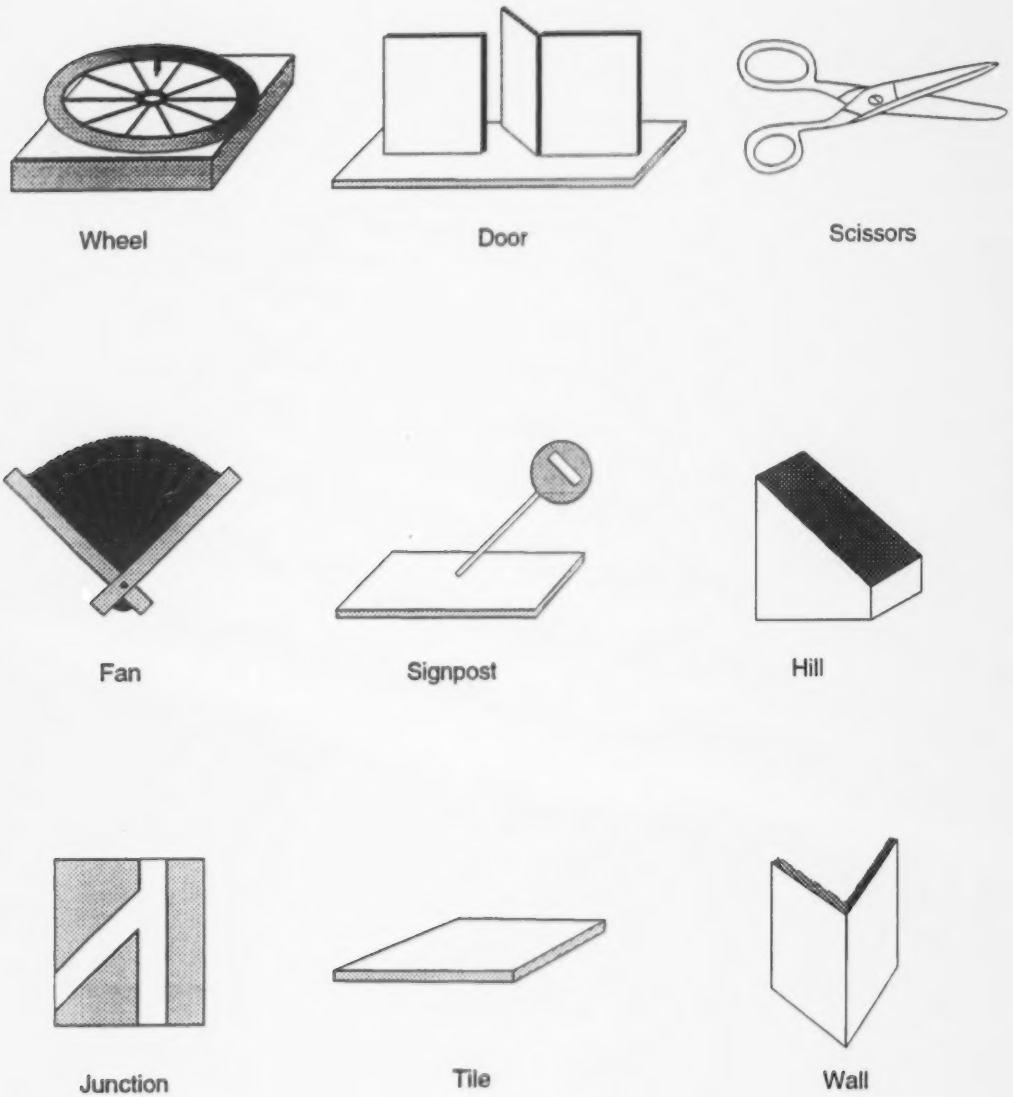


Figure 1: Models used to represent nine physical angle situations

Task 1: Students were asked to show how the abstract angle model could represent the given situation. For example, the student was asked to use the flexible drinking laid on the table in front of her/him to show the turning wheel or a corner of the tile.

Task 2: Students were asked to demonstrate an angle of 45° on the physical model. (The angle of 45° was defined for the student by opening the flexible drinking straw by about 45° and then replacing it by the fixed straw.) For example, students were asked to turn the wheel through 45° or to select the tile (from three possibilities) whose corner had an angle of 45°. For the movable models an error of ±15° was allowed.

Task 3: Students were asked to place the abstract angle model on the physical model to explain how the 45° angle was formed. In scoring the data, some imprecision was ignored. For example, standard modelling was inferred when the vertex was not exactly on the pivot for the scissors or the lines were not exactly parallel to the edge of the roads of the junction.

Each of the nine situations was administered to 48 students in each of grades 2, 4 and 6. (Each student responded to three situations.) When the analysis of these data showed that the wheel, door and hill were still causing difficulty to grade 6 students, these three situations were also administered to 48 students from grade 8.

Results

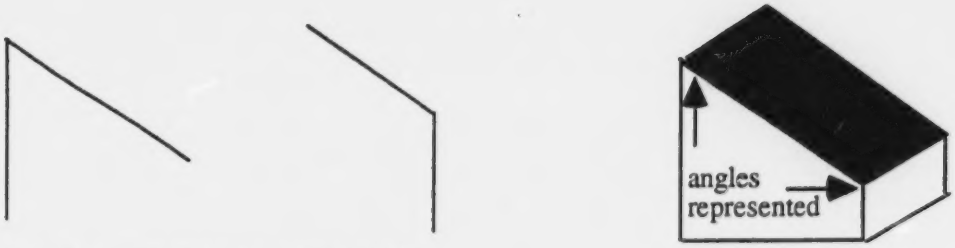
Task 1

The percentages of students who used the abstract model to represent the angular features of each situation in the standard manner are shown in Table 1.

Table 1
Percentage of sample using abstract angle model to represent physical angles in standard manner

Grade	Wheel	Door	Scissors	Fan	Signpost	Hill	Junction	Tile	Wall
2	17	90	100	98	24	48	77	96	96
4	90	96	90	88	67	50	85	100	98
6	94	92	94	92	83	67	96	100	100
8	94	94	—	—	—	59	—	—	—

Mitchelmore and White (1996a) previously noted the difficulties caused by the wheel, signpost and hill for grade 2 students. The difficulties with the wheel largely disappeared by grade 4, and with the signpost by grade 6. However, the hill was still difficult in grade 8. The tendency to represent the hill by a single sloping line (common in grade 2) was rare in grades 4-8. The most common incorrect configuration showed both lines sloping differently in an inverted V-shape. Furthermore, only 19% from grade 4, 27% from grade 6 and 17% from grade 8 represented the angle between the horizontal and the slope of the hill (as most mathematics teachers probably would). The rest of the students who correctly represented the slope using a standard angle preferred to use one of the vertical edges of the hill, as shown in Figure 2. That is, most students opted for a line actually present in the physical model for their second reference line.



Abstract representations for the hill

Hill model

Figure 2

Task 2

The percentages of students who correctly demonstrated an angle of 45° on the physical model are shown in Table 2.

Table 2

Percentage of sample correctly demonstrating size of abstract 45° angle on physical models

Grade	Wheel	Door	Scissors	Fan	Signpost	Hill	Junction	Tile	Wall
2	49	77	87	87	47	43	60	64	64
4	85	100	100	100	88	58	92	96	98
6	98	98	98	98	96	65	96	98	100
8	94	100	—	—	—	75	—	—	—

Most of the difficulties grade 2 students had with this task (as reported in Mitchelmore & White, 1996a) disappeared by grade 4. The exception was the hill, where almost all the errors consisted of choosing the hill with a slope of 22.5°. Students seemed sure that the 45° abstract model represented a sloping hill, but were rather uncertain about how to represent the amount of slope.

Task 3

The percentages of students who could explain how the abstract model matched the physical model are shown in Table 3.

Table 3

Percentages of sample explaining how abstract 45° angle matched physical models

Grade	Wheel	Door	Scissors	Fan	Signpost	Hill	Junction	Tile	Wall
2	21	44	87	79	43	45	94	100	92
4	31	50	89	81	75	60	98	98	92
6	56	65	92	85	88	69	100	96	100
8	42	65	—	—	—	81	—	—	—

By grade 6, over 85% of students could explain how the abstract model matched the angle on the physical model for the scissors, fan, junction, tile and wall. In all of these situations, both arms of the angle are clearly visible. For the signpost, students appear to have learnt between grade 2 and grade 6 to use the horizontal ground as the

second arm of the angle. (Very few students modelled the angle between the post and the vertical.) However, even by grade 8, the abstract angle model was not easily matched to the 45° angle in the wheel or door. Performance on the hill improved markedly between grade 2 and grade 8, with most explanations again using visible lines on the physical model (see Figure 2).

Discussion

For each of eight of the nine physical models, more than 80% of grade 6 students used an abstract model to give a standard representation of the implicit angle and more than 95% demonstrated an angle of 45° correctly. The percentage who could explain how the abstract model matched the angle in the physical model was considerably lower in three of the nine situations. It seems safe to make two conclusions:

- by the end of grade 6, the majority of students can *globally* recognise abstract angles (Tasks 1 and 2) in almost all physical situations, but
- even by grade 8 many students cannot *analytically* recognise abstract angles (Task 3) in a number of physical situations—namely, those where the arms of the abstract angle do not match visible lines in the physical model.

The hill appears to be an anomaly. The data for the other eight situations either show a consistent increase across all three tasks (scissors, fan, signpost, junctions, tiles, walls) or a high percentage of students who recognised the abstract angles globally but not analytically (wheel, door). The hill, on the other hand, shows relative low rates of global recognition and relatively high rates of analytic recognition. We argue that this anomaly is caused by a focus on visible lines which accidentally suggest standard modelling. Responses to Tasks 1 and 2 indicate that, although most students have some global concept of slope, many do not quantify slope by relation to a fixed reference line. When such students respond to Task 3 by placing one line of the abstract model on the vertical edge, they could simply be choosing an available visible line to fit the abstract model. Such responses do not necessarily imply any awareness of the significance of the vertical edge as an appropriate reference line. We conjecture that, had the model of the hill been supported by edges which were either not vertical or not visible, far fewer students would have analytically recognised the implicit angle.

Conclusion

The data strongly support the claim that students up to grade 8 use visible lines as the basis for recognising abstract angles in familiar physical situations. Table 4 shows the data in Table 3 rearranged according to the number of visible lines, namely no lines visible (wheel), one line visible (door, hill), or two lines visible (scissors, fan, junction, tile, wall, signpost).

Where the two lines are obvious, children at all age levels have little difficulty using an abstract angle to model the situation. When only one line is visible, students have difficulty imagining the appropriate reference lines to make up an angle. Many are still having problems in grade 8. When no line is visible, students' difficulties are even greater.

We note that two of the three classes in Table 4 contain both movable and fixed situations. Whether a situation is movable or fixed does not seem to be a crucial determinant of the difficulty students have in identifying the abstract angle.

Table 4

Percentage of sample explaining how abstract 45° angle matches physical model, by number of visible lines

Grade	0 lines visible	1 line visible	2 lines visible
2	21	44	82
4	31	55	89
6	56	67	94
8	42	73	—

Implications for teaching

Our results question the argument that teaching the angle concept should be based on the particular physical relationship of turning between the two arms of the angle (Wilson & Adams, 1992). The data suggest turning is not spontaneously conceived as an angle and so it seems unlikely that a focus on turning is a good strategy. A more obvious strategy is to help children recognise the two lines needed to make up the angle and the significance of the relation between them (turning, sharpness, opening or whatever).

The results suggest that it should be possible to start teaching about angle as early as grade 2, provided the examples are limited to situations where both arms of the abstract model are clearly present. Other situations involving angles need to be placed in a hierarchy according to the number of visible lines. The following points are some suggested learning activities. A creative teacher will think of many more.

Situations with two visible lines

- Identify the two lines on scissors, fans, junctions, tiles etc.
- Identify the two lines on leaning objects such as signposts etc.
- Identify angle-related similarities between these situations. For example, place scissors around the corner of a tile or along the two roads which meet at a junction to show that each of these situations involves the same idea.
- Discuss the precise position of the vertex and the lines in some situations. For example, the pivot point on a pair of scissors is not where the cutting edges meet.
- Discuss features peculiar to each physical situation. For example, the sharpness of a tile corner is related to how pointy the corner is, not how thin or fat the tile is.

Situations with one visible line

- Identify the second (imaginary) line in doors and hills.
- Discuss what "horizontal" and "vertical" lines are.
- Link blackboard compasses with a door by showing how the one arm of the compasses is like the closed position of the door and the other arm like the opening door. Thus the imaginary line of a door matches a visible arm in the compass.
- A tile can be compared with a hill. A blackboard ruler can be used as a hill and tiles made or drawn so that some have corners which are not 90°. The angle between the hill and the horizontal is like a corner of a tile. The similarity is that both situations consist of two lines joined at a point.

Situations with no visible lines

- Start with a rotating fan to model the idea of a rotating radius and identify the beginning and finishing position of the radius. Identify also the centre of rotation.

- Model other rotations (wheels, knobs, body turns). Identify how many full turns are executed (two, one, half) and interpret these as a movement from the beginning to the finishing position.
- A flexible abstract model (e.g., using geostrips) can be used to model turns of a wheel by holding one arm fixed and rotating the other.

In conclusion, turning is easy to see in movable angle situations, but the two lines are not obvious. The two lines are more obvious in some fixed situations, but the turning is not easy to see. Students will only have a firm basis for further work on angle if they can identify both the lines and the amount of turn in a variety of familiar angle situations. The aim of early teaching should be to establish this foundation. Without it, it is no wonder that leaning to use a protractor is such a formidable task for many grade 7 and 8 students.

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Students' understanding of hypothesis testing: the case of the significance concept

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Throughout introductory tertiary statistics subjects, students are introduced to a multitude of statistical concepts and procedures. One such term, *significance*, has been given considerable emphasis in the statistical literature with respect to the topic of hypothesis testing. However, systematic research regarding this concept is very limited. This paper investigates students' conceptual and procedural knowledge of this concept through the use of concept maps and standard hypothesis tests. Eighteen students completing a first course in university-level statistics were interviewed twice during a 14-week semester.

Significance is perceived as an important concept in hypothesis testing, given its attention in the related statistical literature. In this literature, statistical significance is said to be frequently misinterpreted by users of statistics (see Menon, 1993; West, 1990). The practice of characterising significance with descriptions (e.g., "very significant" or "clearly significant"), statements (e.g., there is more evidence against the null hypothesis at the 0.01 than the 0.05 significance level), or symbols (e.g., ** to indicate significance at the 0.01 level), has been criticised for reinforcing the apparent objectivity of significant results (Falk, 1986). Furthermore, the significance testing literature is critical of those who perceive that obtaining statistical significance is the main goal of statistical tests (e.g., Clements, 1993); that decision making is black-and-white, solely based on the significance level (e.g., Oakes, 1986); or that practical aspects are not worthy of consideration (e.g., Moore & McCabe, 1989). Given that even experts have been known to misinterpret significance (McCloskey, 1990), and that journal editors have at times advocated an emphasis on statistical, rather than practical, decision making in their publications (Menon, 1993), it is not surprising that errors persist.

The study described here begins to fill a gap in a vast literature in which almost no empirical research has been performed. This paper reports on a qualitative study investigating students' understanding of the significance concept in hypothesis testing, where hypothesis testing is confined to one- and two-sample t and z tests. More specifically, this study aims at investigating the conceptual and procedural knowledge of significance by students undertaking an introductory course in statistics at the tertiary level. Conceptual knowledge is defined here as the knowledge of concepts and their interrelationships. Knowledge of a concept involves being able to define it, as well as make other statements about it. The latter may entail being able to summarise the issues pertaining to it, provide an example, or describe its features, uses, or limitations. Procedural knowledge is the knowledge of symbols, rules, algorithms, and procedures. In particular, procedural knowledge involves the use of the concept in an applied situation such as an hypothesis test. Understanding the concept therefore means possessing both conceptual and procedural knowledge.

For the purposes of this investigation, significance is defined more succinctly, but aligned with discussion in the statistical literature. Significance has two aspects (i) the decision to reject the null hypothesis after a statistical test has been performed, and (ii) the evaluation of the decision. The first aspect is called statistical significance, the rejection of the null hypothesis when at least one of three conditions hold, namely: (a) the observed value arising from the test is more extreme than the critical value obtained from the appropriate tables, (b) the p-value is less than the level of significance, or (c) the p-value is sufficiently small. The second aspect of significance involves the evaluation of the result, the consideration of factors which are statistically relevant (e.g., sample size, power, effect size,

likelihood of replication), or practically relevant to the particular area being investigated. These factors may lead to practical significance, if the decision to reject is maintained. For first year students with limited knowledge, the evaluation step may be limited to considering the relevance of the statistical result to the question posed, and perhaps to considering the sample size.

Method

Data reported in this paper were collected through two clinical interviews conducted with 18 volunteer students enrolled in a first course in university-level statistics. Students were interviewed several weeks after the topic of hypothesis testing was introduced to them, and again after the final exam in the subject. They were asked to talk aloud as they completed a concept mapping task and standard hypothesis tests, and answer questions after their completion. Student responses were analysed in terms of the conceptual and procedural knowledge exhibited.

In the concept mapping task, concept names associated with hypothesis testing were typed on separate labels. Students were requested to place the labels on an A3 sheet of paper in such a way as to show the relationships between the concepts. Subsequent questioning drew discussion on these relationships, and on the concepts themselves. Given the above definition of conceptual knowledge, this task aimed mainly at investigating students' conceptual knowledge. The hypothesis tests were standard text book exercises with the question clearly defined and numerical information provided. The two main hypothesis tests, named the light bulb task (a two-tailed one sample z test), and the night shift task (a one-tailed two sample independent t test) aimed mainly at investigating students' procedural knowledge.

In the statistics class in which the above students were enrolled, the term *significance* was not used very frequently by the lecturer. However, the lecturer gave the students advice on the significance concept in a number of ways. The need for assessing the evidence to support or reject a null hypothesis was emphasised, noting, for instance, that the smaller the p-value, the stronger the evidence was against the null hypothesis and in favour of the alternative hypothesis. Further, the lecture notes stated that a small p-value may mean (i) that the null hypothesis is false; (ii) that the null hypothesis is true and an unusual event has occurred; or (iii) that the model being tested is not applicable. In addition, the lecturer advised students that it was good practice to quote the p-value with the decision, so that the reader may also evaluate the conclusion; and that a result may be statistically significant but not practically important. In the Study Guide for the subject, *significance* was listed as one of the key words associated with the topic of hypothesis testing. However, the main text book for the course made little reference to the term, stating in only one example that the z value could be significant after the rejection of the null hypothesis at a particular level of significance.

Analysis

Analysis of students' responses during the first interview (beginning Week 11 of the semester) is followed by an analysis of students' responses during the second interview (after the final exam). For each interview, the concept map responses are examined prior to the hypothesis test responses. In this analysis, a student was considered to have demonstrated conceptual knowledge of the significance concept if the concept was defined or explained correctly, or linked with other concepts correctly. Demonstration of procedural knowledge of statistical significance was acknowledged when a student (i) reached a decision to reject or not reject the null hypothesis after performing a statistical test, and recognised significance or non-significance, (ii) discussed the steps used to obtain significance or non-significance (e.g., significance occurs when the p-value is less than the significance level), or (iii) evaluated the statistical decision. When at least one of these three steps was performed, yet the term *significance* was not mentioned, it was acknowledged as a demonstration of implicit procedural knowledge.

INTERVIEW 1 - WEEK 11

Concept Mapping Task: In Interview 1, 5 students used the significance label on their concept maps; 13 did not.

Definitional statements: Efforts to explain the significance concept during the concept map task fell into four main types: correct use, vague description, confusion with significance level, and incorrect use.

First was the correct use of the significance concept. Only Rhys was in this category. He suggested that rejection of the null hypothesis occurred with low values of the p-value, and that a p-value "closer to .5" was non-significant. His explanation, procedural as well as conceptual in nature, referred to the p-value method of hypothesis testing, and was the most advanced explanation offered by a student using the term.

Second, students used descriptions in the form of adjectives, phrases, or sentences, which gave a sense of where significance fitted into hypothesis testing. In general, these were vague or inadequate in demonstrating understanding. For instance, two students (Cheryl, Lisa) used *significance* to describe testing, decision, or results. Cheryl made fleeting reference to "significance in the testing" and "significance of decision". Lisa mentioned the need for "significant results." Neither provided additional information. Further, two students (Elvie and Koby) attempted more prolonged descriptions of significance. Elvie remarked, "I was just talking about like whether your results are statistically significant, because you can get results and like put in a probability and stuff and they look good but they might not be worthwhile kind of thing." Her interpretation of statistical significance hinted at finding a p-value. Koby, having stated that the z or t value had a certain significance, continued, "an amount of significance, yeah, and it affects your decision." Explanation of how significance affected the decision was not forthcoming. While the above students acknowledged the role of significance in testing, decision, results, or z or t values, none offered a clear explanation.

Third, students (e.g., Margaret, Phillip, Rhys, Kylie, Lorraine) interchanged the term with significance level, sometimes resulting in the interpretation of significance as a probability, or as a level for rejection. Several examples demonstrate such misinterpretations. Margaret stated:

> well significance is the, that's the probability that you want to be right, like if you want to be 99 percent sure that you're going to be correct, you need an alpha value of, you look up your little table and find nought point nought nine five or something like that.

Kylie explained, "significance is the, it's sort of interconnected with the significance level, it's the um level that you're prepared to reject it with, sort of the same thing." Consistent with his interpretation, Phillip wrote the alpha symbol α, next to the significance label on his concept map.

Fourth, significance was completely misinterpreted. As an illustration, Michael thought that significance referred to the significant figures in scientific notation for large and small numbers.

Thus, while most students mentioned the term *significance*, little understanding of the concept was manifested in their explanations. Only 1 student offered a correct explanation. Alternatively, one of three things occurred: either inadequate and vague connections were made with testing, decisions, results, or z or t values; the term was interchanged, in both name and meaning, with significance level; or it was completely misinterpreted.

Other statements: In addition to the above explanatory statements, several statements made during the concept mapping task described the steps involved in concluding rejection of the null hypothesis. Kylie's and Lorraine's comments illustrate. Kylie noted, "if your p-value is less than .05 this means you can reject your null hypothesis with 95 percent significance level." Without mention of the word *significance*, and despite using a numerically incorrect value for significance level, she conveyed an understanding of one of the conditions leading to statistical significance. Referring to a p-value falling in the critical rejection region, Lorraine stated, "then if it's significant, go to decision, then you make a decision." Given Lorraine's responses on other tasks, her statement, though not as succinct as Kylie's, portrayed a similar meaning. Kylie's protocol described a comparison of the p-value and the significance level, whereas Lorraine's implicitly compared areas represented by p-value and critical region. Both students, as well as Rhys, displayed procedural knowledge of the significance concept. In the case of Kylie, this knowledge was implicit.

Hypothesis Tests: During the performance of the hypothesis tests, the term *significance* was not mentioned by a single student, and there was little overall evidence of procedural knowledge. Only Phillip, Kylie and Lorraine completed an hypothesis test, the light bulb task, during this interview. Phillip rejected the null hypothesis using the critical value and confidence interval methods, and Kylie used the critical value method, rather than the p-value method suggested by her previous quote. Lorraine used the p-value method. Only Phillip and Lorraine related their statistical conclusion back to the original question, and all 3 students had to be prompted for further discussion. Their procedural knowledge of significance was therefore implicit. Rhys, who had previously demonstrated procedural knowledge of significance, could not use it in the hypothesis tests. Thus, it appears that procedural knowledge of significance was evident in only a few students, and even then it was implicit.

The evidence above suggests that, at the time of the first interview (beginning Week 11), significance was not a well known term. Only 11 students mentioned the term during the course of the interview. In general, the concept was either linked vaguely to testing, decisions, results, or z or t values, or was used synonymously with significance level. Several students demonstrated implicit procedural understanding of significance when they described the conditions for decision making, or rejection of the null hypothesis. Only 2 students reached a conclusion to reject the null hypothesis during the performance of their hypothesis tests. This, too, was classed as demonstrating implicit procedural knowledge. Hence, conceptual knowledge and procedural knowledge were limited overall in both the explicit and implicit senses.

INTERVIEW 2 - AFTER FINAL EXAM

Concept Mapping Task: At the second interview, 9 students (compared to 5 previously) used the significance label on their concept maps; 9 did not.

Definitional statements: Explanations of the significance concept were classified as in Interview 2: correct; vague; interchanged with significance level; and misinterpreted. No statements fitted the first or fourth classifications.

Explanations of the second type (5 students) were illustrated through the brief or inadequate descriptions by Cheryl, Lorraine, and Koby. For example, Cheryl stated that significance related to significance testing. As before, she could elaborate no further. Lorraine reasoned that significance was "how significant the test is." When asked how that could be judged, she was unable to explain. Later she stated that, "if it lies in the accepted or rejected region and then you, if you want to test the significance of it there." While the meaning of both remarks remains unclear, the latter appears to imply that significance has something to do with acceptance or rejection regions. Koby said, "the statistic might be very significant if you have a high p-value, which means the null hypothesis is true." When asked to explain "very significant", she continued, "well if it's high, the p-value's high, there's a high significance level ... [like] .8, anything over .5 is reasonably high." Koby had interpreted the relationship with p-value in the wrong direction, but perceived p-value as

similar to significance level. Thus, explanations of this type, though generally connected to the final decision making concepts, were usually inadequate.

In explanations of the third type (3 students), the term *significance* was again used interchangeably with significance level, resulting in interpretations of significance as a level, a probability value or an area. Each interpretation is demonstrated respectively in the protocols of Dominic, Kylie and Phillip. For example, Dominic said, "it's [significance] the value or level I think it was, to determine the level at which you reject or keep your null hypothesis." Kylie explained, "that's [significance] just basically at 5 percent significance, it means you've got a 5 percent chance of a Type I or Type II error." More expansively, but similarly, Phillip stated:

> I think the significance area is just an area so much to either side of the bell-shaped curve, if you're using a two-sided z statistic then it's 5 percent significance or 10 percent significance, you wouldn't go past 20 percent.

Phillip's explanation went further than Dominic's or Kylie's, by including a connection with the regions at the ends of the distribution curve, which Phillip had previously referred to as rejection regions. Again, with consistency, he wrote the alpha symbol, α, next to the significance label. Summarising students' explanatory statements about significance, none clearly showed a full understanding of the concept. Students continued to provide definitional statements which were vague or inadequate, or confused the terminology and meaning of significance with significance level. However, several did make the connection with tests, values, or rejection regions.

Example statements: In Interview 3, one attempt was made to illustrate significance by an example. Intending to explain the link between significance and probability, Rhys stated:

> significant or not, if you're looking at the male female ratio and seeing if it's dependent on AIDS or something like that, you can test to see if it's significant or not, making the stratified allocation thing to see if it'd be worthwhile.

His meaning was unclear, and the example failed to achieve its purpose.

Other statements: During the concept mapping task, in addition to those statements summarised above, several described the steps involved in obtaining rejection or non-rejection of the null hypothesis. As before, this was acknowledged as demonstrating implicit procedural knowledge of the significance concept, as no mention was made of the term *significance*. Seven students explained how to reject the null hypothesis using the critical value method, and eight offered explanations via the p-value method. Examples of these statements are reflected in the following extracts. Pointing to the tails on the distribution graph, Margaret stated that, "you reject your null hypothesis if it [value from test statistic] falls inside the critical region [defined by the significance level]." Karl remarked, "reject it if the p-value is less than the significance level of .01 or .05." In the hypothesis tests, six of these students could not apply this knowledge. A summary of responses on these tasks follows.

Hypothesis Tests: During the performance of the hypothesis tests, Kylie was the only student to mention the word *significance*. While completing the light bulb task, she referred to "5 percent significance", "significance of 5" and "35 percent significance level", thus maintaining consistency in her exchange of the two terms, *significance* and *significance level*. Furthermore, after standard statistical procedures, 7 students (compared to 3 previously) reached a statistical conclusion. Five used only the critical value method, one only the p-value procedure, while the seventh student eventually reached his conclusion via the confidence interval method on the light bulb task, and via the critical value method on the night shift task. These conclusions were recognised as implicitly establishing statistical significance. In addition, only 5 of the 7 students continued on to relate their conclusion (rejection or non-rejection) back to the original question. Invariably, further evaluation of the decision was forthcoming only after prompting by the researcher, implying that practical

significance was not a normal consideration for these students. Thus, for the hypothesis tests, students' procedural knowledge of the significance concept was recognised only implicitly in the few students who reached a statistical decision. Little emphasis was placed on practical considerations without prompts from the researcher.

Summarising the findings of Interview 3, it is clear that even at the end of the semester, students' understanding of the concept of significance remained poorly developed in the explicit sense. Several did not even recognise the concept name, and use of the significance label on the concept maps was not an indication of understanding the concept. Several students still thought of significance as significance level.

The major findings in these interviews were first, that several students did not recognise the concept name, and few students could adequately explain significance, or its role in hypothesis testing. Students' conceptual knowledge was confined to brief, vague, inaccurate, or inadequate descriptions, or loose connections with rejection or non-rejection. Through misinterpretation, incorrect links were sometimes made to significance level. Second, most students (12 of 18), given their omission of the *significance* term, exhibited implicit procedural knowledge of the significance concept, either through their correct explanations of the p-value or critical value methods or through their statistical conclusion at the end of a standard hypothesis test. However, many who were able to describe the process to obtain a statistical conclusion could not actually perform the process. Third, students placed more emphasis on statistical measures (statistical significance) than practical ones (practical significance). Even the statistical conclusion was not always related back to the original question that was posed. Therefore, in general, evidence of explicit conceptual and procedural knowledge appeared to be limited by the widespread lack of familiarity with the term itself, and the inability to perform hypothesis tests.

Discussion

Each of the major findings described above will be discussed in order.

The first finding consisted of three parts. First, several students did not recognise the concept name, and few could explain the concept. This finding is not of major importance, because the actual use of the term is not essential to obtaining or interpreting a statistical result. It is the ideas behind obtaining significance that are important, not necessarily the term itself. In any case, those who attempted an explanation appeared to have had some idea of the concepts with which significance was related. Furthermore, in this study, the students' text books barely mentioned the term, and the lecturer did not emphasise it often. Second, students' conceptual knowledge was limited to vague descriptions, or loose connections. Throughout both interviews, students appeared to have difficulty expressing themselves statistically. Either they lacked the statistical language, or the statistical knowledge to convey the intended meaning. Most probably the cause was a combination of the two. Third, students confused significance with significance level. Three reasons are offered for this occurrence: the concept names are similar; the interpretation of statistical significance through the p-value and critical value methods involves the significance level; and the low emphasis given to this concept in lectures may explain students' inability to distinguish between them.

The second finding was that students' procedural knowledge was implicit, given their omission of the term *significance*. As indicated above, more students could actually describe the process leading to rejection or non-rejection than could actually successfully perform it. Hence, it must be concluded that reaching a statistical decision is difficult for introductory students. Yet, hypothesis testing is a topic in most introductory statistics subjects, and these students had completed a first course in statistics. Several factors may provide a rationale for this difficulty. First, hypothesis testing is an involved procedure. The translation of data into statistical terms, interpretation of formulas or symbols, and the actual techniques are possible sources of confusion. Second, hypothesis tests performed by hand are tedious and time-consuming, and many mistakes of a mathematical or statistical nature

may be made along the way. Third, given the timing of the first interview, a few weeks after the introduction of hypothesis testing, students' inability to perform a test could be explained by a lack of practice. This would support the finding that several students could explain the process, yet not actually do it. Given the timing of the second interview, after the final exam in the subject, lack of performance could be explained by the fact that there was time to forget the procedure (if it was ever known). Forgetting was aided by the fact that later in the semester, emphasis was on the interpretation of computer printouts using the p-value (e.g., multiple regression). Hand, rather than computer, calculations were necessary for the completion of the hypothesis tests in this study.

The third finding was that statistical conclusions were apparently more important than other considerations. Evaluation of the statistical decision is an important step in hypothesis testing, and several reasons are offered for the failure of students to either link the statistical conclusion back to the question posed, or evaluate the decision. These reasons follow the previous rationale, namely that the complexity of the process and the tedium of hand calculations override additional considerations. The biggest hurdle is reaching a statistical conclusion, and the real meaning of the original question may be forgotten in the process. In addition, typical text book examples do not always stimulate deep reflective and interpretive thinking. Furthermore, many lecturers do not encourage it.

Teaching should attempt to overcome the problems outlined above. If the term itself is used, then it must be mentioned frequently by lecturers in conjunction with rejection and interpretation of statistical results. The encouragement of group discussions, or oral presentations, may foster students' use of statistical language. If the term is not emphasised, then it is the integral ideas of significance that must be accentuated. First, with respect to the difficulties in reaching a statistical conclusion, there are many good statistical computer packages available, which perform speedy numerical calculations, deemphasising the use of confusing statistical symbols and formulas. Once an understanding of the process is formed, students may be curious to learn the statistical rationale and techniques behind it. Computers also facilitate the exploration of statistical conclusions under different conditions, for example sample size. However, the success of computer packages always depends on the lecturer's ability to integrate the important ideas. Second, with respect to the evaluation of statistical results, interpretive skills can be developed through well-designed projects and examples that are applicable to students' fields of interest. Typical text book questions limit students' statistical experience. The ICOTS conferences, in particular, provide ideas for classroom use.

Many of the above comments have been made before by others. However, this study is the first of its kind to provide empirical, rather than anecdotal, evidence of the problems students have with the significance concept.

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Mathematics at two schools, one Australian, the other Japanese, both located in Australia: their Philosophies and Practices

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This paper contains observations of the mathematics curriculum in action of two schools, one Australian and one Japanese, both situated in Victoria, Australia. While school structure and mathematical content in both schools were substantially similar, there were notable differences, for example, continuous school based and government testing in the Australian school compared with the relatively few but highly significant examinations set by Japanese institutions as student entrance requirements. Permeating these differences were divergent mathematical histories, and social and cultural factors in the two countries.

Emphasis placed on the skills of mathematical and related scientific and technological fields in its national education system has been recognised as one of the major contributing factors to the superior commercial and technological expertise reached by Japan with astonishing speed after the Second World War. The introduction of an American education system after the War perhaps may have resulted in the development in Japan of a school education similar to that of Australia which, in its own education system, used ideas largely from Great Britain and America. This study attempted to interpret differences and similarities observed in the mathematics programs of the two schools in terms of the values governing the historical development of mathematical knowledge and social and cultural factors existing in Australia and Japan. The content of this paper is derived from the observer's Masters Project (Wotley, 1993).

Some Historical and Cultural Perspectives

Japan is the only country where oriental mathematics continued undisturbed by outside influence for a long period of time.This can be attributed to the virtual sealing off of its boundaries in the 16th century after the execution of Jesuit missionaries who had converted some of the Japanese population to Christianity. Even China, which had contributed the ancient foundations of mathematical thought in Japan, was excluded (Spence, 1985). Japanese mathematicians were secretive and mathematical treatises were passed on to star pupils and eldest sons. This may have been due to the relegation of mathematics to commercial activities by the noble Samurai class who held it in some contempt. It is recognised Confucianism which spread from China to the East Asian countries 2500 years ago exerted a major influence which continues today. Its values include diligence and harmony in the workplace, strong family ties, and a passion for education (Adam Smith, 1988). The tenets of Taoism were also influential. They included the belief that there was an inherent self-sufficient order in nature and there were few regularities in the "mundane sphere" to which mathematics could be profitably applied.(Needham, 1956). By the 18th century mathematics had arrived at practical application. Japanese mathematicians who were adept at the solution of higher degree equations before the West, did not access Calculus, which revolutionized 17th century mathematics and science in Europe. Instead, the Japanese used Infinite series in meticulous and exhaustive calculations in attempting to find the formula of the circle theoretically (Yenri), much of which was applied to packing problems involving spheres. Smith and Mikami (1914) liken early Japanese mathematics to the execution of a red lacquer box which required exquisite work, minuteness, patience and everlasting pains. When Japan's isolation was ended in the 19th century its scholars still considered this quality of "minuteness" to be superior but this did not prevent the acceptance of Western ideas in Japan (Needham, 1956). In the latter part of the 20th century at the end of World War 2 when Japan was occupied by the allied powers, the American school system was established with its curriculum content and physical environment. While Japanese schools appeared to accept this system, in reality, they evolved differently (Rohlen, 1983), (Howson, 1991). Balanced with the American ideals of individualism and democracy

were the ideals of Confucianism and the prewar elitist system of education, none of which entirely dominated the resulting educational system. Today's Japanese national school curriculum is an amalgam of these influences.

Australia's colonial history which had close ties with Anglo-Celtic ideals looked early to the universities of Great Britain for its school mathematical structures and curriculum. Australia's Western heritage rather than its geography, turned its interest firmly in the direction of Europe and America and its school mathematics throughout most of the twentieth century reflects this connection. In the latter part of the twentieth century new structures and curriculum have some Australian character. Whether this is a recognition of the diversity of its population today (it now has the second most diverse population in the world) is a matter for further investigation. While all states in the country have similar patterns of history and social organisation, there is no national curriculum and a number of different procedures operate in the state education systems.

Two Schools in Australia

School A is an Australian school (years K - 12) designated as an Independent School. Within the largely homogeneous cohort of Australian students of Anglo-Celtic origin at School A, Japanese students constitute about 5% of the total population of the school. Most Japanese students attend School A for approximately 4 years. The progress of these students through school, while of interest, proved to be very difficult to track in practical terms because of continuous arrivals and departures often during term time.

School B is also an Independent or Private School (years K - 9) owned by a Japanese consortium. Ostensibly it provides a continuity of education for children of Japanese professional men appointed to positions in Australia. Employing teachers imported from Japan on a contract basis, it administers the Japanese National Curriculum using imported texts and materials designed for that curriculum.

Communication with Japanese mathematics teachers in School B who had limited English skills, and the lack of Japanese language skills of the writer, did not preclude useful exchange of materials and discussion. A limited amount of translation was available.The international notation used in texts allowed understanding of material printed both in Japanese and English.

Observations of Practice

Are there differences in general structures?

School A has a structure common to the majority of schools in Victoria. School B is the common model in Japan but student numbers at each level and in each classroom are smaller because of its location in Australia. Table 1 summarizes the major features of these structures in the two countries.

Table 1: General School Structures Japan	Australia (State of Victoria)
Elementary School - Years K-6 Lower Secondary School Years 7 -9 Upper Secondary School Years 10-12	Primary School - Years K-6 Secondary School - Years 7-12
Attendance at Upper Secondary level is not compulsory. 94% students proceed further.	A minority of students leave at the end of Year 10. Majority of students proceed further.
National Curriculum defines school programs.	Schools based programs in years K-10. Vic Board of Studies administers the VCE (Vic Certificate of Education) in Years 11-12
Written examinations set by Senior High Schools or Provincial Prefectures and Universities as entrance exams at the end of years 9 and 12 levels.	Continuous school based testing together with government diagnostic testing during Years K-10. Combinations of VCE results used for entrance into higher institutions and universities.

How comparable were the hours of teaching?

In the year observed(1993) the Japanese school day was approximately 8 hours compared with a 7 hour day in the Australian school. The Japanese school year was 187 days which was less than the average of 240 days for Japanese schools in Japan. The time spent in school however still exceeded the Australian times (180 days) and this added to the longer school day meant that the Japanese students had more face-to-face teaching than the Australian students. Vacation periods in the two schools bore little resemblance to each other. The four term structure punctuated by three fortnightly breaks and a longer break during summer in the Australian school differed greatly from the two concurrent half-yearly semesters and one long break of the Japanese school.

In contrast to the student situation, Japanese teachers did not spend proportionately as much time in face-to-face teaching as the Australian teachers. The proportion of preparation time in the allotment of the Japanese teachers was significantly greater.

What were the main features of teaching material?

Japanese National Curriculum texts issued to students were small compact books which contained mainly collections of examples. Within each book a small selection of single topic areas were covered. Several of the text books were issued during the year. They were used by students for practice and homework. It was interesting to note that illustrations in the small Japanese texts contained some photographs of Japanese children of the relevant age participating in activities and small cartoon figures drawn in a very simple way. The figures invariably wore cheerful expressions and were interspersed among problems and indexes. These features, combined with the small size of these books with brightly coloured covers contributed a certain "user friendliness" in the Japanese texts up to Year 9 level. They could not easily be used for self teaching purposes. The Japanese teacher had a central role in introducing and discussing topics and arrived in the classroom with extensive amounts of material prepared before its commencement.

The Australian texts were large, cumbersome, soft-covered volumes which contained enough work to fill the whole year. A text appropriate for each level was chosen by the Mathematics Department of the school from a wide range of commercially available material. Each student purchased a copy for use throughout the year. "User friendliness" seemed to be missing from these heavy, unavoidably dog-eared texts. To a certain extent, these books could be seen as self teaching as they contained instructions, worked examples and further examples for practice usually graded in increasing levels of difficulty. This self teaching nature of the Australian texts allowed individual students to move ahead of the class sometimes to another topic or to extension exercises usually provided at the end of the book. Some teachers tended to follow texts closely, using them as aids.While teachers did prepare other materials it was possible for a teacher to enter the classroom with little preparation and to closely follow the text in teaching a topic which was quite different from previous topics.

What teaching strategies were used in the classroom?

While the Japanese class lessons were highly structured and teacher centred, much discussion between teacher and individual students or groups of students preceded arrival at and application of a concept. The Australian classroom had less reliance placed on the teacher, the text serving as instruction as well as practice and therefore allowing more individual student progress. The Australian school also used techniques of setting or streaming over a year level in years 7 -10 whereby students were grouped very broadly into achievement level groups over a year level. Within these groups there were differences in pace and the use of extension exercises for higher achievers. Exercises were modified for lower achievers. Students were able to move between levels if testing procedures indicated changed levels of understanding. In the Japanese school at levels up to year 9, classes were of mixed ability. Unfortunately while many students in the Japanese class were involved in discussion, language difficulties prevented the observer from gaining a true understanding of the quality of their contributions.

In keeping with the practice of moving from one book to the next during the year, Japanese texts lacked revision exercises. Once a topic was taught thoroughly it did not

reappear. This was in contrast to the constant reappearance of a topic in increasing difficulty over successive year levels in the Australian School. For example, Linear Graphs and Equations as a topic appeared in Year 8 only in the Japanese school but was treated in sections over 4 years in the Australian school. Table 2 illustrates this contrast.

Table 2: Linear Graphs/Equations

Japanese School (from text of Japanese National Curriculum)	**Australian School** (from syllabus documents of teachers at different levels of the school)
Year 8	Year 7
• Practical examples, tables of values, plotting points	•Plotting Points, using Cartesian grids
	Year 8
• y =ax + b, transformations altering a and b.	•Algebraic solution of linear equations and inequations
•gradients and intercepts, restricting domain and range	$x + 9 = 15$ (at first using the technique of filling in gaps)
	$3x < 21$
•applied problems using tables of readings, plots	$x + 5 > 13$
	$3x + 7 = 25$
•ax + by = c and y = mx +c	$\frac{3x}{4} + 13 = -31$
	$3(x + 7) = 4(x + 9)$
•graphical solution of simultaneous linear equations.	$9 + 6(x - 7) > 5x$
	$\frac{8x - 3}{4} = 7$
•Linear programming	
	Problems using linear equations.
•solving inequations eg $x - 8 > 3x$ $3x + 2 < 20$ $6x - 1 < 3(4x + 1)$ including use of number lines for simultaneous inequations.	Year 9
	Graphs using intercepts, points and gradients, y = mx + c Graphing inequations Simultaneous equations
	Year 10
	Plotting, sketch graphs, linear modelling, equations and inequations, problem solving eg fixed and variable costs, lines of best fit.

The number of topics studied in a half-yearly period in the Japanese school exceeded that of the Australian School as illustrated in Table 3. The problem solving time listed in the Australian School syllabus was not aligned to topics in the course. Problems were designed to encourage creativity through the exploration of different ways of solving number puzzles, spatial types of exercises and everyday life situations such as traffic survey, factory production. Students usually worked in groups for these exercises and different interpretations sometimes resulted in different answers. Shorter hours of face-to-face teaching in the Australian School and the time set aside for problem solving exercises contributed to the smaller number of skill-based topics covered by the Australian School.

Table 3: **Japanese School** **Year 8 (1 semester)**	**Australian School** **Year 8 (1 semester)**
1. Algebraic simplifications - collection of like terms including use of positive and negative numbers , use of brackets, including indices and fractions, all operations, applied examples. 2. Co-ordinate pairs, techniques of algebraic solution of linear and simultaneous equations. 3. Linear equations and Graphs including simultaneous equations. 4. Inequalities - algebraic solutions of inequations. 5. Straight line geometry, transformations, similar and congruent triangles. 6. Geometry Theorems and proofs, triangle and quadrilateral geometry. 7. Applied problems using similarity and congruence with various polygons 8. Statistics, bar graphs, frequency polygons, frequency tables, using intervals of measurement, mean, grouping data (2 pages at the back of the book give calculator instruction) •Total number of chapters = 8 approximately depending on how overlap of some chapters is considered.	1. Directed number - introduction of all operations, applied problems 2. Algebra - collecting like terms, multiplication by an integer, multiplication and division with pronumerals, building up symbolic expressions, the Distributive Law. 3. Measurement - perimeter and circumference, area of simple and complex shapes, area of parallelogram and trapezium, area of circle and annuli, volume of prism. 4. Statistics - Interpretation of graphs, bar, pie and other pictorial graphs 5. Equations and Inequations using algebraic solutions. There were 3 problem solving sessions built into the semester which use about 6 to 7 hours of class time during the semester. •Total number of topics = 5. in addition to 6-7 hours of problem solving activity spread throughout the semester.

Assessments - at what stage did they occur and for what purposes?

As can be seen from Table 2 the major concepts of linear equations and graphs were taught at an earlier stage in the Japanese curriculum. This characteristic is consistent throughout all levels and can be equally illustrated in the areas of arithmetic skills at Primary level. In general, at the secondary level, the Japanese students operated at a skill-based standard about two years ahead of the Australian students.

Assessment observed in the Australian School occurred about three times in the term usually at the end of each topic and these results were tabled four times in the year, and an average result together with teacher comment was sent home to the parents on each of these occasions. Formal examinations commenced at year 10 level. Government-based diagnostic testing is currently in the process of introduction into schools over K - 10 levels At year 11 level (in some cases year 10) the two year VCE is commenced. This has 2 assessment pathways : one which stipulates that the Work Requirements of each study taken by a student
must be fulfilled and the other is an achievement mark allocated to the combination of various parts of the study (Common Assessment Tasks or CATs) which may be

examination testing, presentation of a project, fulfilment of practical aspects etc. Student scores are arrived at through various combinations of these marks and the results of a General Achievement Test (GAT). Scores are used by Universities and Higher Institutions as entrance qualifications.

Since the Japanese school operated up to year 9 level only, the assessment patterns of the few Japanese students at VCE level in the Australian school were investigated. Table 4 records the.assessment pattern of the only Japanese student completing year 12 in 1992 in the Australian school.

Table 4 : VCE Results of a Japanese Student at the Australian School.

Student	CAT 1 - Project	CAT 2 Problem Solving	CAT 3 Facts and Skills Multiple choice exam	CAT 4 Analysis External Exam
Student X	C+	B+	B+	A

There was evidence, according to the experience of the mathematics teachers in the Australian school, that Japanese students in general found assignments and forms of assessment which were not directly skill based extremely difficult. In 1992, the Australian students who were checked on a random basis of every fourth student in an alphabetical list achieved better results in CAT 1 and CAT 2 than in CAT 3 and CAT 4 in the ratio 2.5 : 1. The numbers are far too low to draw fair conclusions but the pattern does support the anecdotal evidence of the teachers. Other factors and controls such as language skills would be necessary to test, as problem solving and project exercises often contain more words than the skill-based exercises.

In Japan, written examinations are set by universities at the end of year 12 and these are of paramount importance to teachers, parents and students. These examinations emerged as the driving force in the Japanese school where, in discussions about aims and objectives with teachers, success in preparing students was a pivotal objective. In both Australia and Japan a huge amount of attention is devoted to this level by teachers, students and parents. Competition is intense. A coaching industry in Japan addresses itself to placement of students in the institutions of their choice. Unlike Australian students however, Japanese students may sit again, sometimes in the same year, if they fail to qualify on their first attempt . In Victoria the the multiplicity of tasks in each study and a penalty system act as strong deterrents to second attempts.

The progress of mathematics students in the Australian school therefore seemed to be punctuated by a myriad of testing procedures of different types and at different times of the year which provided constant feedback to students, teachers and parents. In contrast examinations in the Japanese school were driven by entrance to institutions at two major focal points - entrance to Senior Secondary Schools at the end of Year 9 and entrance to Universities at the end of year 12.

Interpretation of the different mathematical pathways taken by the two schools: what impact did the Australian location of the Japanese school have on its mathematical program?

Differences were observed in the practice of both schools despite some similarity in mathematical content and structure. Interpretation of these differences could be made through linkages between them and the histories and cultural and social characteristics of Japan and Australia.

In the Japanese school the meticulous planning of lessons and the exhaustion of a whole topics in their entirety have echoes in the mathematics.of old Japan. The teacher centred classroom with all students taking part in mastering a concept has the familial ring of Confucianism which regards the school as a family where teachers have their own familial roles to play.(Rohlen, 1983). The text books which were addressed to the students age level through their graphic design reinforced the place of the student in the family as well as providing a volume of skill-based examples to be diligently completed. The Confucianist belief that the difficulty of the task is not significant in the child's level

of performance (Robitaille and Travers, 1991) means that poor performance is attributed to lack of effort. Assessment does not reinforce failure but is a screening device for movement to another level. Success is directly proportional to effort.

The multiplicity of testing observed in the Australian school perhaps resulted from the onslaught of constant change in school curriculum design in the last years of the twentieth century as Australia moves to a different identity.The lack of uniformity of mathematics teaching between states and schools within each state also may have played some part in this.

The striving for creativity and individual participation through problem solving activities are directions commonly followed in Western school mathematics. A belief in progress at an appropriate individual ability level also accounted for ability groupings at each level of the Australian school and the high frequency of diagnostic testing. However the effect of these activities, together with less teaching time in the classroom resulted in lower levels of skill in the Australian school, particularly in secondary levels.

Finally, did the Japanese school located on "alien soil" adopt any of the mathematical philosophies or practices of mathematics schooling in Australia? Did interchange take place with the Australian school which had in its own cohort of Japanese students living in Australia for the same reasons as those in the Japanese school.? Were any general characteristics of Australian education applied in the Japanese school? No evidence of this was observed. The Japanese school's semester dates and time schedules, in particular, indicated its separate existence as did its wholly imported teaching staff and curriculum materials. No non-Japanese students attended the school at the time of observation. While the writer found the Japanese staff helpful and friendly and text materials were generously provided, virtually no elements of Australian school mathematics policy were observed in the practices.of the Japanese school.

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Goal Orientation Beliefs and Mathematics Achievement: A Longitudinal Study

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The influence of goal orientation beliefs upon mathematics achievement was investigated over almost three years. Task involvement, ego orientation and achievement in mathematics to achievement.were measured with an sample of 243 primary and lower secondary students. While task involvement correlated with achievement at a low but significant level, regression analyses revealed that this factor did not predict achievement over and above the prediction from the earlier *Progressive Achievement Test in Mathematics* score. At no stage did ego orientation relate

Student school performance has been found to be related to prior achievement, attitudes to specific aspects of school learning, and motivational factors (Keeves, 1972). Within motivational psychology, the importance of self-efficacy (Schunk, 1996), self regulation (Pintrich & Garcia, 1991; Zimmerman, 1990), self determination (Deci & Ryan, 1991), and causal attributions (Graham, 1991) have been emphasized. Goal theory has been advanced to explain the relationship between students' beliefs about the causes of school success, and their engagement and persistence in academic learning (Dweck, 1986). Adaptive patterns of achievement orientation have been variously termed mastery oriented goals (Ames, 1992), learning goals (Dweck, 1986) or task involvement (Nicholls, 1984), while maladaptive patterns have been referred to as performance oriented goals, performance goals , or ego orientation. The terms task involvement and ego orientation are used in this paper to exemplify goal orientation.

Task involvement goals have been distinguished from ego oriented goals in terms of students' conceptions of success, different reactions for approaching and engaging in achievement activity, and different ways of thinking about the self, the task and the task outcomes (Ames, 1992; Nicholls *et al.*, 1989). Task involvement goals are akin to "motivation to learn" as students focus on mastery and understanding content. Ames (1992) has suggested that effort and outcome covary, leading to stronger achievement directed behaviour over time. Students' attention is more likely to be focused on the intrinsic value of learning (Nicholls, 1984; Butler, 1988; Meece & Holt, 1990), and on effort utilisation, with the belief that effort leads to success and that mastery is intrinsic to self-efficacy. Task involved students are oriented towards the development of new skills, trying to understand their work, improving their level of competence or achieving a sense of mastery based on self referenced standards. Within this mental frame, students perceive ability as being improvable and incremental, and are more confident in investing or expending effort (Schunk, 1996).

Ego orientation, however, entails a focus upon ability as a fixed attribute which determines a sense of self-worth (Covington, 1984; Dweck, 1986; Nicholls, 1984). Ability is evidenced by doing better than others, by surpassing normative-based standards, or by achieving success with little effort (Covington, 1984). Ames (1992) has suggested that central to ego orientation is the need for public recognition, to be better than others, or to perform in a superior manner. Learning is viewed as a way to achieve normatively defined success. Effort becomes a double-edged sword as the self concept can be threatened if trying does not lead to immediate success (Covington & Omelich, 1979b). Over time, effort is seen as counterproductive, with increased effort interpreted as an indication of lack of ability.

Task involvement and ego orientation are not necessarily fixed characteristics, as they are influenced by conditions in school environments (Ames, 1992). Dweck (1986, 1989) considers that the nature of achievement goal orientation may change in relation to subject-matter areas, but Duda and Nicholls (1992) found that high school students' causal explanations of success were generalized across subject areas. Differences between ego oriented and task involved students were found in the amount of time students spend on learning tasks, persistence in the face of difficulty, quality of engagement in learning,

and use of adaptive mental strategies (Butler, 1987; Elliott & Dweck, 1988; Meece *et al.*, 1988; Nolen, 1988; Nolen & Haladyna, 1990; Graham & Golan, 1991). Task involved students responded to impending failure by remaining task focused, while ego oriented students chose simpler tasks, used inefficient strategies, or preserved their self image by adopting an attitude of academic alienation (Dweck & Leggett, 1988). The adoption of task involvement goals could be expected to lead to long term achievement motivation in students, but the extent to which this relates to actual achievement is not clearly defined within the existing literature.

Factor analytic studies have determined that task involvement and ego orientation are independent dimensions of both personal academic goals and beliefs about the causes of school success (Nicholls *et al.*, 1989; Nicholls *et al.*, 1990), with the third dimension of work avoidance being found in mathematics achievement (Nicholls *et al.*, 1990). All three dimensions were only slightly correlated with perceived ability (Nicholls *et al.*, 1990). Goal theory researchers suggest that learning and performance goals are orthogonal rather than opposing ends of a continuum. (Maehr & Pintrich, 1991; Meece & Holt, 1993; Miller *et al.*, 1993). Four dichotomous goal configurations are thus possible, as any given student may be high on both dimensions, low in both or high on one and low on the other.

Within the field of mathematics, achievement has been examined in relation to school-type factors, curriculum considerations, student characteristics and background, and gender differences, but few studies have examined the relationship between goal orientation beliefs and achievement in mathematics (Bong, 1996). In an earlier study, Yates, Yates & Lippett (1995), found that ego and task goal orientation measures failed to correlate significantly with concurrently measured achievement in mathematics. The present study, part of a larger data set concerned with motivational variables in mathematics over a period of almost three years, investigated the extent to which achievement gain in mathematics could be predicted by goal orientation beliefs.

Method

Subjects: The study commenced in Term 1, 1993, with 328 students from Years 3, 4, 5, 6 and 7 in two primary schools in metropolitan Adelaide in South Australia. In 1995, 133 of these students were traced to 26 primary schools and 110 to 24 secondary schools in both the government and nongovernment sectors. Complete data was available for 243 students who then ranged from years 5 to 7 (primary) and 8 to 9 (secondary). The analyses of the *Feelings in Mathematics: A Questionnaire* was thus conducted on the initial 328 students, with the sample of 243 being used for the relational analyses.

Instrumentation: Progressive Achievement Tests in Mathematics: *Tests 1, 2, 3 (ACER, 1984.)* The Progressive Achievement Tests of Mathematics were adapted by the Australian Council for Educational Research (ACER) from the *Progressive Achievement Tests: Mathematics* developed by the Test Development Division of the New Zealand Council for Educational Research. The series, which utilised a multiple choice format, consisted of three tests at different levels of difficulty. Test 1 designed for Years 3 to 5 contained 47 items measuring number, computation, fractions, measurement and money, statistics and graphs, and spatial relations. The same topics with logic and sets added were covered by 57 items in Test 2, suitable for Years 5 to 8. Test 3, designed for Years 6 to 8, contained 55 items sampling the topics of number, computation, measurement and money, statistics and graphs, spatial relations, relations and functions, and logic and sets. Within each test, items were grouped by topic and arranged in increasing order of difficulty within each topic. The difficulty level was determined by the ACER from the Rasch analysis of responses from the standardisation sample tested in November, 1983.

Feelings in Mathematics: A Questionnaire. This 25 item questionnaire, a variant of the *Motivation Orientation Scales* developed by Nicholls (Nicholls *et al.*, 1990; Duda and Nicholls, 1992) was designed to measure task involvement and ego orientation. Each item commenced with the stem *"Do you really feel pleased in maths when ..."* followed by a statement reflecting either task involvement or ego orientation, with some filler items

in random order. Students were required to rate each statement on a five point scale ranging from strongly agree to strongly disagree.

Administration of the Test and Questionnaire

At the commencement of Term 1, 1993, the *Progressive Achievement Tests in Mathematics* were administered to intact classes by a male researcher in one school, while the second school administered the tests as part of the normal procedures for the start of their academic year. The *Feelings in Mathematics: A Questionnaire* was then administered to intact classes in both schools by a male researcher. When the students were traced in 1995, the test and questionnaire were administered in Term 4 either by a male or female researcher. A standardised administration of Form A of Test 1, 2 or 3 of the *Progressive Achievement Tests in Mathematics* was given, with the level of the test chosen in accord with the student year level. All students in Year 9 took Test 3. Students recorded their responses in pencil on the ACER computer-scoring answer sheets.

Results

Achievement in Mathematics

The raw scores from the *Progressive Achievement Test in Mathematics* for 1993 and 1995 were converted to scaled scores, using the table from the *Teachers Handbook* (ACER, 1984). During the standardisation process, the scaled scores had been calibrated for difficulty with the Rasch model calibration program BICAL3, using the common-items linking procedure. The scores for the Forms 1, 2, and 3 were thus able to be placed on a single scale for both 1993 and 1995. Mean achievement scores for 1993 and 1995 were then calculated and compared in relation to year level and gender. When the scores were analysed with one way analysis of variance, a significant relationship between year level and achievement in both 1993 and 1995 was apparent (see Table 1). However there was no significant relationship between gender and mathematics achievement.

Table 1 Analysis of variance for year level and gender in relation to mathematics achievement in 1993 and 1995

1993 Mathematics achievement n = 243	df	F	Significance of F
1993 Year level	4	51.66	<0.00
Gender	1	1.50	ns
1995 Mathematics achievement			
1995 Year level	4	18.62	<0.00
Gender	1	1.93	ns

The predictive relationship between achievement in 1993 and 1995 was substantiated with multiple regression using direct entry of the variables, although the year level and gender variables were not significant (see Table 2).

Table 2 Multiple regression analysis of year level, gender and mathematics achievement in 1993 on mathematics achievement in 1995

n = 243	Beta	r	t	Significance of t
1993 Mathematics achievement	0.75	0.74	13.37	<0.00
1993 Year level	0.20	0.43	1.42	ns
Gender	-0.32	-0.06	0.74	. ns
Multiple R = 0.74	R square = 0.55			

The Feelings in Mathematics Questionnaire

Factor analysis was employed to examine the unidimensionality of the questionnaire. Principal components analysis and the oblimin rotation revealed two major

factors of task involvement and ego orientation, with eigen values of 7.47 and 2.36 respectively. The two factors correlated at 0.40. As items 2, 7, 11 and 25 did not contribute to either of these factors they were deleted. On the basis of the factor analysis results, the remaining items were divided into two unidimensional scales. Each scale was Rasch analysed separately with the rating scale model of the QUEST program (Adams & Khoo, 1993), using the data from the 328 students (see Tables 3 and 4).

Task Involvement in Mathematics: Items 16, 20 and 21 were deleted from the 15 items task involvement scale as their item infit mean square values were outside the acceptable range of 0.83 to 1.20. Thus the final scale for task involvement was composed of 12 items (see Table 3).

Table 3 Task involvement in mathematics
Item analysis of the 15 item 1993 rating scale

n=328		First analysis		Final analysis	
Item No.	Do you really feel pleased in maths when...	Infit mean square	Discrim. index	Infit mean square	Discrim. index
1	you really get busy with the work	1.06	0.55	1.12	0.54
3	you really understand things	0.97	0.51	0.94	0.54
5	you learn new things about mathematics	1.06	0.52	1.04	0.56
6	what the teacher says makes you think hard	0.94	0.66	0.96	0.66
9	the problems make you think hard	0.96	0.66	1.03	0.66
10	you are making good progress in learning difficult things	1.04	0.52	1.03	0.55
13	you find a new way to solve a problem	1.02	0.52	1.02	0.54
15	something you learn makes you want to find out more	.95	0.62	1.02	0.62
16	you solve a problem by working hard	.73	0.70	Deleted	
17	something you find out really makes sense	1.01	0.55	1.00	0.55
18	you work hard all the time	0.92	0.63	0.94	0.65
20	the teacher looks at your work	1.16	0.53	Deleted	
21	the teacher says its time for a test	1.32	0.52	Deleted	
22	you try your hardest	0.91	0.58	0.90	0.61
24	the teacher says you are doing excellent work	0.95	0.46	0.93	0.48

Estimated scores for each student were then calculated on the basis of these 12 items, with the case estimates being derived by the concurrent equating method. The 1993 and 1995 data files were combined, the case estimates calculated from the 486 cases and the scores determined for each subject for the two occasions on the basis of these estimates. Concurrent methods have been found to yield stronger estimates than equating based on anchor item equating procedures (Mohandas, 1996).

Ego Orientation in Mathematics: When the six items that composed the ego orientation scale were analysed with the Rasch procedure, item 14 was deleted as it had an infit mean square greater than 1.20 (see Table 4). The final scale for ego orientation was thus composed of five items. Estimate scores for each student were then calculated on the basis of these five items, with the case estimates being derived from the same concurrent equating method that was used for the task involvement scale.

Table 4 **Ego orientation in mathematics**
Item analysis of the 1993 rating scale

n=328		First analysis		Final analysis	
Item No.	Do you feel really pleased in maths when...	Infit mean square	Discrim. index	Infit mean square	Discrim. index
4	you know more than the others	0.91	0.75	0.94	0.79
8	you do better than the other children	0.85	0.77	0.88	0.80
12	you are the only one who can answer a question	1.05	0.70	1.11	0.73
14	you can see others making mistakes	1.49	0.59	Deleted	
19	you finish before your friends	1.03	0.74	1.17	0.75
23	you score better on the test than others	0.81	0.77	0.89	0.79

Stability of the Task Involvement and Ego Orientation Measures: When measured with intraclass and interclass correlations, neither task involvement nor ego orientation between 1993 and 1995 was particularly stable, although the task involvement scale was more robust over time (see Table 5).

Table 5 **Intraclass correlations between task involvement and ego orientation in 1993 and 1995**

n = 243	F	r (rho)	r
Task involvement	1.92	0.32	0.34
Ego orientation	1.45	1.18	0.20

The Relationship between Mathematics Achievement and Goal Orientation in 1993 with Achievement in and Goal Orientation towards Mathematics in 1995

The relationships between the measures of achievement and goal orientation towards mathematics in 1993 were examined by bivariate correlations (see Table 6), and by multiple regression (see Tables 7 and 8). There was a weak correlation between mathematics achievement and task involvement for both 1993 and 1995. Ego orientation failed to correlate significantly with achievement in either year (see Table 6).

Table 6 **Correlations between achievement, task involvement and ego orientation in mathematics in 1993 and 1995**

n = 243	2	3	4	5	6
1 1993 Maths achievement	0.74***	0.13*	•	•	•
2 1995 Maths achievement		0.18**	0.13*	•	•
3 1993 Task involvement			0.39***	•	•
4 1995 Task involvement				0.27***	0.26***
5 1993 Ego orientation					0.20**
6 1995 Ego orientation					-

* p <.05, ** p <.01, *** p <.001 , • correlation not significant

The relationships between achievement, task involvement and ego orientation was then examined with direct entry multiple regression for both 1993 and 1995 (see Table 7). Mathematics achievement was most strongly predicted by prior performance in 1993, but

neither task involvement nor ego orientation measured in 1993 significantly added to the prediction of achievement in 1995.

Table 7 The influence of mathematics achievement, task involvement, and ego orientation in 1993 on mathematics achievement in 1995

n - 243	Beta	r	t	Sig. of t
1993 Mathematics achievement	0.73	0.73	16.39	0.00
1993 Task involvement	0.08	0.18	1.78	ns
1993 Ego orientation	0.02	-0.03	0.51	ns
Multiple R = 0.74 R square = 0.55				

The data were then analysed by multiple regression to determine the effects of the three measures in 1993 on both task involvement and ego orientation respectively in 1995 (see Table 8). While there was no significant relationship between mathematics achievement and the two indices of goal orientation, there was an interesting relationship between the measures of task involvement and ego orientation. Both task involvement and ego orientation in 1993 influenced task involvement in 1995, while ego orientation in 1995 was predicted by ego orientation in 1993 only.

Table 8 The influence of mathematics achievement, task involvement and ego orientation in 1993 on task involvement in 1995 and ego orientation in 1995

1995 Task involvement	**Beta**	**r**	**t**	**Significance of t**
1993 Mathematics achievement	0.02	0.04	0.39	ns
1993 Task involvement	0.29	0.34	4.70	0.00
1993 Ego orientation	0.20	0.27	3.31	0.00
Multiple R = 0.39 R square = 0.15				
1995 Ego orientation				
1993 Mathematics achievement	-0.04	-0.06	-0.64	ns
1993 Task involvement	-0.06	-0.02	-0.87	ns
1993 Ego orientation	0.19	0.20	2.93	0.00
Gender	1.45	-0.16	2.29	0.02
Multiple R = 0.24 R square = 0.06				

Discussion

1. Mathematics Achievement: Achievement in mathematics in 1993 was strongly predictive of achievement in mathematics in 1995. Analysis of variance indicated that achievement in both 1993 and 1995 was significantly related to the year level of the students but not to their gender.

2. Goal Orientations and Achievement in Mathematics: Weak but significant correlations were found between task involvement and concurrent achievement ($r = 0.13$ in both cases). Ego orientation did not correlate with achievement in either 1993 or 1995. Overall, goal orientation in mathematics as measured by the task involvement and ego orientation constructs was not related to year level or gender.

3. The Influence of Achievement in Mathematics, Task Involvement and Ego Orientation in 1993 on Achievement in Mathematics, Task Involvement and Ego Orientation in 1995: Once prior achievement was included in the regression equation, task involvement failed to add to the prediction of subsequent achievement. Task involvement in 1995 was predicted by both task involvement and ego

orientation in 1993, while ego orientation in 1995 was predicted only by the same measure in 1993.

Further evidence was found for the remarkably strong impact of past achievement on current achievement, a relationship borne out despite the fact that the majority of students were tested on different forms of the *Progressive Achievement Test in Mathematics*, with the scores equated with the common Rasch-derived scale. These data thus make a further contribution to the known validity of this mathematics achievement instrument.

Goal orientation data, in the form of task involvement and ego orientation questionnaire measures, failed to add to the prediction of achievement over time. Task involvement significantly correlated with achievement across both time phases, but failed to account for additional variance in the 1995 achievement data once the effect of prior achievement was taken into account. Ego orientation similarly did not add to the prediction of achievement gain over time. Overall, the current data suggest that it would be unwise to make predictions of future performance changes in achievement domains from simple questionnaire measures of dispositional goal orientation.

The two goal orientation measures were based on a trait theory assumption that it is possible to measure goal orientations in order to infer enduring dispositions. However, as other goal theory researchers have regarded goal dimensions as situationally induced states, the extent to which students can be meaningfully assigned to dispositional categories such as "ego oriented" and "task involved" is unknown. It is very likely that goal orientation measures relate meaningfully to achievement gain, but uncovering the nature of this relationship will require more complex designs and measures. Linkages between dispositional goal orientations and achievement gain could possibly be mediated by environmental conditions such as classroom climate and perceived competitiveness that were not addressed in this study.

Significance of the Study

1. The strong influence of prior performance was evident for mathematics over a three year period for students from primary to lower secondary school.
2. While task involvement did correlate with the contiguous measure of mathematics achievement in 1993, it did not predict subsequent achievement over and above the effects of prior achievement. Ego orientation failed to correlate with achievement at any time. This finding however, must be tempered by the evidence that the measures were only weak to moderately stable across time, and the ego measure in particular was based on only a small number of items.
3. The study has made a significant contribution to the goal orientation literature, particularly given the longitudinal nature of the design.
4. While the present findings failed to give strong support to the notion that dispositional goal orientations were predictive of achievement gain, the extent to which goal orientation measures can be regarded as having trait-like qualities is a matter as yet undecided.

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From Research Tool to Classroom Assessment Device: The development of *Checkout/Rapua*, a shopping game to assess numeracy at school entry

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This paper describes the development of a procedure for assessing numeracy at school entry. What began more than decade ago as a research tool consisting of a collection of tasks to assess children's numeracy, has developed into a classroom assessment device in which various numeracy tasks are embedded within a shopping game, with the potential to assess a broad range of mathematical understandings in addition to those included in the game itself.

The New Zealand Curriculum Framework identifies Numeracy Skills as one of the eight groupings of essential skills which all students need to develop throughout the years of schooling (Ministry of Education, 1993). These skills are thought to be important for ensuring that students achieve their potential and participate fully in society, including the world of work.

If teachers are to identify children's learning needs and plan programmes which will meet those needs, they need to find out about the knowledge and understanding which children bring with them to school. The New Zealand Curriculum Framework has identified school entry at age five as one of the key transition points for students entering the schooling system, and numeracy as one of the skills to be assessed at this point (Ministry of Education, 1993). The purpose of gathering information at a point such as this is to enable resources to be targeted more effectively by teachers and the government.

Research on children's numeracy skills

Over the past couple of decades, research on children's mathematical thinking has increased rapidly. Such studies provide a detailed picture of the kinds of numeracy skills which are within the repertoire of many five-year-olds (see Visser & Bennie, 1996; Wylie, Thompson & Kerslake Hendricks, 1996; Young-Loveridge, 1987a, 1987b, 1988, 1989a, 1991a, 1993, 1995). Several features about the numeracy skills of five-year-olds are particularly striking.

1. The numeracy skills of five-year-olds are often extraordinary

Research done over the last couple of decades shows that many five-year-olds have considerably greater understanding and skill with numbers than their teachers realise (see Young-Loveridge, 1987a, 1987b, 1988, 1989a, 1991a, 1993). In a study undertaken in New Zealand schools ten years ago, teachers were asked to make judgements about which numeracy skills and concepts their new entrant children possessed. These judgements were then compared to children's actual performance on particular tasks (Young-Loveridge, 1987b). The study found that many teachers underestimated their pupils' numeracy skills, and this was particularly noticeable for operations (ie, addition and subtraction), both with real and with imagined objects (see Young-Loveridge, 1987b, 1988, 1989a). Consequently, many children were being given lots of practice doing mathematics which they already knew, rather than being challenged to develop more advanced concepts and skills. British researchers found a similar pattern (Bennett, Desforges, Cockburn & Wilkinson, 1984). As a result, a great deal of time and opportunity for learning more advanced mathematical skills was probably being wasted and achievement may have been lower (Bruner, 1960). Higher ability students in particular seem to be hurt by the slow pace at which teachers move through instructional materials (Barr, 1974, 1975).

2. *The numeracy skills of five-year-olds are highly variable*

Research shows that there are huge differences between the *most* and the *least* competent five-year-olds (see Young-Loveridge, 1987b, 1988, 1989a, 1991a). For example, some five-year-olds can rote count to 100, form a group of 9 objects, and add and subtract mentally, whereas others have little or no knowledge of the number sequence, or of how to form small groups of objects. A longitudinal study of New Zealand children from ages five to nine found that the least capable children took about two years to reach the point which the most capable children had already reached on entry to school (see Young-Loveridge, 1991a). But, after two years at school, the "experts" had made further progress and by then were substantially better than the "novices" at other skills such as counting forwards in multiples of 5 of 10, or backwards in ones. They were also better at naming 3- and 4-digit numerals, explaining the meaning of equations, explaining the place value meaning of 1 in the tens position and 2 in the hundreds position, and recalling addition and subtraction number facts with single digit numbers. Two years further on, the "experts" were substantially better than the "novices" at multiplication and division problems (including the use of more sophisticated strategies, such as multiple counting or number fact retrieval, in preference to a simple "counting by ones" strategy), as well as operations involving time and money, and fractions of sets.

3. *The numeracy skills of five-year-olds are extremely stable*

Research shows that there is a strong tendency for children who are the *most* competent initially to be the *most* competent several years later (see Young-Loveridge, 1991a). Similarly, the children who are the *least* competent initially, tend to be the *least* competent several years later. Research with New Zealand children shows that more than half of the differences in mathematical skill and understanding at age nine could be explained by differences in children's numeracy skills at age five when they entered school (Young-Loveridge, 1991a). This pattern varied as a function of gender, with the numeracy skills of girls being substantially more stable than those of boys (ie, for girls, more than two-thirds of the differences at nine could be explained by differences at five, whereas for boys, just under a third of the differences at nine could be explained by differences at five). Although this is good news for girls who arrive at school with extensive numeracy skills, it is not so good for those girls who begin school with relatively few numeracy skills.

The picture is not all gloomy, however. An intervention study with five-year-olds (Early Mathematics Improvement Study: Five-year-olds - EMI-5s, Young-Loveridge, 1993) showed that it may be possible to change some of these patterns. Children who were selected for intervention participated in a programme which used number books and games to enrich the mathematical experiences of five-year-olds, either in a daily withdrawal session to play number games and read number stories (school-based programme), or one which combined home visiting with parent workshops and a lending library of number books and games (home-based programme). The pattern of results suggests that the intervention programmes may have been responsible for weakening the relationship between early and later numeracy (and hence in the proportion of explained variance), by bringing about improvement in many children's numeracy levels, but at varying rates. It was interesting to note that it was the girls for whom the differences as result of intervention were the greatest (the variance in later numeracy which could be explained by early numeracy was virtually halved for the intervention children compared with their contrasts).

Problems with Current Assessment Practices for Mathematics at School Entry

It is difficult to know just exactly what assessment procedures teachers are currently using at school entry. A survey of school entry practices undertaken late in 1991 (Thackery, Syme & Hendry, 1992) found that almost two thirds of schools surveyed used Beginning School Mathematics (BSM) day-to-day recording sheets and/or checkpoints (Ministry of Education, 1985). Examples given of the range of mathematical concepts which teachers looked for in new-entrant children include classification, size, order, pattern, colour, shape, position and movement - all concepts from BSM Cycle 1.

Less than ten percent of the sample mentioned using forms of assessment from a topic-based approach, including such numeracy skills as rote counting, enumeration and numeral recognition. Data collected in April 1992 showed that the majority of teachers sampled (35 out of 40) monitored their pupil's progress in mathematics using BSM checkpoints and/or day-to-day recording (Visser & Bennie, 1996), and that virtually all of the children in their classrooms were working on Cycle three or earlier. By October 1993, almost all of the teachers in Visser and Bennie's study (37 out of 39) were using some combination of BSM checkpoints and/or day-to-day recording to monitor their students' knowledge and skills in mathematics.

A major problem with using tasks from the first few cycles of BSM for school-entry assessment is that, unless a deliberate effort is made to sample skills at a variety of levels, information about a child's understanding of concepts presented in later cycles will probably be missed. The likely result is that many children's abilities will be underestimated and programmes will not be sufficiently challenging for more able pupils.

Furthermore, some of the tasks provided in BSM checkpoints are, like Piaget's own interview method, dependent on children's understanding of particular vocabulary. Children who fail a particular task may really understand the concept but are simply confused about what the interviewer means by the words in the question (Miller, 1982). In the conservation of number task, a transformation in the arrangement of the objects followed by further questioning may mislead the child into thinking that the reason for the adult asking the conservation question again is that the quantity has changed (McGarrigle & Donaldson, 1974). Take for example, the task for Checkpoint 5, Number, Objective 1. Seven cubes are dropped in a disordered array in front of the child and the child is then asked to count the objects and say how many there are. The same objects are scattered again, and the child is asked "how many are there now?". The implication of this latter question is that the number of objects has changed.

Another related problem with BSM checkpoints is that some of the language used to give the tasks for the checkpoints is of the "guess what's in my head" variety. For example, in Checkpoint 4, Number, Objective 4, the child has in front of him or her a stylised number pattern card and a set of the same number of objects. The child is asked "what is the same about that set and that card?". The acceptable answer is "both got four" or "same amount" or "same numbers". However, there might be other (irrelevant) similarities between the objects in the sets, such as shape or colour on which children comment. Research shows that, of the very high proportion of teacher utterances that are questions, very few are questions to which the teacher does not already know the answer, and often there is only one answer that is really acceptable to the teacher (eg, Wells, 1983). Wells suggests that certain children may be at a greater disadvantages than others in their unfamiliarity with this kind of "talk". When such "talk" is used to find out about children's mathematical understanding, it may create difficulties for certain children who don't know the acceptable answers to the questions. Asking children to *demonstrate* their understanding of a particular concept using objects might provide a more valid measure of their understanding than getting them to explain their reasoning.

Another problem with many of the BSM checkpoint tasks is that they are special "school" tasks which bear little resemblance to things people do in the world outside of school. This may be particularly problematic for children who have just arrived at school, and even more so for those who have not attended an early childhood centre prior to school entry.

Research shows that New Zealand teachers of new entrants can make quite accurate judgements about what their pupils know based on observations, although they do not necessarily use that information to improve their classroom programmes (Young-Loveridge, 1987b, 1989a). Similarly, Bennie and colleagues (1990) found that, despite the fact that most teachers said they used evaluation in order to establish the point at which children should enter BSM, only 16 percent of teachers at the new-entrant and J1 levels were using BSM levels beyond cycle 4. Data collected in April 1992 for another study showed that the majority of new entrant children sampled (91%) were taught mathematics using BSM, and that virtually all of these children were working on Cycle three or earlier (Visser & Bennie, 1996).

Another problem with the way that assessment tasks for early mathematics are used which Visser and Bennie (1996) identified was that almost all of the monitoring by teachers in their study appeared to be retrospective (backward looking) in nature, and measured what the students had learned (or already knew) by the end of a particular lesson or group of lessons. They found that teachers appeared to do little, if any, monitoring of children's *existing* knowledge and skills *prior* to planning a new topic (or cycle/module) in mathematics.

It is difficult to know just how much teachers have changed the ways they assess and teach mathematics since these surveys were done (eg, Bennie et al, 1990; Thackery et al, 1992; Visser & Bennie, 1996), and hence whether there is still a marked mismatch between what children know when they enter school and the kinds of activities which are being provided for them in their classroom mathematics programmes. Anecdotal evidence suggests that the development of a national curriculum statement for mathematics (MiNZC, Ministry of Education, 1992), in which the learning outcomes for all students are specified, has led to the development of programmes which may be more relevant to the learning needs of the children. Schools are certainly more accountable now for the programmes they provide, and are obliged to show how their programmes meet the needs of *all* learners.

What numeracy skills should be assessed at school entry?

There is now considerable research, both in New Zealand and overseas, which show the kinds of mathematical knowledge and skills which children entering school are likely to have (see Young-Loveridge, 1987a, 1987b, 1991a, 1993). Over the past decade or so, the numeracy skills of more than 500 New Zealand four- and five-year-olds have been studied (see Young-Loveridge, 1987b, 1988, 1989a, 1991a, 1993, 1995). In a longitudinal study of children between the ages of five and nine, correlations over time show which particular numeracy skills at school entry were most strongly predictive of children's success in mathematics later on in school (see Young-Loveridge, 1991a). The best predictors of children's later success in mathematics were forming sets, numeral recognition, pattern recognition, rote counting, sequence forwards and backwards, and addition and subtraction with imaginary objects. These seven types of tasks together accounted for between half and two thirds of the variance in levels of numeracy at the ages of six through nine.

When the findings for children who were more and less capable were examined separately, a slightly different pattern of predictors emerged (Young-Loveridge, 1991a). For the less capable children, the best predictors of latter performance were those shown by the sample as a whole (forming sets, rote counting, numeral recognition, pattern recognition and sequence forwards). However, for the more capable children, the best predictors of later performance were addition and subtraction with imaginary objects, identifying the number just before another number in the sequence (ie, sequence backwards), as well as numeral recognition, rote counting, sequence forwards, and forming sets.

The consequences of not assessing children's numeracy

Some writers argue that tests encourage teachers to adjust their teaching to match what is on the tests (eg, Kamii, 1990). Taken to its absolute extreme, this argument means never trying to find out what children actually know for fear of discovering that the classroom programme might need to be modified to match the needs of certain children better. What is important is not whether or not children are assessed, but the nature of those assessments and the purposes for which they are used. There is a delicate balance to be found between doing no assessment and subjecting children to a barrage of inappropriate procedures.

If children's numeracy skills on entry to school are not assessed appropriately, then less capable children may continue to struggle with mathematics because their particular needs are not identified (see Young-Loveridge, 1991a). They may fall further and further behind their peers in mathematics (particularly if they are girls), and develop negative attitudes towards maths which may make it impossible for them to benefit from help that

might be offered later in their school years. More capable children may also lose interest in mathematics because their need for challenge is not being met.

The Development of the Shopping Game *Checkout/Rapua*

Checkout/Rapua was developed as an assessment tool for mathematics at the school entry level (see Young-Loveridge, 1994, 1997; Ministry of Education, 1997). It is based on a collection of tasks which have been used as research tools in three large studies by the author and by several other researchers (eg, Peters, 1991; Visser & Bennie, 1996; Wylie et al, 1996; Young-Loveridge, 1987b, 1989a, 1991a, 1993, 1995). Over the past decade or so, some of the tasks have been changed slightly, but each of the main types of tasks (numeral recognition, pattern recognition, forming sets, rote counting, sequence forwards and backwards, and addition and subtraction with imaginary objects) has been included in the various versions of the task-based interview.

In 1991, at the completion of a longitudinal study of children in Christchurch, an abbreviated "test" version called the School Entry Numeracy Skills Interview (SENS) was developed and made available to interested teachers (Young-Loveridge, 1991b). In 1994 a "game" version was developed (The Supermarket Maths Game) in which the tasks were embedded within the context of a board game based on supermarket shopping (see Young-Loveridge, 1994). Since then the game version has undergone further development, with the help of teachers in Hamilton and Wellington, resulting in the most recent version of the Game, described in this paper (*Checkout/Rapua*).

The current version of *Checkout/Rapua* is now somewhat different from the version that was originally proposed (see Young-Loveridge, 1994). A decision was made to eliminate the board and the dice throws, because the "moves" were difficult and frustrating for some children and slowed down the assessment process substantially. *Checkout/Rapua* now consists primarily of cards, objects, and shopping baskets. At each turn, the child picks up a card and follows the instructions on the card (depicting either numerals and/or pictures of the objects, or words which the teacher reads out to the child), getting a certain number of objects from the supermarket to put into the shopping basket in the process. Numeracy skills which are revealed by *Checkout/Rapua* include getting a certain number of objects (*forming sets*), naming numerals (*numeral recognition*), naming stylised number patterns (*pattern recognition*), saying the sequence of number names in order (rote counting), and identifying the number just after or just before another number in the counting sequence (sequence forwards and backwards, together with rote counting referred to *number sequence knowledge*), and joining and separating groups of imaginary objects (addition and subtraction with imaginary objects, together referred to as *mental operations*). The objects used in the game (many of which are miniature versions of objects found in real life) have high levels of familiarity and appeal for children. They include carrots, bananas, sausages, birthday cards and candles, sponges, pegs, buttons, nuts, dice, and notebooks. These have been selected because of their varying properties of shape, colour, texture, and weight, and because they can be used to assess mathematical concepts other than those included in the game. For example, the nuts might be put in order of size, or weighed. Buttons can be sorted in various ways, including by size and colour. Sandwiches can be put together to form a square or oblong. The dice can be ordered by set size from one to six. The names and properties of differently shaped objects can be discussed (including circle, square, oblong, hexagon, triangle). Pegs or candles can be used as non-standard units of measurement. The birthday cards can be ordered from 1 to 12, or matched with the appropriate number of candles, and so on. These concepts can be explored before, during, or after playing the game itself. These options allow for diversity in assessment practices and encourage a broadening of the curriculum. The numeracy record sheet provides space for the teacher to note how the child goes about the task. A list of faulty procedures is included on the record sheet to alert teachers to common problems they might need to look out for.

An important development with *Checkout/Rapua* has been the introduction of alternative ways to play the game. These include two children playing together while the teacher observes, and one child playing against the teacher (helping the teacher with her turn while the teacher observes). (Note: a version of the game in which one child goes through all the activities alone while the teacher observes was trialed during the

development phase, but is not recommended because some of the important "game" elements, such as turn-taking, are missing and it is more like a test than a game.) The version in which one child plays against the teacher, but in helping the teacher with her turns actually completes all of the activities, is substantially quicker than the two-child version, and does not require that a pair of children be available to play. This one-child version has obvious advantages for rural schools where new entrants often arrive at widely spaced and irregular intervals, and in other schools where there is difficulty pairing new entrants who have comparable skills.

The current version of *Checkout/Rapua* takes about 10 minutes to use with one child, and abut 20 to 25 minutes with two children at a time. *Checkout/Rapua* yields valuable diagnostic information as well as information of predictive significance, including information about performance and strategies used for individual tasks. Scores for each type of task can be aggregated, allowing an examination of strengths and weaknesses in particular areas (eg, forming sets, numeral recognition, pattern recognition, number sequence knowledge, mental operations). It is also possible to calculate a total score for the purpose of comparing children within a class in terms of overall number skills. Data from earlier versions has been used to provide broad indicators, with children in the upper quartile described as experts in using numbers for their age, children in the two middle quartiles described as quite competent in using numbers, and those in the lower quartile described as novices in using numbers. *Checkout/Rapua* was designed to yield a large amount of information, including things about personal and social development, as well as other aspects of mathematics such as measurement, geometry, statistics, algebra, and mathematical processes, in addition to numeracy.

Important Features of *Checkout/Rapua*

1. Child-centred

Checkout/Rapua gives the children the opportunity to collect up a variety of objects with high levels of appeal. The children dictate the nature and pace of the game. Their enjoyment of the game takes precedence over the information-gathering purpose for which the game has been developed.

2. Meaningful context

Because *Checkout/Rapua* is set within a supermarket context, it should be a reasonably familiar and meaningful context to most children. The items in the supermarket have been chosen as objects which most, if not all, children have experienced in their everyday lives.

3. Process as well as product

Children's strategies and approaches to particular problems in *Checkout/Rapua* are just as important as whether or not they give the correct responses. Valuable diagnostic information, which can assist teachers in planning programmes, is provided by the game.

4. Usefulness to teachers

Checkout/Rapua has the flexibility to allow the inclusion of additional concepts and skills which teachers regard as important. The objects in the game have been deliberately chosen because of their potential for assessing other mathematical concepts and skills besides those included in the game itself. Teachers will be able to make decisions about what other concepts and skills are important, and use objects in the game to assess those.

5. Based on research

The five main types of tasks used in *Checkout/Rapua* have been developed on the basis of extensive research, both in New Zealand and overseas. There is substantial data about the appropriateness and usefulness of each of the tasks from more than 500 New Zealand children in the four- to five-year-old age group, including data on the levels of difficulty, discrimination, and predictive validity of almost all of the tasks (see Young-Loveridge, 1994). The reliability of the tasks used in the Game is high (.95 to .96: for details, see Young-Loveridge, 1987b, 1989, 1991a, 1993).

6. Unobtrusive

The recording process is intended to be as inconspicuous as possible. Teachers are encouraged to record as much as possible of what the children do and say, and to avoid putting ticks until after the activities have been completed.

At the time of writing, the professional development programme designed to train teachers in the use of *Checkout/Rapua* (as part of the School Entry Assessment package) had only just begun. The distribution of the resource to all primary schools in New Zealand in conjunction with the professional development programme will take some time. Further research is needed on the usefulness of the game for teachers and the ways they adapt it to suit their purposes.

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SHORT COMMUNICATIONS (& POSTER PRESENTATIONS)

Mary Barnes & Gaye Williams, University of Melbourne

"Magical" moments in mathematics: insights into the process of coming to know

Students in a senior mathematics classroom were videotaped, and some were later interviewed, as they worked collaboratively on challenging problems. The paper's title comes from a remark by one student about what is sometimes called the "Aha"!" experience. We present examples of "magical" moments, and conclude with some hypotheses about why such moments appear to happen fairly frequently in this particular classroom.

George Booker, Griffith University—Nathan

Number sense vs number skills: children s use and interpretation of mathematics in the upper primary school

This paper reports on research which investigated classroom contexts and activities that might promote the development of mathematics as problem-focussed and sense-making activity. Situations were presented which were judged likely to foster the communication of mathematical ideas, provide processes which might initiate and guide the growth of problem-centred learning, and bring about changes in student's mathematical thinking from a procedural and answer dominated focus into one in which making sense of. about and with mathematics come to the fore.

Gillian Boulton-Lewis, Tom Cooper, Bill Atweh, H Pillay & Lynn Wilss, Queensland University of Technology

A cognitive perspective on the transition from arithmetic to algebra

This paper is a discussion of development of algebra knowledge, a mathematical sequence for learning complex equations, strategies used to teach early algebra, the understanding of a sample of 7th grade students of arithmetic and prealgebra, the strategies they used and the implications for teaching algebra.

Coralie Daniel & John Curran, University of Otago

A profile of top achievers in a primary school mathematics problem solving scheme

A survey of the top 114 Form 1 achievers in the New Zealand 1994 Problem Challenge competition indicated that these students came from a range of economic and educational backgrounds; were found in most types of schools; enjoyed problem solving and challenging maths more than their teachers thought they did; were competent readers; had relatively wide interests; and were cooperative, enthusiastic, and thoughtful in the classroom but did not exhibit noticeable confidence or the desire to work independently.

Freda Goodall, Victoria University of Wellington

Mathematics education in New Zealand

By overseas standards our mathematics education results are unsatisfactory. (IEA 1987, Second International Mathematics Survey 1991). Research over the past fifty years points to the weakness being due to the dearth of qualified mathematics teachers in both primary and secondary schools (Werry B, 1980; Goodall F G, 1993). In order to help teachers and give them a tool so they may diagnose a student's weakness and strength in mathematics the researcher has created individualised diagnostic tests in mathematics which range in an orderly progression from 7 years to 14 years.

Evan Harris, Australian Catholic University—McAuley

Constructing initial algebraic understanding: a methodology

Many studies into the development of initial algebraic understanding have used a methodology that consisted of testing large groups of students. This paper presents a proposed methodology to be used as part of a study which is exploring how students construct initial algebraic understanding. The in-depth analysis of students' thinking is intended to complement the findings of such large scale research projects. The methodology employs a naturalistic inquiry in which the case study approach is used.

Georgina Herbert, Deakin University—Geelong

Assessment of young children in mathematics

The educational climate in Australia is increasingly concerned with, and driven by, issues of accountability and results or *student outcomes*. My investigation into the assessment of lower primary children in mathematics seeks to gain a picture of assessment practices that are in use, and to explore the ideas and influences that have shaped teachers' decisions in this area. During this session I will discuss my research plan and the early stages of the project and seek comments and discussion from participants.

Ruth Hubbard, Queensland University of Technology

The tasks we give students and their impact on learning

This paper is a wish list of aspects of a topic I'd like investigated. The topic is the use of large numbers of repetitive exercises at all levels of mathematics instruction. Why are they so widespread? Are they essential for learning? Do they improve conceptual understanding? What do they achieve? Do they hinder understanding? What effect do they have on attitude to and feelings about maths. Do they distort the view of what maths is? Do they motivate students to use maths outside the classroom? Finally, can we find other activities and demonstrate that they are more effective at achieving our goals

Anthony Jones, La Trobe University

Young children graphing with technology: producing and understanding

Computer technology has given primary school students access to powerful graphical tools. Some mathematics curricula now encourage student use of spreadsheets and other graphical software. This paper examines one primary teacher's attempts to provide her grade 1 and 2 students with experiences in collecting, recording and graphing data using a spreadsheet program.

W T Kaleva, Monash University

Secondary teacher beliefs and practices about mathematics in the PNG context

Official Papua New Guinea (PNG) Government education policies encourage a "community oriented and culturally based" education and curriculum. This paper reports on the research which examined Papua New Guinea secondary teacher beliefs about various aspects of mathematics and culture. The research shows that even though teachers' expressed beliefs are clearly in line with government policies on the need for a culturally based curriculum, these beliefs are not necessarily translated into classroom practice.

Francis Kari, Monash University

Bridging mathematical cultures for adult distance learners: a case study concerning beliefs from Papua New Guinea

This paper describes some of the results of a survey conducted with mathematics students who are enrolled in the matriculation programme offered by the University of Papua New Guinea Institute of Distance and Continuing Education. The particular issue discussed in this paper concerns the existence of mathematical ideas in home cultures, and whether or not these ideas are rationally related to formal mathematics learning situations.

Sergiy Klymchuk, University of Waikato

A comparative analysis of mathematical courses taught to university economics students in the former USSR and New Zealand

This paper discusses some advantages and disadvantages of 5 aspects of the teaching and learning of mathematics courses for university economics students in the former USSR and NZ. The aspects are: curricula, professional development, assessment, language, ethics and evaluation of the lecturing. The author spent 16 years teaching mathematics to university economics students in Ukraine, visited universities in Eastern Europe, spent sabbatical at the University of Waikato, and is now researching effective teaching in such courses.

Tom Lowrie, Charles Sturt University

The use of imagery in primary school mathematics: a case study

Visual imagery in thought and memory has been viewed from a variety of perspectives in education. In this paper the view is taken that a subject is more likely to use visual means in information processing if they are unfamiliar with a task and have no symbolic procedures readily available to handle the task. During three years the types of imagery evoked by primary school children were examined. It was found that different types of imagery serve different purposes for students solving mathematics problems.

Pramote Markshoe, Naresuan University, Thailand

Strategies in solving mathematics problems of the ninth-grade students in the Lower Northern Region of Thailand

This study investigated the computation strategies in solving mathematics problems of grade 9 students in five provinces of the lower Northern region of Thailand. Four hundred students in 15 public secondary schools were selected for the study. A subjective test composed of 10 verbal problems adapted from EMQ test, a video-taped recorder, and interviews were employed to gather the data. The findings were: 14.95 percent of the students preferred to use definable strategy, 47.31 percent preferred to use propertied strategy, 37.37 percent preferred to use definable and propertied strategy, and 0.37 percent preferred to use independent strategy.

Beth Marr, Royal Melbourne Institute of Technology

It's not what you do, it's the way that they view it - differences between adult learners in mathematics/numeracy classes.

This presentation will explore the differences observed in TAFE adult mathematics/numeracy learners. Research aimed at studying differences in mathematical talk generated by a variety of learning activites, has instead revealed differences more dependent on the individual learner's view of mathematics, learning and their own self perception, than on the style of mathematical activity. Characteristics of the learners' participation tended to remain consistent through problem solving and discussion in groups and pairs, writing exercises, individual practice worksheets and hands on experiences.

Colleen McMurchy-Pilkington, Auckland College of Education

Maori students and pedagogical practices in mathematics classes implications for all maths teachers

Maori women teacher trainees in this study recall their schooling experiences and relate how the pedagogical practices contradict the socialisation of their home and preferred way of learning. This study suggests, in line with Walkerdine (1989), that the pedagogical and social practices of teachers contribute to the positioning of women, particularly Maori women, as not mathematically competent or confident.

Ramaiya Naidu, Te Kauwhata College

Ways that graphics calculators might contribute to learning and understanding of calculus concepts

Calculus has long been a difficult subject for many students. Graphic calculators offer a way for mathematics teachers to improve their students' involvement in constructing their own understandings of calculus concepts. Participant observation with interviews, conversations, field notes and journals will be the major means of collecting data. In the first phase of the study a unit of course in year 13 calculus using graphics calculators will be prepared and trialed. In the second phase the program will be implemented and data collected.

Stephen Norton, Queensland University of Technology
Leonie Rennie, Curtin University of Technology

Sex-role stereotypic attitudes of secondary mathematics students in single-sex and coeducational schools

The Fennema-Sherman Mathematics as a Male Domain Scale was used to examine the attitudes of secondary school boys and girls from grades 8 to 12 in single-sex and coeducational schools. There were clear attitude differences between girls and boys, with girls being less stereotyped in their perceptions than boys. There were also differences relating to the school environment with girls in a single-sex schools being less stereotyped than girls in the coeducational schools.

Tamsin Roberts, Maningrida CEC

Mathematics curriculum for indigenous students

The mathematics curriculums used in remote Aboriginal communities in the Northern Territory are the same as those for mainstream urban students. My PhD will involve the trialling of a mathematics curriculum framework which could be used by individual Aboriginal communities to develop a curriculum which reflects their needs and aspirations. The framework consists of a range of issues that each community must consider. These include deciding on what maths to include, what methodology to use, how to assess, how to include teacher professional development, what resources to use and the choice of the language of instruction. The research will involve interviewing participants in the appropriateness of the issues chosen and their presentation within the framework.

Deborah Scott, University of New South Wales—St. George Campus

Decreasing the incidence of the I'm hopeless at maths syndrome: is number sense the key?

There increasing body of literature (both research and popular) highlights the failure of the traditional school system to develop effective numeracy skills for the majority. Despite a growing awareness of the important role of attitudes in mathematics learning, at best, a fairly small minority of students enter high school with mathematical confidence. Recent research suggests that this trend can be reversed but the question of how to marry innovative approaches with long standing traditions remains.

Dianne Siemon, Royal Melbourne Institute of Technology

Representing decimals - an unconscious marriage of convenience

A discussion with teacher education students about their performance on a mathematics test revealed what appeared to be a relatively common misconception in relation to the ordering of decimal fractions (approximately 10% of commencing students in 1988/9). Since then a number of experienced teachers have admitted that they too shared the same construction. This is worth reporting for two reasons. Firstly, the means by which the misconception was identified has important implications for traditional constructivist research methodologies, and secondly, the nature of the construction, a marriage of unrelated models on the basis of a perceptual similarity, has implications for future research on the understanding of place-value and the use of number lines in the teaching, learning and application of number ideas

Dianne Siemon, Royal Melbourne Institute of Technology

Visual learning the contribution of part-part-whole ideas to an understanding of number

Much post neo-Piagetian research on children's early number ideas has focussed on counting and children's use of symbols to indicate the numerosity of a collection. While this has expanded our understanding of the difficulties involved in counting and coming to an understanding of number in terms of countable units, it's focus on conscious, verbal behaviour has tended to overshadow other forms and modes of learning such as unconscious, perceptual learning. This paper will explore the role of visual text in relation to young children's emerging ideas about number.

Ron Smith, Peter Grover & Russell Tytler, Deakin University

Professional development: a case study of teachers involved in a change program in the topic of probability

This research compares two Victorian teacher professional development programs: Learning in Primary Science (LIPS) and Mathematics in Schools (MIS). Coding of responses showed that LIPS teachers valued the development of activities and discipline knowledge more than the MIS teachers. However, NUD•IST coding of interviews has shown that the schools focussing on probability have taken an approach similar to that of science, where teachers responded favourably to specific curriculum content knowledge and activities for the classroom. Possible reasons are considered.

Tan Boon Tee, Universiti Brunei Darussalam

Preservice primary teachers conceptual difficulties with mathematics in physics

Numerous preservice primary student teachers complain that they find physics numerical problems difficult. Apparently, it is not the incorrect choice of algebraic formula that is the root of the difficulty, but rather the inability or weakness in manipulating the algebra leading to the answer. To find the extent that student teachers have mastered the basic maths skills required in solving physics problems, a test consisting of 19 mathematical cognitive tasks was administered to 108 student teachers. The result left much to be desired. Many still lacked the skills to solve numerical problems in physics

Steve Tobias & Sally Oakes, La Trobe University—Bendigo

A classroom teacher's perceptions of a collaborative teaching practicum

This paper discusses a teacher education program that promotes teaching and learning approaches consistent with current reforms in mathematics education. It aims to encourage a reconceptualization of teaching and learning mathematics via intensive and fundamental classroom experiences centred on social-constructivist approaches. Collaborative interaction, as a 'classroom-team', is expected between the classroom teacher and two preservice teachers while planning, implementing and reflecting on classroom teaching sessions. The experiences and incidences that cause one classroom teacher to reconceptualize her beliefs about teaching and learning mathematics are reported.

Gurudeo Anand Tularam, Queensland University of Technology

Interaction effects between algebraic knowledge, higher-order thinking, and affective factors in algebraic word-problem solving

This study examined the interaction effects between algebraic knowledge, higher-order thinking and affective factors in novel algebraic word-problem solving. The findings indicated that the 3 factors were related to performance and higher-order thinking seemed to have the most effect. Algebraic knowledge appeared to be a stronger predictor of success than affective factors, but students' beliefs, attitudes, persistence and confidence played an important role in problem solving. It seems that for students to perform well they need to demonstrate high levels in all the factors studied.

Gurudeo Anand Tularam, Queensland University of Technology

The role of algebraic knowledge, higher-order thinking, and affective factors in algebraic word-problem solving

This study examined the effect of algebraic knowledge, higher thinking and affective factors on performance on novel algebraic word-problem solving. The performance of the year 11 high school students' seemed unsatisfactory. Three factors appeared to influence performance—algebraic knowledge was one predictor of success; metacognition and critical thinking were related to performance; and students' beliefs, attitudes, persistence and confidence played a role. For students to solve algebraic word-problems satisfactorily or better, they need high levels in all three factors.

Boguslawa Welna, Murdoch University

Errors, history and humour as a tool in learning and teaching maths

There are several reasons for humorous component in mathematics lessons. One of them is that humour has a power to dispel fear and anxiety about mathematics which are still among our students. Fear and anxiety in most cases occur when there is limited or no understanding of mathematical ideas. The natural question arises. Does humour facilitate the understanding of mathematical ideas? The poster presents one part of a larger study of the nature and significance of errors for learning and teaching mathematics. It is illustrated how errors, history and humour altogether can be used as a powerful aid in overcoming cognitive and epistemological obstacles in order to attain a deeper understanding of mathematical ideas.

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